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Some remarks on twin primes

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Abstract

By taking advantage of a recursion formula due to Euler for $\sigma(n)$, the sum of the divisors of n, we deduce an explicit expression for $\Delta = \sigma(n+1) - \sigma(n-1)$ which involves only quantities that can be readily evaluated. If $\Delta = 2$, then $n \pm 1$ are twin primes.

1. Some generalities

If two integers, such as

$$p_1 = n - 1 \quad \text{and} \quad p_2 = n + 1 , \tag{1}$$

are prime numbers, then p_1 and p_2 are called twin primes. Examples are 5 and 7, 29 and 31 or 101 and 103. We first want to show that twin primes (for $p_1 > 3$) are always of the form

$$p_{2,1} = 6k \pm 1 , \tag{2}$$

with k = 1, 2, 3, ...

From the product

$$(n-1)\, n\, (n+1) = \binom{n+1}{3}\, 3!$$

it follows, since n-1 and n+1 are assumed to be prime numbers and binomial coefficients are integers, that n can be divided by 6. This proves the form (2) for twins. In addition, numbers of the form

$$n_{2,1} = 6k \pm 3 \tag{3}$$

cannot be prime as they are divisible by 3. Hence, "triplet" primes of the type n, n+2, n+4 do not exist. The only exception (3, 5, 7) can be discarded by requiring that $n > 3$.

Relation (2) has a more general meaning since in fact any prime number can be written in this way. To see this, we continue our decomposition by considering numbers of the form

$$6k \pm 5 = \mp 1 \pmod 6. \tag{4}$$

These may be prime, but are already covered by (2). Since the forms (2) and (4) include all odd numbers modulo 6, there are no other possibilities to consider and (2) turns out to hold for any prime number. Obviously, not all integers k produce primes (or even twins), but all primes (and twins) are of the form (2).

2. Euler's surprising relation

For a natural number n it is usual to denote by $\sigma(n)$ the sum of its divisors, including 1 and n. Thus, for n = 8, with the divisors 1, 2, 4 and 8, we have $\sigma(8) = 15$. For a prime number $p \geq 2$, since it is divisible only by 1 and p, we have

$$\sigma(p) = p + 1 . \tag{5}$$

It is practical to define $\sigma(1) = 1$; the meaning of $\sigma(0)$ will be discussed later.

In 1747 Euler [1], thanks to his unique sagacity, detected that the numbers $\sigma(n)$ actually follow a recurrence formula (see also [2] or [3]). It is rather complicated and can be written as

$$\begin{aligned}\sigma(n) = {} & \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) \\ & + \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) \\ & + \sigma(n-35) + \sigma(n-40) - \sigma(n-51) - \sigma(n-57) \\ & + \quad \dots \quad - \quad \dots \; .\end{aligned} \tag{6}$$

Euler himself rightly called this expression "une loi tout extraordinaire".

The sum in (6) ends if the argument of σ becomes negative, since $\sigma(n) = 0$ for $n < 0$. However, if the last term is $\sigma(0)$, then, as Euler has shown, we have to put

$$\sigma(0) = n , \tag{7}$$

if the decomposition starts at n.

The sequence of integers occurring in the recurrence formula (6), namely

$$\eta = 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, \dots , \tag{8}$$

becomes less mysterious if we form the first differences

$$1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, \dots ,$$

in which Euler has seen an alternate sequence of natural numbers (1, 2, 3, ...) and of odd numbers (3, 5, 7, ...). This allows us to write, for j = 0, 1, 2, ... ,

$$\begin{aligned}\eta_j &= \sum_{k=1}^{\left[\frac{j+1}{2}\right]} k + \sum_{k=1}^{\left[\frac{j+2}{2}\right]} (2k-1) \\ &= \frac{1}{2}\left[\frac{j+1}{2}\right]\left[\frac{j+3}{2}\right] + \left[\frac{j+2}{2}\right]^2 ,\end{aligned} \tag{9}$$

where [x] denotes the largest integer not exceeding x. All integers η_j are pentagonal numbers, thus of the form (m/2) (3m $\mp$ 1), with m = 1, 2, ...

By means of the quantities η_j Euler's recursion can be brought into the form

$$\sigma(n) = \sum_{j=0}^{J} S_j \; \sigma(n - \eta_j) \, , \tag{10}$$

where $S_j = (-1)^{[j/2]}$ gives the required sequence (+ + - -) of signs, an abbreviation we shall also use later. The upper limit J is such that $\eta_J \le n$, but $\eta_{J+1} > n$. Note that (10) is just a more compact version of (6).

At first sight, Euler's recursion formula, as stated by (6) or (10), seems to have little to do with twin primes. However, relation (5) gives a hint to the contrary. It is in fact difficult to understand why Euler, who used to follow the remotest indications, apparently did not pursue this track any further.

Of course, as given above the relation is of no direct use, but the recurrence itself may allow us to "work down" the sum until there remain only terms of the form "$\sigma(0)$". All which follows is based on (6) and is an attempt to draw some useful conclusions from this remarkable formula, especially for prime numbers.

3. A first approach to the twin problem

If we consider the difference

$$\Delta(n) \equiv \sigma(n+1) - \sigma(n-1) \, , \tag{11}$$

then twin primes of the form p_1 = n-1 and p_2 = n+1 will yield

$$\Delta(n) = n+2 - n = 2 \, . \tag{12}$$

Of real interest for us is the inverse relation, namely the conclusion that $\Delta = 2$ implies the presence of a pair of primes. This is likely to be true, and supported by an extended inspection of listed values of $\sigma(n)$; a complete formal proof is not yet available, but should be possible. In what follows we assume this inverse relation to hold.

Let us first show by an example how the recurrence can be used to arrive at a numerical result. This is cumbersome, but illustrates the procedure. We choose the simplest case when, according to (2), we can hope to find a twin prime, namely k = 1, thus n = 6. By repeated application of (6) and (7) we then find

$$\begin{aligned}
\Delta(6) &= \sigma(7) - \sigma(5) \\
&= \{\sigma(6) + \sigma(5) - \sigma(2) - 7\} - \sigma(5) \\
&= \{\sigma(5) + \sigma(4) - \sigma(1)\} - \{\sigma(1) + 2\} - 7 \\
&= \{\sigma(4) + \sigma(3) - 5\} + \{\sigma(3) + \sigma(2)\} - 2 - 9 \\
&= \{\sigma(3) + \sigma(2)\} + 2\{\sigma(2) + \sigma(1)\} + \sigma(1) + 2 - 16 \\
&= \{\sigma(2) + \sigma(1)\} + 3\{\sigma(1) + 2\} + 3 - 14 \\
&= \sigma(1) + 2 + 4 - 5 = 2 \, .
\end{aligned}$$

Similarly, one can verify (by lengthier calculations) that

$$\Delta(12) = 2 \quad \text{and} \quad \Delta(18) = 2\,. \tag{13}$$

All this is in agreement with expectation, since (5, 7), (11, 13) and (17, 19) are prime pairs.

It should be noted, however, that numerical values of $\sigma(n)$ are not used in the derivation of the results (13). All is based solely on Euler's recurrence.

It is clear that this direct approach, illustrated by the case of $\Delta(6)$, is too cumbersome for practical use. Some progress can be made by paying attention to the way the final result is obtained, namely

$$\begin{aligned} \Delta(6) &= 6\cdot 1 + 4\cdot 2 - 1\cdot 5 - 1\cdot 7\,, \\ \Delta(12) &= 35\cdot 1 + 26\cdot 2 - 11\cdot 5 - 6\cdot 7 + 1\cdot 12\,, \\ \Delta(18) &= 154\cdot 1 + 121\cdot 2 - 58\cdot 5 - 35\cdot 7 + 8\cdot 12 + 3\cdot 15\,. \end{aligned} \tag{14}$$

Equation (14) gives us a first glimpse at the enigmatic structure of the unknown general expression for $\Delta(6k)$ that we would like to know.

4. A graphical method

Since the algebraic "reduction" method illustrated above becomes rapidly too laborious, we have to look for a simpler alternative. Such a possibility is offered by a graphical method. It can be illustrated in the following way.

To leave open the question of when to stop the process, the numbering is reversed, that is, we begin at $n = 0$ and end at $n = N$. Instead of (6) one then uses a scheme with a symbolic location quantity τ, which, in analogy with σ, has the property

$$\begin{aligned} \tau(0) = {} & \tau(1) + \tau(2) - \tau(5) - \tau(7) \\ & + \tau(12) + \tau(15) - \tau(22) - \tau(26) \\ & + \ \ldots \ - \ \ldots\,, \end{aligned} \tag{15}$$

where the arguments are the numbers η_j.

Figure 1 shows the beginning of this scheme in graphical form. In the first column, with start (denoted by x) at n = 0, we indicate for all terms $\tau(n)$ their location n. Positive contributions are shown by a full dot • and negative contributions by a circle.

For n = 1 we proceed as for n = 0, putting a cross (x) on top of the second colum. We thus develop $\tau(1)$ according to

$$\begin{aligned} \tau(1) = {} & \tau(2) + \tau(3) - \tau(6) - \tau(8) \\ & + \tau(13) + \tau(16) - \tau(23) - \tau(27) \\ & + \ \ldots \ - \ \ldots\,, \end{aligned}$$

and note this graphically using the symbols • and o in the second column, as before.

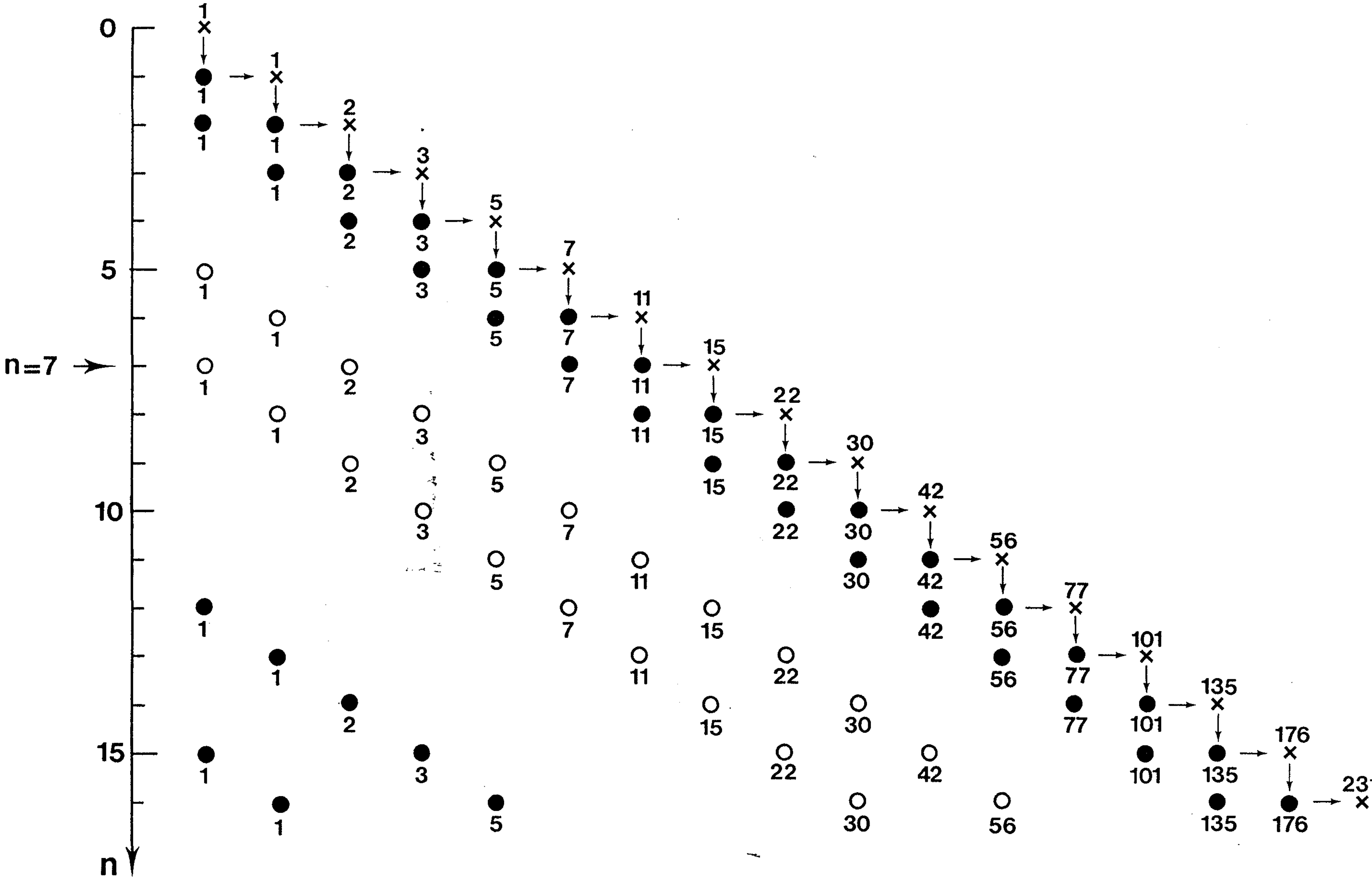

Figure 1. Scheme for the graphical evaluation of the coefficients $\alpha_\eta(n)$ appearing in (17).

For n = 2 we have two contributions (one from n = 0, the other from n = 1) and hence we put a number 2 at the starting cross on top of column three. This is developed as previously, always by using the formal recurrence (here for n = 2)

$$\tau(n) \;=\; \sum_{j=0}^{\infty} S_j \;\tau(n + \eta_j)\,. \tag{16}$$

This process is continued till we arrive at the number n for which we want to know $\sigma(n)$. We now have to remember Euler's rule that $\sigma(n)$ has to be put equal to n' if this location results from a development that started at n'. What counts is the "height of the fall" or the "size of the jump" which, in our graphical arrangement, is given by the difference between the start (noted by x) and the final position n. All the "decays" that "miss" the value n are ignored.

Thus, we see immediately by looking, for example, at the horizontal line given at n = 7, that there are four intersections, with corresponding "heights" 7, 5, 2 and 1. With their respective indicated multiplicities -1, -2, 7 and 11 we find that, returning to σ,

$$\sigma(7) \;=\; -\,1{\cdot}7 - 2{\cdot}5 + 7{\cdot}2 + 11{\cdot}1 \;=\; 8\,,$$

as expected. In this way it is easy to obtain the coefficients α_η necessary for the evaluation of

$$\sigma(n) \;=\; \sum_{\eta} \alpha_\eta(n)\;\eta\,. \tag{17}$$

The sum extends over all integers η given in (8) or (9). The coefficients $\alpha_\eta(n)$ found by this graphical method are listed in Table 1.

Table 1 - Coefficients $\alpha_\eta(n)$ appearing in (17), for $n \leq 16$.

n	$\eta = 1$	2	5	7	12	15
1	1					
2	1	1				
3	2	1				
4	3	2				
5	5	3	-1			
6	7	5	-1			
7	11	7	-2	-1		
8	15	11	-3	-1		
9	22	15	-5	-2		
10	30	22	-7	-3		
11	42	30	-11	-5		
12	56	42	-15	-7	1	
13	77	56	-22	-11	1	
14	101	77	-30	-15	2	
15	135	101	-42	-22	3	1
16	176	135	-56	-30	5	1

However, relation (17) is of limited utility as long as the coefficients have to be taken from a table. This would change if a formula were available for their evaluation.

5. An arithmetic approach

A look at Table 1 immediately shows its simple structure: the coefficients in all columns are apparently the same (apart from sign); they are just shifted by η-1 positions (with respect to the column with $\eta = 1$), where the numbers η are those we first met in (8). The problem of the coefficients $\alpha_\eta(n)$ is thus reduced to the explanation of a single new basic sequence, denoted $\mu(n)$, which is given numerically (for $n \leq 20$) in Table 2.

Table 2 - The first numerical values of the sequence $\mu(n)$.

n	$\mu(n)$	n	$\mu(n)$	n	$\mu(n)$
0	1	7	15	14	135
1	1	8	22	15	176
2	2	9	30	16	231
3	3	10	42	17	297
4	5	11	56	18	385
5	7	12	77	19	490
6	11	13	101	20	627

A closer look at Table 1 also reveals the relations

$$\alpha_\eta(n) = S_j \; \mu(n-\eta_j) \,, \qquad \sum_\eta \alpha_\eta(n) = \mu(n) \,. \tag{18}$$

This leads directly to the basic recurrence

$$\begin{aligned} \mu(n) = \; & \mu(n-1) + \mu(n-2) - \mu(n-5) - \mu(n-7) \\ & + \mu(n-12) + \mu(n-15) - \mu(n-22) - \mu(n-26) \\ & + \quad \ldots \quad - \quad \ldots \,. \end{aligned} \tag{19}$$

A comparison with (6) shows that (19) is identical with Euler's recurrence relation for $\sigma(n)$. This is quite surprising. The only difference lies in the initial condition which is now $\mu(0) = 1$. In fact, the two series $\sigma(n)$ and $\mu(n)$ are quite different. In particular, we note that $\mu(n)$ is a monotonously increasing function of n, a feature which is not shared by $\sigma(n)$. It will turn out that the derivation of (19) is the decisive step in our reasoning.

It follows from (17) and (18) that $\sigma(n)$ can also be given in terms of the sequence μ by writing

$$\begin{aligned} \sigma(n) = \; & \mu(n-1)\cdot 1 + \mu(n-2)\cdot 2 - \mu(n-5)\cdot 5 - \mu(n-7)\cdot 7 \\ & + \mu(n-12)\cdot 12 + \mu(n-15)\cdot 15 - \mu(n-22)\cdot 22 - \mu(n-26)\cdot 26 \\ & + \quad \ldots \quad - \quad \ldots \,. \end{aligned} \tag{20}$$

Using the notation of (10), this can be expressed in the more condensed form

$$\sigma(n) = \sum_j S_j \, \mu(n - \eta_j) \; \eta_j \,. \tag{21}$$

6. A direct attempt at locating twins

It is known from (11) that to check the presence of twin primes we have to evaluate

$$\Delta(n) = \sigma(n+1) - \sigma(n-1)$$

and to see if it is equal to 2. For this purpose it should be possible to use the graphical method described above. This requires the determination of the coefficients which apply to the situation $\sigma(N+1) - \sigma(N-1)$, with $N = 6k$. This has been done for some values of k. We do not reproduce the graphical plot (which is similar to Fig. 1, although more involved). Instead, Table 3 lists the coefficients $\beta_\eta(k)$ which allow us to evaluate

$$\Delta(6k{=}N) = \sum_\eta \beta_\eta(k)\ \eta\,, \tag{22}$$

which is a development similar to that given in (17) for $\sigma(n)$.

Table 3 - Coefficients $\beta_\eta(k)$ appearing in (22), for $k \leq 6$.

k	$\eta = 1$	2	5	7	12	15	22	26	35
1	6	4	-1	-1					
2	35	26	-11	-6	1				
3	154	121	-58	-35	8	3			
4	573	463	-242	-154	45	20	-2		
5	1 886	1 555	-861	-573	193	96	-15	-4	
6	5 667	4 740	-2 745	-1 886	703	375	-75	-26	1

This makes it possible to extend the results given in (14), but unfortunately still no simple structure in the coefficients becomes visible. To reach this goal, a more systematic approach is required.

7. A stepwise procedure

To evaluate $\Delta(6k)$, we first determine from (21)

$$\sigma(n \pm 1) = \sum_j S_j\ \mu(n-\eta_j \pm 1)\ \eta_j\,. \tag{23}$$

Since it is known from (2) that twin primes are necessarily of the form $6k \pm 1$, we readily obtain from (23), replacing n by 6k, the general expression

$$\Delta(6k) = \sum_{j=0}^{J} (-1)^{[j/2]} \{\mu(6k-\eta_j+1) - \mu(6k-\eta_j-1)\}\ \eta_j\,. \tag{24}$$

This is our final result in the search for twin primes. In principle, its application is simple but we must first evaluate and store the two sequences η_j and $\mu(n)$.

For η_j we can use relation (9). If the surmised primes are located around n, it is sufficient to determine η_j to $j_{max} = \sqrt{8n/3}$. For $\mu(n)$, evaluated by recurrence (19), we have to carry on to n. Once these two sequences are available, we can, for increasing values of k, evaluate $\Delta(6k)$ by means of (24). Whenever $\Delta = 2$, we have a pair of twin primes.

In retrospect, we can also explain the values listed in Table 3 simply by

$$\beta_{\eta_j}(k) = S_j \{\mu(6k-\eta_j+1) - \mu(6k-\eta_j-1)\} . \tag{25}$$

Thus, for k = 3 and j = 4, we find

$$\beta_7(3) = -1 \{\mu(18-7+1) - \mu(18-7-1)\} = -35 ,$$

as expected.

It is useful to know that some values of k are not worth trying. Thus, in order to exclude the cases where 6k+1 or 6k-1 would be a multiple of 5, it can easily be seen that k must not end (in decimal notation) in the digits 1, 4, 6 or 9 (only k = 1 is allowed). This already eliminates 40 % of the trials. More sophisticated choices would be possible, but are hardly worth while.

8. Some complements

Among the many arrangements of prime numbers, twins are certainly the most popular ones, but there are others. Thus, the occurrence of primes in pairs of the form

$$p_2 = p_1 + 2m , \qquad m = 1, 2, \dots ,$$

is always possible. In addition, one may wish to single out some particular arrangement with three or more primes. Apart from twins (m = 1), pairs of the form

$$p_2 = p_1 + 4 , \tag{26}$$

here called "cousins", may be of some interest. Pairs of the form (26) can also be written as

$$p_{2,1} = 3(2k+1) \pm 2 . \tag{27}$$

Note that the "centres" of cousins, 6k+3, always lie in the middle between subsequent centres of possible twins, given by 6k.

By analogy with (24), one arrives for cousins at the expression

$$\Delta'(6k+3) = \sum_j S_j \{\mu(6k-\eta_j+5) - \mu(6k-\eta_j+1)\} \eta_j . \tag{28}$$

Whenever $\Delta' = 4$, we conclude that $6k+3 \pm 2$ are cousin primes. In this case, the restrictions on k (due to the divisibility of a prime candidate by 5) are different: checks can now be omitted if the final digit of k is 0, 4, 5 or 9.

Some numerical calculations to check the above conclusions are planned; they will no doubt require special programming techniques. However, the main interest of the present study is not the numerical evaluation of prime pairs, but the very existence of an algebraic method which, at least in principle, makes it possible to find prime numbers without recourse to any kind of sieve method.

This brief excursion into the territory of number theory is dedicated to the memory of my dear friend Herbert Gross (1936-1989) who, in 1961, became Associate Professor of Mathematics at the Montana State University, Bozeman, and, in 1967, Full Professor at the University of Zurich. Although best known to insiders for his deep results on quadratic forms in infinite-dimensional spaces, he also possessed an extended general knowledge, had a cunning humour and was always ready to take an active interest in new ideas. We lost with him an excellent mathematician and a wonderful man.

References

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[3] A. Weil: "L'oeuvre arithmétique d'Euler", in "Leonhard Euler 1707-1783, Beiträge zu Leben und Werk" (Birkhäuser, Basel, 1983), p. 111.

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