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Generalized binomial coefficients

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Abstract

A numerical problem which has occurred in numismatics is taken as a stimulus to consider binomial coefficients with negative arguments, a case rarely treated in textbooks.

1. Introduction

Binomial coefficients, usually written in the form $\binom{n}{k}$, are part of the standard equipment available for treating combinatorial problems. They have many remarkable properties and are usually defined for integer arguments $0 \leq k < n$. As is well known, they can then also be interpreted as the number of certain combinations of elements.

Binomial coefficients can be generalized in various ways. Recalling the binomial theorem, which says that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k , \tag{1}$$

for $x^2 < 1$, it is tempting to extend the exponent n first to negative integers, then to fractions and finally to real numbers. Another kind of generalization, denoted by

$$\binom{n}{k}_h = \frac{n\,(n-h)(n-2h)\,\ldots\,(n-(k-1)h)}{k!} ,$$

with h = 1, 2, ... , is discussed in [1].

Alternatively, one may wonder what happens if k is allowed to assume negative integer values. One would no doubt expect that such a case, though perhaps of some theoretical interest, would have no applications. Curiously enough, it is a practical problem which has led us to consider this situation. We try to explain the context.

2. A calendric problem

In ancient lunar-solar calendars, a period of 19 years, called the Metonic cycle, was used to combine month and year, for it had been observed that this time interval corresponds almost exactly to 235 (synodic) months. The resulting 7 additional months - for 235 - 19x12 = 7 - were inserted as regularly as possible, e.g. in the years with rank R = 3, 6, 8, 11, 14, 17 and 19, which then became intercalary years (with 13 months). More details will be given in [2].

At present, usually only a few of these intercalations are known to us by documents, for a specific city or region. The problem then arises to determine how far it is possible to reconstruct the whole cycle when only N of the 7 intercalations are known by their years.

It is not difficult to see that N known intercalary years can be arranged in $\binom{7}{N}$ different ways. However, a specific series of year intervals may correspond to several possible sequences of intercalary years. Thus, for example, the intervals $D_1 = 5$ and $D_2 = 8$ (in this order) are compatible with three sequences of intercalary ranks, namely R = 6, 11, 19, or 14, 19, 8, or 17, 3, 11, considering the periodicity of 19 years. The indicated interval sequence thus corresponds to m = 3 possible Metonic cycles (with different time origins).

In this context, various questions can be asked, for example: How often can a given multiplicity m occur, if exactly N intercalary years are known? Let us denote this number by $M_N(m)$. The direct enumeration of all possible arrangements has allowed us to obtain values of M for various arguments N and m. It appears that, at least for $2 \le N \le 7$ and $1 \le m \le 6$, the numbers found can all be described by the binomial coefficient

$$M_N(m) = \binom{6-m}{N-2} . \qquad (2)$$

This is quite a surprising empirical result. Indeed, such a simple relation could not be expected since M depends on the exact form of the Metonic cycle as defined previously. For any other choice - for instance with year rank R = 5 instead of R = 6 as intercalary - formula (2) becomes invalid, although the 7 intercalations would still be separated by 5 intervals of 3 years and 2 intervals of 2 years.

Let us discuss briefly some specific examples. The case N = 2 leads to $M_2(m) = \binom{6-m}{0}$, which can be shown to be 1, for $1 \le m \le 6$; m = 7 does not occur. What happens if we apply the formula to N = 1? Use of (2) leads to

$$M_1(m) = \binom{6-m}{-1} , \qquad (2a)$$

the interpretation of which is not obvious.

3. Search for a mathematical issue

For questions involving binomial coefficients, a natural reaction is to look for advice from the writings of John Riordan. In [3], already on page 2, one finds a table of binomial coefficients $\binom{n}{k}$ which includes negative values of n (till -5). Unfortunately, no negative values of k are considered (always in our present notation). However, there is a remark (on page 1) saying that

"... the boundary relations with combinatorial priority are the following. First

$$\binom{n}{0} = 1, \quad n = 0, \pm 1, \pm 2, \ldots ,$$

and then $\binom{0}{k} = \delta_{0,k}$, with ... . These equations entail

$$\binom{n}{-k} = 0, \quad n = 0, \pm 1, \pm 2, \ldots, \quad k = 1, 2, \ldots\text{''}. \qquad (3)$$

This "boundary convention" is repeated below the inserted table of binomial coefficients. This seems to imply that all coefficients of the form $\binom{-n}{-k}$, with n, k > 0, vanish - and would explain why they are not listed.

However, something is strange. From our practical example given above, and specifically for $M_1(m)$, we would expect that $\binom{-n}{-1}$ is equal to zero only for n > 1, but that $\binom{-1}{-1} = 1$.

Indeed, if only a single intercalary year is known (N = 1), it may have any of the 7 possible ranks. Since each rank corresponds to another sequence of intercalary years, we are in the situation m = 7; other values of m are excluded.

In addition, this expectation is in line with a formula which the same author gave previously in [4] (on page 5), namely that

$$\binom{-n}{-k} = (-1)^{n-k}\binom{k-1}{n-1}. \tag{4}$$

This expression can be easily derived and has in fact been known for a long time. We also find it for example in [1], the original edition of which goes back to 1939. The conclusion must be that (3) is due to an oversight and should be corrected. This then allows us to extend the table in [3] to negative arguments. The results are assembled in Table 1.

4. Some final remarks

a) A careful look at Table 1 shows that the basic recurrence

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{5a}$$

still holds for all positive or negative integers n and k, but not for n = k = 0. In addition, for n < 0 the binomial coefficients may be negative. In order to be applicable to any argument, (5a) has to be generalized to

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} - \delta_{n,0}\,\delta_{k,0}\,. \tag{5b}$$

This is certainly an unexpected result.

b) Some special cases may be worth mentioning. Thus

$$\binom{n}{k} = 1 \quad \text{whenever} \quad k = n \quad \text{or} \quad k = 0. \tag{6}$$

$$\binom{n}{k} = 0 \quad \text{if,} \quad \text{for } n \geq 0,\ k < 0 \text{ or } k > n \quad \text{or,} \quad \text{for } n < 0, \text{ if } n < k < 0. \tag{7}$$

Table 1: Some numerical values of the generalized binomial coefficients $\left[{n \atop k}\right]$, for $|n|, |k| \le 8$. Blanks indicate vanishing values.

	k=-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
n=-8	1								1	-8	36	-120	330	-792	1716	-3432	6435
-7	-7	1							1	-7	28	-84	210	-462	924	-1716	3003
-6	21	-6	1						1	-6	21	-56	126	-252	462	-792	1287
-5	-35	15	-5	1					1	-5	15	-35	70	-126	210	-330	495
-4	35	-20	10	-4	1				1	-4	10	-20	35	-56	84	-120	165
-3	-21	15	-10	6	-3	1			1	-3	6	-10	15	-21	28	-36	45
-2	7	-6	5	-4	3	-2	1		1	-2	3	-4	5	-6	7	-8	9
-1	-1	1	-1	1	-1	1	-1	1	1	-1	1	-1	1	-1	1	-1	1
0									1								
1									1	1							
2									1	2	1						
3									1	3	3	1					
4									1	4	6	4	1				
5									1	5	10	10	5	1			
6									1	6	15	20	15	6	1		
7									1	7	21	35	35	21	7	1	
8									1	8	28	56	70	56	28	8	1

In particular, for $n = -1$ we have

$$\binom{-1}{k} = \begin{cases} (-1)^k , & \text{for } k \geq 0 , \\ (-1)^{k-1} , & \text{"} \quad k < 0. \end{cases} \tag{8a}$$

The two cases can be combined in the expression

$$\binom{-1}{k} = (-1)^K , \tag{8b}$$

where $K = \text{integer} \left| k + \frac{1}{2} \right|$.

c) Finally, let us come back to the surmised formula (2). While the generalization of binomial coefficients proposed above nicely supports our expectation that $M_1(7) = 1$, it does not explain why $M_N(7) = 0$ for $N > 1$. Obviously, one could avoid the difficulty by limiting all the multiplicities to $m \leq 6$, but this would then also have removed our motive for considering binomial coefficients with negative arguments. Alternatively, and remembering that (2) still is not really understood, one could think of "amending" it, for instance by putting

$$M_N(m) = \binom{6-m}{N-2} + (-1)^{N-1} (1 - \delta_{N,1}) \, \delta_{m,7} \; .$$

Although such a formula would lead in all cases to the expected values, it looks too artificial. A more natural and simpler way may be to limit the values of m in Eq. (2) to $1 \leq m \leq 8\text{-}N$, in agreement with the actual practical requirements.

Perhaps we should learn from this innocent excursion to binomial coefficients the lesson that interesting numerical questions may also occur in contexts which seem far away from the world of mathematics or physics. We are well advised to take them seriously and listen carefully to their message.

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References

[1] C. Jordan: "Calculus of Finite Differences" (Chelsea, New York, 1965[3])

[2] J.W. Müller: "A new look at the Metonic cycle" (in preparation)

[3] J. Riordan: "Combinatorial Identities" (Wiley, New York, 1968)

[4] J. Riordan: "An Introduction to Combinatorial Analysis" (Wiley, New York, 1958)

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