A simple derivation of the Takács formula*

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Abstract

Using the approach indicated by Feller to determine the distribution of the effective time of paralysis produced by an extended dead time, we show how the output rate for a generalized dead time of the Albert-Nelson type can be readily obtained for a Poisson input.

1. Introduction

Going back in the scientific literature to the birthplace of a new development is always rewarding - and sometimes full of surprises. As for the notion of a generalized dead time, it is usually considered that this concept made its first appearance in a paper published by Albert and Nelson [1] in 1953. In fact, it may be suspected that similar or equivalent ideas have been considered before, but that the novelty has not been clearly noted. In a way, series arrangements of two dead times of different type can well be taken as a valid generalization - at least in retrospect -, and such cases have been treated long before. However, these approaches were not made for the purpose of generalizing the traditional dead-time types, nor has the link been clearly recognized at that time.

A closer look at [1] is interesting, but ultimately rather disappointing, in spite of the length and the mathematical appearance of the paper. Thus, more than half of the contents deals with confidence intervals for the original count rate (with several numerical examples), a subject that is now considered to be of quite marginal interest. Hence, the only point of more than transitory value seems to be the clear description of a model, but its practical application has not been pursued in any detail.

Since generalizations can usually be made in many ways, it is of interest to dispose of criteria permitting one to compare their relative merits. Among them much weight is justly given to the mathematical structure which should be simple enough to permit an exact evaluation of the quantities that are of main practical interest. Some elementary

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* This report is dedicated to Wilfrid B. Mann (NBS) on the occasion of his eightieth birthday.

considerations would show that in our case this requirement leaves the Albert-Nelson model as the only serious competitor in the field. However, it seems that the authors were essentially led to their proposal by intuition.

Real progress with this model was only achieved some years later by Takacs who, in a series of sophisticated papers (often in Hungarian, which hardly helps matters) further developed the theory of a generalized "counter" (Fig. 1). He was, in particular, the first to obtain the Laplace transform for the interval density of the pulses after a generalized dead time. From this result the average time interval, i.e. the reciprocal of the output rate R, can be readily obtained. Unfortunately, Takács' papers are written in a way that puts high mathematical demands on the reader. In view of the basic role that the formula for the generalized output rate

$$R = \frac{\theta\rho}{e^{\theta\rho\tau} + \theta - 1} \tag{1}$$

has played in recent years (and is likely to continue to play), we thought that many potential users might be pleased to have available a simple derivation of (1). This seems to be possible indeed, namely by having recourse to an idea described a long time ago by Feller [2] in the context of an extended dead time. As we shall see below, this approach can be readily generalized.

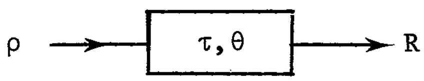

Fig. 1 - Notation for the count rates used in connection with a generalized dead time.

The traditional way to explain the effect of an extended dead time on a pulse train uses the fact that in this case each input event is followed by a dead time τ. Since all pulses arriving during a dead time are lost, this is equivalent to saying that for the input process, described by the interval density f(t), all events which have a distance of less than τ from their predecessor are eliminated. Hence, this concerns the fraction

$$\Pi = \int_0^\tau f(t)\, dt \,. \tag{2}$$

If the input is Poissonian, with count rate ρ and interval density

$$f(t) = \rho\, e^{-\rho t} , \qquad \text{for } t > 0 , \tag{3}$$

then

$$\Pi = \rho \int_0^\tau e^{-\rho t}\, dt = 1 - e^{-\rho\tau} .$$

The count rate of the "surviving" pulses at the output is therefore

$$R = \rho(1 - \Pi) = \rho\, e^{-\rho\tau} , \qquad (4)$$

which is the well-known expression valid for a Poisson input count rate ρ and an extended dead time τ. As (2) is quite general, output rates can also be determined for more complicated input processes, provided $f(t)$ is known.

We may note that in contrast to the above situation, where the dead times, and as a consequence the losses, can be associated with the input events, the case of a non-extended dead time calls for a treatment where only the output pulses are followed by a dead time which can produce losses. This is an essential difference which seems difficult to bridge in the case of a generalized dead time. It is the merit of W. Feller to have shown a way out of this problem, although this was not at all the purpose of his approach. For an extended dead time, as is well known, the period of actual paralysis is not constant, but has a certain random distribution that Feller found worth while to determine.

2. The Feller mechanism

Let us briefly describe in which way Feller [2] has evaluated the effective time of paralysis. Once an event has been registered and has initiated a dead time τ, an extension can be produced by a subsequent pulse, provided it arrives within τ, and this process may be repeated. For a Poisson input, the conditional probability density for an extension by T due to the next event is given by (Fig. 2)

$$h(T) = \frac{1}{q}\,\rho\, e^{-\rho T} , \qquad \text{for} \quad 0 > T > \tau , \qquad (5)$$

where

$$q = \int_0^\tau f(T)\, dT = 1 - e^{-\rho\tau}$$

is the probability for a first extension to take place. However, since additional extensions may occur subsequently, their total length is given by the random sum

$$T = T_1 + T_2 + \dots + T_n , \qquad (6)$$

where n is the last extension. All contributions are independent and follow (5).

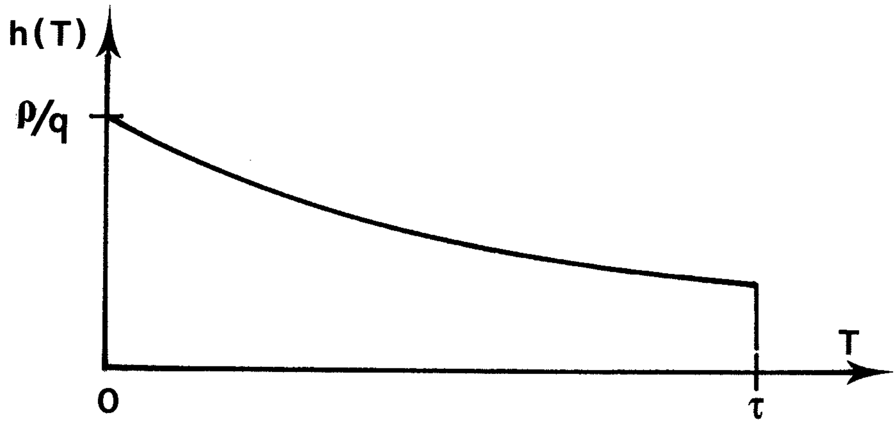

Fig. 2 - The probability density h(T) causing an extension of a dead time.

It is practical to use Laplace transforms from now on. We readily find that

$$\tilde{h}(s) \equiv \mathcal{L}\{h(T)\} = \int_0^\infty e^{-sT}\, h(T)\, dt$$

$$= \frac{\rho}{q} \int_0^\tau e^{-(s+\rho)T}\, dT = \frac{1}{q}\,\frac{\rho}{s+\rho}\left[1 - e^{-(s+\rho)\tau}\right] . \qquad (7)$$

The density $h_n(T)$ for the arrival of event number n has the transform

$$\mathcal{L}\,\{h_n(T)\} = \tilde{h}_n(s) = \left[\tilde{h}(s)\right]^n . \qquad (7')$$

We now use the fact that the probability for n to be the last pulse arriving within the dead time of its predecessor is given by the geometric law

$$P_n = q^n\,(1-q) , \qquad \text{for} \quad n = 0,\ 1,\ 2,\ \ldots ,$$

where the case $n = 0$ corresponds to no extension.

The unconditional density for the arrival time T of the last event capable of extending the dead time is thus

$$H(T) = P_0\, h_0(T) + P_1\, h_1(T) + P_2\, h_2(T) + \ldots ,$$

with the transform

$$\tilde{H}(s) = \sum_{n=0}^{\infty} P_n\, \tilde{h}_n(s) = p \sum_{n=0}^{\infty} q^n \left[\tilde{h}(s)\right]^n . \qquad (8)$$

where we have put $p = 1-q = e^{-\rho\tau}$. One can see that $h_0(T)$ corresponds to the delta function $\delta(T)$ since, according to (7'), its transform $\tilde{h}_0(s)$ is unity.

Substitution of (7) leads to

$$\tilde{H}(s) = p \sum_{n=0}^{\infty} \left\{\frac{\rho[1 - e^{-(s+\rho)\tau}]}{s + \rho}\right\}^n$$

$$= \frac{p}{1 - \frac{\rho}{s+\rho}\left[1 - e^{-(s+\rho)\tau}\right]}$$

$$= \frac{(s+\rho)\, e^{-\rho\tau}}{s + \rho\, e^{-(s+\rho)\tau}} , \qquad (9)$$

which is identical with Feller's result (eq. 42 of [2]). It is possible to determine the exact probability density [3] corresponding to (9), but this result is not needed here. The only quantity relevant for what follows is the mean value of the extension T defined in (6), and this can be obtained directly from its transformed density (9). Indeed, for any density f(x), with transform $\tilde{f}(s)$, the ordinary moments

$$E\{x^r\} \equiv m_r(x)$$

are obtained by forming the derivative

$$m_r(x) = (-1)^r \left.\frac{d\tilde{f}(s)}{ds^r}\right|_{s=0} .$$

In our case, we thus have

$$\bar{T} = -\left.\frac{d\tilde{H}(s)}{ds}\right|_{s=0} \equiv -\tilde{H}'(0) \qquad (10)$$

and some elementary manipulations with (9) lead to the relation

$$\tilde{H}'(0) = \frac{1}{\rho}(1 + \rho\tau - e^{\rho\tau}) . \qquad (11)$$

Remembering that the last arrival time T is followed by a dead time of length τ, the average total time of paralysis following a pulse at the output of an extended dead time is found to be

$$\tau_{eff} = \tau - \tilde{H}'(0) = \frac{1}{\rho}(e^{\rho\tau} - 1) . \qquad (12)$$

The last step, namely the link with the output rate, is now obvious.

As the effective length of dead time (12) is associated with an output pulse, the count-rate balance between input (ρ) and output (R) events is given by

$$R = \rho - R\,\rho\tau_{eff} , \tag{13}$$

since ρ is the density of original pulses falling in the (extended) dead time and which are therefore lost. Hence, by substitution of (12) we readily obtain the expected relation

$$R = \frac{\rho}{1 + \rho\tau_{eff}} = \rho\, e^{-\rho\tau} , \tag{14}$$

which is well known to hold for an extended dead time. We may conclude from the above reasoning that it is always possible to associate the loss-producing dead times with the (surviving) output pulses, where the price we have to pay for this generalization consists in replacing the nominal value τ by an effective one which will, in general, be longer and become a function of the input rate. This particular point has been considered in more detail in [5].

3. Application to a generalized dead time

The application of Feller's mechanism to a dead time of the Albert-Nelson type is now very simple. It is sufficient to remember that for a generalized dead time only a fraction θ of all the incoming events is capable of triggering an extension of the dead time. Since pulses chosen at random (with probability θ) from a Poisson process are still Poisson distributed, but with mean value $\theta\rho$, the contents of section 2 are fully applicable if we simply replace ρ by $\theta\rho$, and this leads to

$$_{\theta}\tau_{eff} = \frac{1}{\theta\rho}\left(e^{\theta\rho\tau} - 1\right) . \tag{15}$$

For the count-rate balance we only have to keep in mind that the pulses suppressed by the dead times are those of the input process, thus with count rate ρ. Therefore we still have

$$R = \rho - R\; _{\theta}\tau_{eff}\, \rho ,$$

from which it follows that

$$R = \frac{\rho}{1 + \rho \dfrac{1}{\theta\rho}\left(e^{\theta\rho\tau} - 1\right)} = \frac{\theta\rho}{e^{\theta\rho\tau} + \theta - 1} ,$$

which is indeed the correct expression (1) for the output rate of a generalized dead time.

It is easy to verify that this formula for R includes as limiting cases (i.e. for $\theta = 0$ or 1) the traditional types of dead time.

In retrospect, one may even be somewhat surprised to see that Albert and Nelson have not themselves made use of the Feller mechanism (with its simple outcome) in order to render their model quantitative and thus really useful. Since Feller's paper is listed among their references, they must have been familiar with its contents, but apparently the close link escaped them all the same.

4. Concluding remarks

We may first note that the simple derivation of (1), which we call the Takács formula, is closely linked to the definition chosen for a generalized dead time. It can therefore also be considered as an a posteriori justification for the choice made by Albert and Nelson.

Of more relevance is the fact that the Feller mechanism lends itself to other important applications, after suitable modifications, as we hope to illustrate soon for some series arrangements of two dead times.

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The author takes pleasure in dedicating this report to Dr. W.B. Mann. Wilfrid's services rendered to the field of radioactivity are too well known to need commenting upon. At BIPM we gratefully remember him as a prominent, although at times difficult, member of Section II; at its meetings lengthy discussions have often been enlivened by his good jokes. May his collaborators, and his wife Miriam in the first place, go on spoiling him for many years to come.

APPENDIX

Transform of the interval distribution for a generalized dead time

In view of the important role played by the Laplace transform of the modified interval density F(t) in most recent studies on generalized dead times, it seems worth while to derive this expression from scratch (Fig. A1). This is all the more tempting as, with the expression (9) for the dead-time extension available, by far the most difficult part in the complete derivation has already been done.

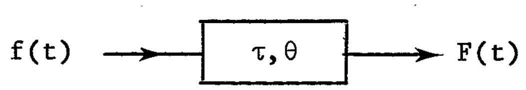

Fig. A1 - The probability interval densities used in conjunction with a generalized dead time.

The time interval t between two consecutive output pulses can be readily subdivided into three additive contributions (Fig. A2)

$$t = t_1 + t_2 + t_3 \,, \tag{A1}$$

where

t_1 is the extension T of the nominal dead time, which we have studied in the main part of this report by means of the Feller mechanism,

t_2 is the constant dead time τ which follows the last event detected within the dead time initiated by the registered pulse and

t_3 is the waiting time, after the end of the paralysis, for the arrival of the next event, which will be registered.

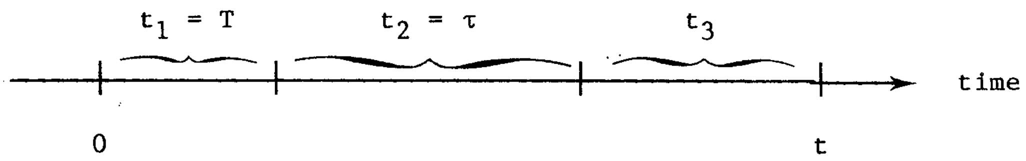

Fig. A2 - Schematic decomposition of the time interval t between two registered pulses into the three components described in the text.

Since the three contributions are independent of each other, the probability density F(t) of their sum, which is the observed interval t, is given by the convolutions

$$F(t) = {}_{\theta}H(t) * \delta(t-\tau) * f(t) \,, \tag{A2}$$

where ${}_{\theta}H(t)$ is identical with H(t), but with ρ replaced by $\theta\rho$, for the reason explained before in the context of (15). The Laplace transform of the interval density looked for can therefore be written as

$$\tilde{F}(s) = {}_{\theta}\tilde{H}(s)\, e^{-s\tau}\, \tilde{f}(s) \,.$$

Since for a Poisson input we have $\tilde{f}(s) = \rho/(\rho+s)$ and as $\tilde{H}(s)$ is known from (9), this leads directly to the final formula

$$\tilde{F}(s) = \frac{(s + \theta\rho)\, e^{-\theta\rho\tau}}{s + \theta\rho\, e^{-(s+\theta\rho)\tau}}\; e^{-\rho\tau}\; \frac{\rho}{\rho + s}$$

$$= \left(\frac{\theta\rho + s}{\rho + s}\right) \left[\frac{\rho\, e^{-(\theta\rho+s)\tau}}{s + \theta\rho\, e^{-(\theta\rho+s)\tau}}\right] . \tag{A3}$$

This is the correct expression for the transformed interval density that a Poisson process assumes after passage through a generalized dead time of the Albert-Nelson type.

The corresponding original probability density, which turns out to be rather complicated, can be found by inverting the transform (A3); this has been achieved in [6] but the result is not needed here. We may mention that (A3) is also the natural starting point for the evaluation of the moments of t beyond first order which are required for a valid characterization of the respective counting statistics.

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(July 1988)