Note on the decay correction required for a radionuclide $^{N}X$ in presence of its metastable state $^{N}X_{m}$.

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INTRODUCTION

When measuring a radioactive source composed of a radionuclide $^{N}X$ together with its metastable state $^{N}X_{m}$ decaying to the ground state, special attention should be given to the decay correction. Indeed the activity $A$ of the radionuclide $^{N}X$ is the summation of the usual decaying activity $A_{D}$ of the initial $^{N}X$ with the growing activity $A_{G}$ of the $^{N}X$ coming from the disintegration of the metastable state to the ground state.

If $t_{0}$ is the time of production of the source (which is often not known by the user), the usual decay correction $A_{D}(t) = A_{D,0} \exp[-(t-t_{0})/\tau]$ allows an easy calculation of the correction $D$ for any time difference $\Delta t = t_{2} - t_{1}$:

$$A_{D}(t_{2}) = A_{D}(t_{1}) \exp[-(t_{2}-t_{0})/\tau] / \exp[-(t_{1}-t_{0})/\tau] = A_{D}(t_{1}) \exp[-\Delta t/\tau] = A_{D}(t_{1}) D(t_{1}, t_{2})$$ (1).

On the other hand, the correction for the growing activity of $^{N}X$ coming from the metastable state is given by

$$A_{G}(t) = A_{m,0} \{ \exp[-(t-t_{0})/\tau_{m}] - \exp[-(t-t_{0})/\tau] \} \tau_{m} / (\tau_{m}-\tau)$$ (2),

where $A_{m,0}$ is the activity of the metastable state at the time $t_{0}$ [1]. The correction for a time difference $\Delta t$ is then

$$A_{G}(t_{2}) = A_{G}(t_{1}) \{ \exp[-(t_{2}-t_{0})/\tau_{m}] - \exp[-(t_{2}-t_{0})/\tau] \} / \{ \exp[-(t_{1}-t_{0})/\tau_{m}] - \exp[-(t_{1}-t_{0})/\tau] \}
= A_{G}(t_{1}) G(t_{0}, t_{1}, t_{2})$$ (3).

Depending on the respective half-lives $\tau$ and $\tau_{m}$ of both states, the equilibrium (constant activity ratio) between $^{N}X_{m}$ and the daughter $^{N}X$ may or may not be reached:

- In case of $\tau$ smaller than $\tau_{m}$ equilibrium is reached after a transition period. At equilibrium, the activity of the daughter $^{N}X$ can be deduced from the activity of the parent $^{N}X_{m}$ and, in consequence, the calculation of $G(t_{0}, t_{1}, t_{2})$ is avoided. Indeed, it can be seen from (2) that when $(t-t_{0}) >> \tau$, the second term $\exp[-(t-t_{0})/\tau]$ tends to zero faster than the first term and, in consequence, the daughter $^{N}X$ is decaying following the simple exponential law with the half-life of the metastable state.

- Obviously, in case of $\tau$ larger than $\tau_{m}$ equilibrium cannot be reached and the time $t_{0}$ of the source production is needed to calculate $G(t_{0}, t_{1}, t_{2})$. In the extreme case of $\tau >> \tau_{m}$, the metastable state rapidly decays to the ground state and the source, then composed of pure $^{N}X$, decays normally following the simple exponential law.

In this short report, it is demonstrated that, perhaps surprisingly, the decay correction for the total activity $A = A_{D} + A_{G}$ is independent of $t_{0}$. No conditions on the respective half-lifes of $^{N}X$ and $^{N}X_{m}$ are imposed.
DEMONSTRATION

By definition, \( \lambda = 1/\tau \) and \( \lambda_m = 1/\tau_m \).

If \( A_m(t) \) is the activity of \( N^X^m \) and the ratio \( R(t) = A_m(t) / A(t) = A_m(t) / (A_D(t) + A_G(t)) \), we have:

\[
\begin{align*}
A_D(t) &= A_{D,0} e^{-\lambda (t-t_0)} \\
A_m(t) &= A_{m,0} e^{-\lambda_m (t-t_0)} \\
A_G(t) &= B A_{m,0} e^{-\lambda_m (t-t_0)} \left( \frac{\lambda}{\lambda - \lambda_m} (1 - e^{(\lambda_m - \lambda)(t-t_0)}) \right)
\end{align*}
\]

(4)

where the index 0 relates to the time \( t_0 \) of production of the source. The last equation is the general relation calculating the activity of a daughter radionuclide from the activity of the parent [1] and is equivalent to equation (2). The factor \( B \) corresponds to a possible branching ratio when the metastable state decays only partially to the ground state.

If the quantities \( A \) and \( R \) are known at a time \( t_1 \), the total activity \( A \) at any time \( t_2 \) is given by \( A(t_2) = A(t_2) + A_G(t_2) \):

\[
\begin{align*}
A(t_2) &= A_D(t_1) e^{-\lambda \Delta t} + B A_m(t_1) e^{-\lambda_m \Delta t} \left( \frac{\lambda}{\lambda - \lambda_m} (1 - e^{(\lambda_m - \lambda)(t_2-t_0)}) \right) \\
&= (A(t_1) - A_G(t_1)) e^{-\lambda \Delta t} \\
&+ BR(t_1) A(t_1) \left( \frac{\lambda}{\lambda - \lambda_m} e^{-\lambda_m \Delta t} (1 - e^{(\lambda_m - \lambda)(t_2-t_0)}) \right) \\
&= A(t_1) e^{-\lambda \Delta t} - BR(t_1) A(t_1) \left( \frac{\lambda}{\lambda - \lambda_m} (1 - e^{(\lambda_m - \lambda)(t_2-t_0)}) \right) e^{-\lambda \Delta t} \\
&+ BR(t_1) A(t_1) \left( \frac{\lambda}{\lambda - \lambda_m} (e^{-\lambda_m \Delta t} - e^{-\lambda \Delta t}) e^{(\lambda_m - \lambda)(t_2-t_0)}) \right) \\
&= A(t_1) \left[ e^{-\lambda \Delta t} + BR(t_1) \left( \frac{\lambda}{\lambda - \lambda_m} (e^{-\lambda_m \Delta t} - e^{-\lambda \Delta t}) \right) \right] \\
&= A(t_1) C(t_1, t_2, R(t_1))
\end{align*}
\]

(5)

Expression (5) shows a first term corresponding to the simple case of pure \( N^X \) decay. The second term may reach non-negligible values as shown in the numerical examples below. This term is independent of \( t_0 \), showing the advantage of using (5) to calculate the decay correction \( C \) for the total activity \( A \), instead of calculating the decay for the initial \( N^X \) only and evaluating a correction for the contribution of the \( N^X^m \) decay.
In conclusion, when measuring a radioactive source composed of a radionuclide $^{N}X$ together with its metastable state $^{N}X^{m}$ decaying to the ground state with a branching ratio $B$, the decay correction for the total activity of $^{N}X$ is given by $C(t_1, t_2, R(t_1))$ for any time difference, i.e. whether equilibrium is reached or not. This expression is independent of the time of source production and is valid for any values of $\tau$ and $\tau_m$. Finally, given that the measured quantity is usually the total activity of $^{N}X$, i.e. $A_D$ and $A_G$ are generally not determined independently, equation (5) is particularly convenient.

NUMERICAL EXAMPLES

1. $^{133}$Xe ($\lambda = 0.1322 \text{ d}^{-1}$) containing some $^{133}$Xe$^m$ ($\lambda_m = 0.3168 \text{ d}^{-1}$):
   
   if $R(t_1) = 10^{-3}$ and $\Delta t = -10 \text{ d}$, the ratio of the second to the first term in (5) is equal to $3.8 \times 10^{-3}$.

2. $^{177}$Lu ($\lambda = 0.1043 \text{ d}^{-1}$) containing some $^{177}$Lu$^m$ ($\lambda_m = 4.321 \times 10^{-3} \text{ d}^{-1}$), $B = 0.217$:
   
   if $R(t_1) = 10^{-3}$ and $\Delta t = 20 \text{ d}$, the ratio of the second to the first term in (5) is equal to $1.5 \times 10^{-3}$.