The relation between the numbers M_r and B_j

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In our recent study [1] on alternate sums of powers of integers, i.e. of sums of the type

$$ {}_rZ_n \equiv \sum_{j=1}^{n} (-1)^j \, j^r , \quad \text{with } r = 1, 2, \ldots , \tag{1} $$

it was shown that they can be expressed in the form

$$ {}_rZ_n = \sum_{j=1}^{r-1} {}_r\mu_j \, t^j + \frac{1}{2} (-1)^n (2t)^r , \tag{2} $$

where $t = \left[\frac{n+1}{2} \right]$ is given by the number of terms in (1), while ${}_r\mu_j$ are coefficients that have been tabulated.

We found that these coefficients can be expressed by the "basic" integers $M_r \equiv {}_r\mu_1$, since

$$ {}_r\mu_j = \frac{2^{j-1} \, r!}{j! \, (r-j+1)!} \, M_{r-j+1} , \quad \text{for } r > j . \tag{3} $$

The alternate sum (1), written in powers of t, thus begins for r even with

$$ {}_rZ_n = M_r \, t + {}_r\mu_2 \, t^2 + \ldots , \tag{4} $$

whereas for r odd there is no term proportional to t.

It appears that the numbers M_r play a role similar to that of the well-known Bernoulli numbers B_j which occur in the study of the sums

$$ {}_rS_n \equiv \sum_{j=1}^{n} j^r . \tag{5} $$

Some similarities between the numbers B_j and M_r have already been noted in [1], but it has not been possible to establish a clear link between the two series. It is the purpose of the present Note to make up for this deficiency. Some familiarity with [1] will make the reading easier.

The idea is to use the fact that the numbers M_r are the coefficients of t, as shown in (4). If we can subdivide ${}_rZ_n$ into a sum of contributions of type ${}_rS_n$ and determine the coefficients of the linear terms t (in which the Bernoulli numbers occur), it must be possible to obtain the required relation by comparison of the respective coefficients.

Let us begin by decomposing the alternate sum (1) in a suitable way - the very rearrangement we tried to avoid in [1]. Since n is even, we can put n/2 = t. Then

$$
\begin{aligned}
{}_rZ_n &= \sum_{j=1}^{t} (2j)^r - \sum_{j=1}^{t} (2j-1)^r \\
&= 2^r \sum_{j=1}^{t} j^r - 2^r \left(\tfrac{1}{2}\right)^r \sum_{j=1}^{t} \sum_{k=0}^{r} \binom{r}{k} (-2)^k j^k \\
&= 2^r\, {}_rS_n - \sum_{k=0}^{r} \binom{r}{k} (-2)^k\, {}_kS_n \\
&= - \sum_{k=0}^{r-1} \binom{r}{k} (-2)^k\, {}_kS_n\,. \qquad (6)
\end{aligned}
$$

Let us now look at the development of a Bernoullian sum ${}_rS_n$. From relation (23) given in [1] we conclude that, for $r \geq 2$ and even,

$$
{}_rS_n = \ldots + \frac{1}{r}\binom{r}{r-1} B_r\, t^{r+1-r} = \ldots + B_r\, t\,. \qquad (7)
$$

We note in passing that ${}_rS_n$ with r odd has no terms proportional to t as the development stops at t^2, exactly as for ${}_rZ_n$.

The two cases with r = 0 and r = 1 appearing in (6) have to be treated separately. One finds

$$
{}_0S_n = t
$$

and (8)

$$
{}_1S_n = \frac{1}{2}t + \frac{1}{2}t^2\,.
$$

A look at (4) shows that the value of M_r can be obtained from (6) by assembling the coefficients of t appearing in ${}_kS_n$. Writing (6) as

$$
{}_rZ_n = -\,{}_0S_n + 2r\,{}_1S_n - \sum_{k=2}^{r-1} \binom{r}{k} (-2)^k\, {}_kS_n\,, \qquad (9)
$$

we find, with (7) and (8),

$$
\begin{aligned}
M_r &= -1 + 2r\frac{1}{2} - \sum_{j=2}^{r-1} \binom{r}{j} (-2)^j B_j \\
&= r - 1 - \sum_{\substack{j=2 \\ \text{(even)}}}^{r-2} 2^j \binom{r}{j} B_j\,. \qquad (10)
\end{aligned}
$$

This is the relation looked for. It shows that the two series of numbers M_r and B_j are indeed closely linked, and relation (10) is even somewhat reminiscent of the recurrence formula (18) found previously in [1]. It may be worthwhile noting that the sum in (10) yields an even integer.

Let us check (10) with three practical applications.

- For = 8 :

$$M_8 = 8 - 1 - \sum_{j=2}^{6} 2^j \binom{8}{j} B_j$$

$$= 7 - \left[2^2 \binom{8}{2} B_2 + 2^4 \binom{8}{4} + 2^6 \binom{8}{8} B_6 \right]$$

$$= 7 - \left[\frac{4\ 28}{6} - \frac{16\ 70}{30} + \frac{64\ 28}{42} \right] = -17 ;$$

- for r = 10 :

$$M_{10} = 10 - 1 - \sum_{j=2}^{8} 2^j \binom{10}{j} B_j = \ldots = 155 ;$$

- for r = 12 :

$$M_{12} = 11 - \sum_{j=2}^{10} 2^j \binom{12}{j} B_j = \ldots = -2\ 073 .$$

All these results agree with the numerical values given in [1].

Obviously, the existence of the new relation (10) does not mean that the numbers M_r become superfluous; their practical usefulness is obvious in [1]. In any case, (10) is a very useful tool for their numerical evaluation, since it is a simple relation making use only of the Bernoulli numbers, which are readily available in tabular form, for example up to B_{60} in [2].

References

[1] J.W. Müller: "Sums of alternate powers - an empirical approach", Rapport BIPM-94/14 (1994)

[2] "Handbook of Mathematical Functions", ed. by M. Abramowitz and I.A. Stegun, NBS, AMS 55 (GPO, Washington, 1964)

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