

## Bureau International des Poids et Mesures

# How to define the base units of the revised SI from seven constants with fixed numerical values

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See Reference 7

February 2018

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## 1. Introduction

Preparations for the upcoming revision of the International System of Units (SI) began in earnest with Resolution 1 of the 24th meeting of the General Conference on Weights and Measures (CGPM) in 2011 [1]. The 26th CGPM in November 2018 is expected to give final approval to a revision of the present SI [2] based on the guidance laid down in Ref. [1]. The SI will then become a system of units based on exact numerical values of seven defining constants,  $\Delta\nu_{\text{Cs}}$ ,  $c$ ,  $h$ ,  $e$ ,  $k$ ,  $N_{\text{A}}$  and  $K_{\text{cd}}$  exactly as specified in the following bullets:

- the unperturbed ground state hyperfine transition frequency of the caesium 133 atom  $\Delta\nu_{\text{Cs}}$  is 9 192 631 770 hertz,
- the speed of light in vacuum  $c$  is 299 792 458 metres per second,
- the Planck constant  $h$  is  $6.626\,070\,15 \times 10^{-34}$  joule second,
- the elementary charge  $e$  is  $1.602\,176\,634 \times 10^{-19}$  coulomb,
- the Boltzmann constant  $k$  is  $1.380\,649 \times 10^{-23}$  joule per kelvin,
- the Avogadro constant  $N_{\text{A}}$  is  $6.022\,140\,76 \times 10^{23}$  reciprocal mole,
- the luminous efficacy of monochromatic radiation of frequency  $540 \times 10^{12}$  Hz,  $K_{\text{cd}}$ , is 683 lumens per watt.

The hertz, joule, coulomb, lumen, and watt, with unit symbols Hz, J, C, lm and W, respectively, are related to the seven base units: second, metre, kilogram, ampere, kelvin, mole and candela, with unit symbols s, m, kg, A, K, mol and cd, respectively, through the relations  $\text{Hz} = \text{s}^{-1}$ ,  $\text{J} = \text{kg m}^2 \text{s}^{-2}$ ,  $\text{C} = \text{A s}$ ,  $\text{lm} = \text{cd sr}^\dagger$  and  $\text{W} = \text{kg m}^2 \text{s}^{-3}$  (see Ref. [2]). Only so-called “coherent” units [2] are used here and in the following. This means that, just as in the bullets, we will not affix numerical prefixes (such as mega or nano) either to the base units or to combinations of base units that have special names (such as joule or watt).

The numerical values of  $\Delta\nu_{\text{Cs}}$ ,  $c$  and  $K_{\text{cd}}$  given in the bullets have been fixed (defined to be exact) since 1967, 1983 and 1979 respectively [2]. It was premature in 2011 to specify exact values for  $h$ ,  $e$ ,  $k$  and  $N_{\text{A}}$  because their experimentally-determined values were not yet known with sufficiently small uncertainty to assure a smooth transition to the proposed new definitions. That has changed, and the numerical values given above are those recommended in October 2017 by the CODATA Task Group on Fundamental Constants [3]. They have been accepted by the CIPM and are expected to be confirmed by the CGPM at its next meeting, in November 2018.

This note describes an efficient method to convert the information contained in the seven bullets to definitions of the SI base units, which are, not coincidentally, seven in number [2]. The reasons for this particular choice of defining constants are important but have been presented elsewhere [4].

Using a two-step process, we derive the seven combinations of defining constants whose unit is a base unit of the SI. The algorithm used results in an easy-to-read table. The exact numerical values given in the bullets are then introduced to complete the definitions of the base units. The Appendix

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<sup>†</sup> sr is the symbol for steradian, the unit of solid angle. Although  $\text{sr} = \text{m}^2/\text{m}^2 = 1$ , sr is used when needed for clarity [2].

applies the same method to the present SI, illustrating the method’s generality as well as providing a novel contrast to the upcoming revision. Each base unit will be defined independently of the others, although typical derivations take a different approach.

If all seven base units of the SI can be defined in terms of the seven defining constants, an obvious but important corollary follows: **All SI units can be defined in terms of the seven defining constants.** The distinction between base and derived units remains useful, but not essential for many purposes.<sup>‡</sup>

The following method is consistent with a more rigorous analysis provided by Mohr in 2008 [6], which the interested reader is encouraged to consult.

## 2. Defining constants written in terms of base units

We begin with Table 1, which presents much of the information given in the bullets of section 1 in more usable form. Note that the defining constants are shown as labels in the first column. The four new defining constants and the base units which are redefined in consequence are shown in red. The units of the defining constants can be expressed as the product of powers of the base units,  $s^\alpha m^\beta \text{kg}^\gamma \text{A}^\delta \text{K}^\epsilon \text{mol}^\zeta \text{cd}^\eta$  [2], as specified in the bullets of section 1. The exponents that are required appear in the rows of Table 1 for each of the defining constants; for example, the coherent unit of the Planck constant  $h$  is  $\text{J s} = \text{kg m}^2 \text{s}^{-1}$ , so that for the row labelled  $h$  the exponents  $\alpha$  through  $\eta$  are (-1, 2, 1, 0, 0, 0, 0). The columns show whether a unit appears in a particular bullet. We see, for instance, that the second appears in the unit of every constant except  $N_A$ , but with four different exponents.

	<b>s</b>	<b>m</b>	<b>kg</b>	<b>A</b>	<b>K</b>	<b>mol</b>	<b>cd</b>
$\Delta\nu_{\text{Cs}}$	-1	0	0	0	0	0	0
$c$	-1	1	0	0	0	0	0
$h$	-1	2	1	0	0	0	0
$e$	1	0	0	1	0	0	0
$k$	-2	2	1	0	-1	0	0
$N_A$	0	0	0	0	0	-1	0
$K_{\text{cd}}$	3	-2	-1	0	0	0	1

Table 1.

The sequence of the seven defining constants in the left column of labels follows the order in which they are presented in the CGPM Resolution and in section 1. The sequence of base units in the top row of labels follows the order in which these units are defined in Ref. [4]. This results in a table where the all exponents above the diagonal cells are all zero (as are most exponents below the diagonal). The diagonal cells (those with a violet background) associate each defining constant with a unique base unit. The cells with a yellow background show that “helping” units are also needed. For instance, the Planck constant  $h$  is key to redefining the kilogram (violet cell), but the unit of  $h$  also contains the inverse second and the metre squared (yellow cells).

<sup>‡</sup> Even early editions of the SI Brochure remarked that separate classes of base and derived units are “not essential to the physics of the subject” [5], but added that the classifications were useful, considering the goal of “a single, practical, worldwide system [of units] for international relations, for teaching and scientific work”.

### 3. Base units as defined by the “defining constants”

The seven SI base units can be defined in terms of the seven defining constants. To do this, we create a second table, Table 2, that shows the combination of defining constants required to define each base unit. Table 2 is the major contribution of this report.

The numbers in Table 2 are also exponents, this time used to show the combination of defining constants (labelled in the top row) that has the same unit as each base unit (labelled in the left column). Except for an exact scaling factor, which is easily derived as shown below, each base unit is defined by the product  $\Delta v_{Cs}^{\alpha'} c^{\beta'} h^{\gamma'} e^{\delta'} k^{\epsilon'} N_A^{\zeta'} K_{cd}^{\eta'}$ , where the required exponents for each row appear in the table. If an exponent is zero, it means that its constant is not needed, and its cell, though containing zero, has been left blank for visual clarity. Each column shows which defining constants are needed in the definition of the base units. We see that  $\Delta v_{Cs}$  is needed to define six of the seven base units (using three different exponents),  $c$  is only needed to define two base units, etc.

All exponents have been derived from Table 1 in one step, by the following mathematical operation. Note that the cells containing numbers in Table 1 constitute a  $7 \times 7$  matrix. Invert that matrix using, for example, the MINVERSE command in Excel. The inverse obtained is the  $7 \times 7$  matrix of exponents shown in Table 2.<sup>§</sup> Remember that the blank cells actually contain zero in the inverted matrix.

	$\Delta v_{Cs}$	$c$	$h$	$e$	$k$	$N_A$	$K_{cd}$
<b>s</b>	-1						
<b>m</b>	-1	1					
<b>kg</b>	1	-2	1				
<b>A</b>	1			1			
<b>K</b>	1		1		-1		
<b>mol</b>						-1	
<b>cd</b>	2		1				1

Table 2. The revised SI [4]. Blank cells all contain zero (not displayed).

From the “kg” row of Table 2 we may infer that the following combination of three defining constants has the kilogram as its unit [6]:

$$\Delta v_{Cs}^1 c^{-2} h^1 e^0 k^0 N_A^0 K_{cd}^0 = \Delta v_{Cs} c^{-2} h = \frac{\Delta v_{Cs} h}{c^2} . \quad (I)$$

In the revised SI, all mass determinations must ultimately be traceable to this quantity because its numerical value in kilograms has been fixed. The exact values of  $\Delta v_{Cs}$ ,  $c$  and  $h$  given in the bullets of section 1 provide the fixed value. Substituting the information in the first three bullets into the left side of the following equation, which is a combination of physical constants, gives us the right side, which is the value of the combination in the revised SI:

<sup>§</sup> The  $7 \times 7$  squares of numbers in Tables 1 and 2 are called “lower triangular matrices” because all numbers above the diagonal are zero. The inverse of a triangular matrix is triangular as well, provided that the inverse exists. The inverse exists if and only if none of the numbers in the diagonal cells is zero.

$$\frac{\Delta\nu_{\text{Cs}} h}{c^2} = \frac{(9\,192\,631\,770)(6.626\,070\,15 \times 10^{-34})}{(299\,792\,458)^2} \text{ kg}.$$

The numbers in parentheses are obviously the exact numerical values of  $\Delta\nu_{\text{Cs}}$ ,  $c$  and  $h$  specified in section 1. The base units associated with these numerical values cancel out (unit symbols can be treated algebraically)—except for the kilogram! Then by simple arithmetic,

$$1 \text{ kg} = \frac{(299\,792\,458)^2}{(9\,192\,631\,770)(6.626\,070\,15 \times 10^{-34})} \frac{\Delta\nu_{\text{Cs}} h}{c^2} = 1.475\,521\,3997\dots \times 10^{40} \frac{\Delta\nu_{\text{Cs}} h}{c^2}. \quad (\text{II})$$

Any given base unit can be defined similarly, without knowing the definitions of any other base units. Only the exact numerical values of the defining constants are required.

It is irrelevant that the mass  $\Delta\nu_{\text{Cs}} h / c^2$  is so miniscule that it must be scaled up by 40 orders of magnitude to equal one kilogram. It has always been true that “any method consistent with the laws of physics could be used to realize any SI unit” [2] and such methods already exist for the kilogram as it will be defined by Eq. (II) [4]. The Appendix to this note discusses in more detail the condition that assures continuity of the redefined kilogram with the present kilogram, and by extension the continuity conditions for the three other redefined units.

#### 4. Summary and Discussion

Several pictorial illustrations of the revised SI are already available [7,8]. In one case, readers are cautioned that the illustration is not an explanation [7]. By contrast, Table 2 has been derived mathematically from the seven defining constants, knowing only their units. It is easily observed from Table 2 that:

- The violet cells on the diagonal connect a base unit in the left column with the constant which defines it, in the top row. This is loose terminology because in most instances one or two “helping constants” are required, and these are shown in the yellow cells of each row. All other cells contain zero, and these are left blank;
- There are only three helping constants,  $\Delta\nu_{\text{Cs}}$ ,  $c$  and  $h$ , and these also serve as the defining constants for the second, metre and kilogram, respectively. [It is perhaps noteworthy that the second, metre and kilogram are the mechanical units of the old metre-kilogram-second (MKS) system, from which the SI evolved];
- In each row, the product of powers of the constants in the violet cell and any yellow cells form a quantity (which is also a constant) whose unit is the base unit of the row. The exponents needed are shown;
- At most, two helping constants are required to define any base unit. [The appearance of helping constants can be viewed as a mathematical requirement which reconciles continuity of the historical base units with the most useful selection of defining constants. See the Appendix, which shows that the present SI [2] is not very different in this respect];
- Helping constants are not needed to define either the second or the mole.
- The ground state hyperfine transition frequency of the caesium 133 atom  $\Delta\nu_{\text{Cs}}$  is needed in the definitions of all base units except the mole;
- The speed of light in vacuum  $c$  is needed *only* in the definitions of the metre and kilogram;

- The Planck constant  $h$  is needed *only* in the definitions of the kilogram, kelvin and candela;
- The elementary charge  $e$ , the Boltzmann constant  $k$ , the Avogadro constant  $N_A$  and the luminous efficacy of a specified wavelength  $K_{cd}$  are each needed to define a single base unit (ampere, kelvin, mole and candela respectively). They are not used as helping constants.

There is no general requirement that exponents in Tables 1 and 2 must be displayed as lower triangular matrices, although this arrangement makes the tables easier to scan visually and therefore has merit. Because Table 2 is a lower triangular matrix, one can see that the units can also be defined in seven separate steps rather than in a single step, as we have done. The seven-step solution, used in the draft 9th edition of the SI Brochure [4], first defines the SI second from the upper left corner of Table 2. The metre can then be defined from the second row because its helping constant has already been defined. The kilogram can be defined from the third row because its two helping constants have been defined. All helping constants have now been defined and so the remaining four SI units can be defined in any order one wishes, including of course the order found in [4].

The seven unique combinations of defining constants whose unit is a base unit (recall that Eq. (I) shows the combination for the kilogram) were derived in one step by matrix inversion. Since any order of units and defining constants used as labels in Table 1 leads to identical definitions of the base units, we have chosen an order that makes Table 2 visually simple. It is also the order found in the major reference for the revised SI [4].

### Appendix: the present SI

The present SI [2] could also have been formulated in terms of the six defining constants and one defining quantity that had been specified by the CGPM, either explicitly or implicitly, between 1889 and 1983:

- the unperturbed ground state hyperfine transition frequency of the caesium 133 atom  $\Delta\nu_{Cs}$  is 9 192 631 770 hertz, (1967)
- the speed of light in vacuum  $c$  is 299 792 458 metres per second, (1983)
- the mass of the international prototype of the kilogram  $m_K$  is 1 kilogram, (1889)
- the permeability of vacuum  $\mu_0$  is  $4\pi \times 10^{-7}$  newton per ampere squared, (1948,1952)
- the thermodynamic temperature of the triple point of water  $T_{TPW}$  is 273.16 kelvin, (1954)
- the molar mass of carbon 12,  $M(^{12}C)$ , is 0.012 kilogram per mole, (1971)
- the luminous efficacy of monochromatic radiation of frequency  $540 \times 10^{12}$  Hz,  $K_{cd}$ , is 683 lumens per watt. (1979)

The newton (symbol: N) is expressed in terms of base units as  $N = \text{kg m s}^{-2}$  [2]. Expressions for the hertz, lumen and watt in terms of base units are found in section 1. The defining quantity and three defining constants that will be replaced in the revised SI are shown in blue.

Carrying out the same procedure as described in sections 2 and 3 for the revised SI, we start with Table A1, which contains an embedded  $7 \times 7$  matrix. Again, this matrix is a table of exponents inferred from seven bullets, but now they are the bullets found in this Appendix. For ease of comparison with Tables 1 and 2, the order of units in the top row of Table A1 is chosen to be identical to that of Table 1, and the order of quantities in the left column is chosen to produce a lower triangular matrix.

	<b>s</b>	<b>m</b>	<b>kg</b>	<b>A</b>	<b>K</b>	<b>mol</b>	<b>cd</b>
$\Delta v_{Cs}$	-1	0	0	0	0	0	0
$c$	-1	1	0	0	0	0	0
$m_{\kappa}$	0	0	1	0	0	0	0
$\mu_0$	-2	1	1	-2	0	0	0
$T_{TPW}$	0	0	0	0	1	0	0
$M(^{12}C)$	0	0	1	0	0	-1	0
$K_{cd}$	3	-2	-1	0	0	0	1

Table A1.

Now transpose the labels of Table A1 and invert its embedded matrix to arrive at Table A2. As with Table 2, cells containing zero are left blank.

	$\Delta v_{Cs}$	$c$	$m_{\kappa}$	$\mu_0$	$T_{TPW}$	$M(^{12}C)$	$K_{cd}$
<b>s</b>	-1						
<b>m</b>	-1	1					
<b>kg</b>			1				
<b>A</b>	1/2	1/2	1/2	-1/2			
<b>K</b>					1		
<b>mol</b>			1			-1	
<b>cd</b>	1	2	1				1

Table A2. The present SI [2]. The blank cells all contain zero (not displayed).

This table can easily be compared with Table 2. Note that only the first two rows, those for the second and metre, are identical in the two tables.

There is no reason that the exponents must be integers, as this example illustrates. Thus, in the

present SI [2], the ampere is realized by traceability to the quantity  $\left(\frac{\Delta v_{Cs} c m_{\kappa}}{\mu_0}\right)^{\frac{1}{2}}$  which, according

to the information in the first four bullets, has an exact value of order  $1.5 \times 10^{12}$  A. Note that the SI unit of  $(\Delta v_{Cs} c m_{\kappa})$  is the newton and that of  $\mu_0$  is the newton per ampere squared.

The exponents appearing in Table A2 and the exact numerical values of the six constants and one physical quantity listed in the bullets of this Appendix would have been sufficient to define all base units of the present SI. For example, a definition equivalent to the present definition of the ampere [2] would be:

$$1 \text{ A} = \left( \frac{(4\pi \times 10^{-7})}{(9\,192\,631\,770)(299\,792\,458)(1)} \right)^{\frac{1}{2}} \left( \frac{\Delta v_{Cs} c m_{\kappa}}{\mu_0} \right)^{\frac{1}{2}} = 6.789\,687\dots \times 10^{-13} \left( \frac{\Delta v_{Cs} c m_{\kappa}}{\mu_0} \right)^{\frac{1}{2}}.$$

In the present SI,  $m_{\mathcal{K}}$  is the sole defining quantity which is not some kind of constant. Rather, it is the mass of an artefact known as the international prototype of the kilogram,  $\mathcal{K}$ , which has been used since 1889 to define one kilogram [2]. This artefact definition of the kilogram is simple, understandable and independent of the six constants. Unfortunately, since the mass of  $\mathcal{K}$  is not a physical constant, the stability over time of the unit it defines cannot be assured. The same lack of assurance affects, at least in principle, the three units for which  $m_{\mathcal{K}}$  is a “helper”, one of which is the ampere (see yellow cells in the column of Table A2 labelled “ $m_{\mathcal{K}}$ ”). When the SI was first approved by the 11th CGPM in 1960, it was recognized that the artefact definition of the kilogram was a weakness of the International System of Units—to be remedied “sooner or later” [9].

The present definition of the kilogram [2] is contained entirely in the third bullet of this Appendix. In symbols,

$$1 \text{ kg} = m_{\mathcal{K}} .$$

The revised definition of the kilogram [4] is given by Eq. (II),  $1 \text{ kg} = 1.475\,521\,3997\dots \times 10^{40} \frac{\Delta v_{\text{Cs}} h}{c^2}$ .

The value of the prefactor on the right-hand side ensures that there will be no perceptible discontinuity in the kilogram unit when it is redefined [3,4]. The continuity condition requires that the weighted mean of the most accurate experimental values of  $h$  will have been fixed [3] so that, *just after* the redefinition comes into force,

$$x \cdot m_{\mathcal{K}} = 1.475\,521\,3997\dots \times 10^{40} \frac{\Delta v_{\text{Cs}} h}{c^2},$$

where the experimental value of  $x$  is unity to within an uncertainty that is sufficiently small to make the redefinition imperceptible to the vast majority of users. (Subsequently, the experimental value of  $x$  might change simply because  $m_{\mathcal{K}}$  is not a physical constant. Time will tell.) The impact of the revised SI on most users of the present SI has been assessed to be small by international experts [10].

### Acknowledgements

The author acknowledges helpful discussions with Estefanía de Mirandés (BIPM), Richard J.C. Brown (NPL) and Paul Quincey (NPL).

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Access dates are given for references that may be subject to change