Explicit evaluation of the transmission factor $T_1(\theta,E)$

Part I: For small dead-time ratios

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Abstract

By a detailed evaluation of $T_1(\theta,E)$, the transmission factor appearing in the description of a series arrangement of two dead times, it is shown that formulae obtained previously by heuristic arguments are indeed valid, provided that the ratio of the dead times remains within certain limits.

1. Introduction

A simple and useful way of treating series arrangements of two dead times is based on the concept of a so-called transmission factor T_1 (see e.g. [1]). This quantity describes in a quantitative manner the additional effect produced by the first dead time on the observed count rate. In the general situation, sketched in Fig. 1, both dead times are of the generalized type, i.e. described by the two parameters τ and θ.

Fig. 1. Basic notation used for the arrangement of two dead times in series.

For $\tau_1 = 0$, i.e. when there is a single dead time, the relation between input and output count rates is described by

$$R = R_o = T_2 \, \rho. \tag{1}$$

If the input series forms a Poisson process (as we shall assume in what follows), the transmission factor T_2 is a known function of ρ, τ_2 and θ_2.

For a series arrangement we write similarly

$$R = T_1 \, R_o = T_1 \, T_2 \, \rho \, . \tag{2}$$

For a given second dead time, the new transmission factor T_1 is a function of ρ, τ_1 and θ_1.

It will be practical to change the notation slightly by putting

$$\tau_2 = \tau, \quad \tau_1 = \alpha\tau \quad \text{and} \quad x = \rho\tau ,$$

with $0 \leqslant \alpha \leqslant 1$. T_1 then turns out to be a function of x, α, θ_1 and θ_2. For the sake of simplicity, we omit x and α as arguments and write $T_1 = T_1(\theta_1, \theta_2)$.

For the case in which we are now particularly interested, the second (and larger) dead time is of the extended type. We then have $T_2(E) = e^{-x}$ and $T_1 = T_1(\theta, E)$, putting $\theta_1 = \theta$.

For the series arrangements of traditional types, the transmission factors T_1 are known. From them we have recently deduced [2] the first terms of a series development which should be valid for an arbitrary $\theta_1 = \theta$. As the reasoning applied there was based on some assumptions, it may be useful to look for an independent confirmation. This is done in the present report where we derive, for $\theta_2 = 1$, the transmission factor T_1 up to fourth order. The procedure takes advantage of a general feature valid for an extended dead time.

For simplifying the calculations, we shall in what follows always assume that the dead-time ratio $\alpha = \tau_1/\tau_2$ does not exceed the value 1/4.

2. Counting losses due to an extended dead time

Let us start by considering an arbitrary renewal process which is described by its interval density $f(t)$. Then the average time $\bar{t}$ between two successive events of this process is given by

$$\bar{t} = \int_0^\infty t\, f(t)\, dt \,. \tag{3}$$

and its reciprocal $1/\bar{t}$ is called the count rate r.

We now want to describe the effect of an extended dead time of length τ which is inserted in this process. While the distortion produced on the interval density is in general difficult to describe, it happens that the effect on the count rate can be readily expressed in a general manner. This is due to the fact that, for this type of dead time, all events which follow each other by a time interval of less than τ are suppressed. Therefore, the relative count-rate loss L produced by an extended dead time τ is given by the simple relation

$$L = \int_0^\tau f(t)\, dt \,. \tag{4}$$

The output count rate R is therefore reduced to the value

$$R = r\,(1 - L) \,, \tag{5}$$

where r denotes the input rate.

The above relations are of such an elementary form that it should be possible to apply them to relatively complicated input processes. In particular, an important application might be given by the case where the input process is assumed to have already undergone a distortion, namely by a previous "first" dead time τ' which we may suppose to be of the generalized type (involving the parameter $0 \leqslant \theta \leqslant 1$) described previously. In this situation, the evaluation of R would in fact correspond to the explicit determination of an output rate for an original process which has passed through a series arrangement of two dead times as sketched in Fig. 2.

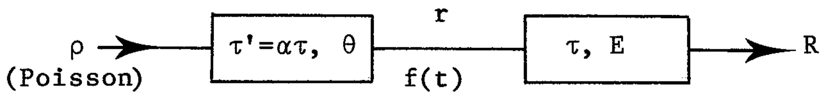

Fig. 2 - Notation used for the series arrangement of two dead times, where the first is of the generalized type and the second is assumed to be extended.

If the original input process, of rate ρ, is taken as Poissonian (an assumption which can be shown to be realistic in many practical situations), then both the count rate r and the interval density f(t) after the first dead time τ' are well known. Indeed, for an original input rate ρ and a dead time of length τ' and parameter θ the output is known to be [3]

$$r = \frac{\theta\rho}{e^{\theta\rho\tau'} + \theta - 1} . \tag{6}$$

The density f(t), valid between the two dead times, is in fact the interval density after a generalized dead time, and this has been derived previously in [4]. It turns out that the later developments can be slightly simplified by using the form given in [5], which is (for $\theta > 0$)

$$f(t) = \sum_{j=1}^{J} \left[\frac{1}{\theta} A_j + \left(\frac{1-\theta}{\theta}\right) B_j\right] , \tag{7}$$

where J denotes the largest integer below t/τ'. The coefficients are given by

$$A_j = \frac{\theta\rho}{(j-1)!} (-\theta T_j)^{j-1} e^{-j\theta\rho\tau'} \quad \text{and}$$

$$B_j = \rho(\theta\, e^{-\theta\rho\tau'})^j \left[e^{-T_j} + (-1)^j \sum_{k=0}^{j-1} \frac{(-1)^k}{(j-1-k)!} T_j^{j-1-k}\right] , \tag{8}$$

with $T_j = \rho(t-j\tau')$. Note that both A_j and B_j vanish if $T_j < 0$, i.e. for $j\tau' > t$.

It will be useful to have some explicit forms available for the first few of the coefficients appearing in (7). In order to simplify the notation we introduce the abbreviations

$$\theta\rho = \tilde{\rho}\,, \qquad \theta x = \tilde{x} \qquad \text{and} \qquad \alpha x = x'\,, \qquad \theta x' = \tilde{x}'\,. \tag{9}$$

This then leads with (8) to

$$\begin{aligned} A_1 &= \tilde{\rho}\, e^{-\tilde{x}'}\,, \\ A_2 &= -\,\tilde{\rho}^2\,(t-2\tau')\, e^{-2\tilde{x}'}\,, \\ A_3 &= \frac{1}{2}\,\tilde{\rho}^3\,(t-3\tau')^2\, e^{-3\tilde{x}'}\,, \\ A_4 &= -\,\frac{1}{6}\,\tilde{\rho}^4\,(t-4\tau')^3\, e^{-4\tilde{x}'}\,, \quad \text{etc.} \end{aligned} \tag{10}$$

Likewise one finds for the first coefficients B_j

$$\begin{aligned} B_1 &= \rho\,\theta\, e^{-\tilde{x}'}\,\{e^{-\rho(t-\tau')} - 1\}\,, \\ B_2 &= \rho\,\theta^2\, e^{-2\tilde{x}'}\,\{e^{-\rho(t-2\tau')} + \rho(t-2\tau') - 1\}\,, \\ B_3 &= \rho\,\theta^3\, e^{-3\tilde{x}'}\,\{e^{-\rho(t-3\tau')} - \frac{1}{2}\,\rho^2(t-3\tau')^2 + \rho(t-3\tau') - 1\}\,, \\ B_4 &= \rho\,\theta^4\, e^{-4\tilde{x}'}\,\{e^{-\rho(t-4\tau')} + \frac{1}{6}\,\rho^3(t-4\tau')^3 \\ &\qquad - \frac{1}{2}\,\rho^2(t-4\tau')^2 + \rho(t-4\tau') - 1\}\,, \quad \text{etc.} \end{aligned} \tag{11}$$

Since the output rate R, given by (5), can be written as

$$R = \frac{\theta\rho}{e^{\theta\rho\tau'} + \theta - 1}\,\left[1 - \int_{\tau'}^{\tau} f(t)\,dt\right]\,, \tag{12}$$

our main problem is now the practical determination of the loss

$$\begin{aligned} L &= \int_{\tau'}^{\tau} f(t)\,dt = \sum_{j=1}^{J}\left[\frac{1}{\theta}\int A_j\,dt + \frac{1-\theta}{\theta}\int B_j\,dt\right] \\ &\equiv \frac{1}{\theta}\sum_j p_j + \left(\frac{1-\theta}{\theta}\right)\sum_j q_j\,, \end{aligned} \tag{13}$$

thus of the quantities p_j and q_j.

3. Evaluation of p_j

In view of (13) and (8) the quantities p_j are given by

$$
\begin{aligned}
p_j &= \int_{j\tau'}^{\tau} A_j \, dt = \frac{\tilde{\rho}}{(j-1)!} (-\tilde{\rho})^{j-1} e^{-j\tilde{x}'} \int_{j\tau'}^{\tau} (t-j\tau')^{j-1} \, dt \\
&= \frac{\tilde{\rho}}{(j-1)!} (-\tilde{\rho})^{j-1} e^{-j\tilde{x}'} \frac{(\tau - j\tau')^j}{j} \\
&= \frac{\tilde{\rho}\,\tau}{j!} (-\tilde{x})^{j-1} e^{-j\tilde{x}'} (1-j\alpha)^j \\
&= -(-\tilde{x}\, e^{-\tilde{x}'})^j \frac{(1-j\alpha)^j}{j!} ,
\end{aligned}
\tag{14}
$$

thus explicitly

$$
\begin{aligned}
p_1 &= \tilde{x}\,(1-\alpha)\, e^{-\tilde{x}'} \\
p_2 &= -\frac{1}{2} (1-2\alpha)^2 \tilde{x}^2 e^{-2\tilde{x}'} , \\
p_3 &= \frac{1}{6} (1-3\alpha)^3 \tilde{x}^3 e^{-3\tilde{x}'} , \\
p_4 &= -\frac{1}{24} (1-4\alpha)^4 \tilde{x}^4 e^{-4\tilde{x}'} , \quad \text{etc.}
\end{aligned}
\tag{15}
$$

Since in (14) the term $1 - j\alpha$ must remain positive, this limits the size of the dead-time ratio α. If we assume $\alpha < 1/4$, all the expressions up to p_4 are meaningful. This remark also holds for the coefficients q_j of the following section.

4. Evaluation of q_j

In a similar way we can determine the coefficients q_j appearing in (13) by means of (8) as

$$
q_j = \int_{j\tau'}^{\tau} B_j \, dt = \rho(\theta\, e^{-\tilde{x}'})^j \left\{ \int_{j\tau'}^{\tau} e^{-\rho(t-j\tau')} dt + (-1)^j \sum_{k=0}^{j-1} \frac{(-1)^k}{(j-1-k)!} \rho^{j-1-k} \int_{j\tau'}^{\tau} (t-j\tau')^{j-1-k} dt \right\} .
$$

Since

$$
e^{jx'} \int_{j\tau'}^{\tau} e^{-\rho t} \, dt = e^{jx'} \frac{1}{\rho} (e^{-jx'} - e^{-x}) = \frac{1}{\rho} \left[1 - e^{-x(1-j\alpha)}\right] ,
$$

we can write

$$q_j = \rho\, \theta^j\, e^{-j\tilde{x}'} \Big\{\frac{1}{\rho}\left[1 - e^{-x(1-j\alpha)}\right] + (-1)^j \sum_{k=0}^{j-1} \frac{(-1)^k}{(j-k)!}\, \frac{x^{j-k}}{\rho}\, (1-j\alpha)^{j-k}\Big\}$$

$$= (\theta\, e^{-\tilde{x}'})^j \Big\{1 - e^{-x(1-j\alpha)} + \sum_{k=0}^{j-1} \frac{[-x(1-j\alpha)]^{j-k}}{(j-k)!}\Big\} . \qquad (16)$$

Explicit expressions are

$$\begin{aligned}
q_1 &= \theta\, e^{-\tilde{x}'} \left[1 - e^{-(1-\alpha)x} - (1-\alpha)x\right] ,\\
q_2 &= \theta^2\, e^{-2\tilde{x}'} \left[1 - e^{-(1-2\alpha)x} + \frac{1}{2}(1-2\alpha)^2 x^2 - (1-2\alpha)x\right] ,\\
q_3 &= \theta^3\, e^{-3\tilde{x}'} \Big[1 - e^{-(1-3\alpha)x} - \frac{1}{6}(1-3\alpha)^3 x^3 \\
&\qquad + \frac{1}{2}(1-3\alpha)^2 x^2 - (1-3\alpha)x\Big] , \qquad (17)\\
q_4 &= \theta^4\, e^{-4\tilde{x}'} \Big[1 - e^{-(1-4\alpha)x} + \frac{1}{24}(1-4\alpha)^4 x^4 \\
&\qquad - \frac{1}{6}(1-4\alpha)^3 x^3 + \frac{1}{2}(1-4\alpha)^2 x^2 - (1-4\alpha)x\Big] , \quad \text{etc.}
\end{aligned}$$

5. Series developments

Since we wish to have the transmission factor T_1 in the form of a series expansion in powers of x and because, according to (13), the determination of the loss L requires a summation over the coefficients p_j and q_j, we now have to evaluate their series developments, for which we decide to proceed as far as order x^4 (or likewise $\tilde{x}^4$, etc.).

For p_j we then obtain from (15)

$$\begin{aligned}
p_1 &\cong (1-\alpha)\left[\tilde{x} - \alpha\,\tilde{x}^2 + \frac{1}{2}\alpha^2\,\tilde{x}^3 - \frac{1}{6}\alpha^3\,\tilde{x}^4\right] ,\\
p_2 &\cong -\frac{1}{2}(1-2\alpha)^2 \left[\tilde{x}^2 - 2\alpha\,\tilde{x}^3 + 2\alpha^2\,\tilde{x}^4\right] ,\\
p_3 &\cong \frac{1}{6}(1-3\alpha)^3 \left[\tilde{x}^3 - 3\alpha\,\tilde{x}^4\right] , \qquad (18)\\
p_4 &\cong -\frac{1}{24}(1-4\alpha)^4\, \tilde{x}^4 , \quad \text{etc.}
\end{aligned}$$

For their sum this leads to

$$\sum_j p_j \cong \tilde{x}\,(1-\alpha) - \tilde{x}^2\left[\alpha(1-\alpha) + \frac{1}{2}(1-2\alpha)^2\right]$$
$$+ \tilde{x}^3\left[\frac{1}{2}\alpha^2(1-\alpha) + \alpha(1-2\alpha)^2 + \frac{1}{6}(1-3\alpha)^3\right]$$
$$- \tilde{x}^4\left[\frac{1}{6}\alpha^3(1-\alpha) + \alpha^2(1-2\alpha)^2 + \frac{1}{2}\alpha(1-3\alpha)^3 + \frac{1}{24}(1-4\alpha)^4\right]$$
$$= \tilde{x}\,(1-\alpha) - \frac{1}{2}\tilde{x}^2(1-2\alpha+2\alpha^2) + \frac{1}{6}\tilde{x}^3(1-3\alpha+6\alpha^2-6\alpha^3)$$
$$- \frac{1}{24}\tilde{x}^4(1-4\alpha+12\alpha^2-24\alpha^3+24\alpha^4)\ . \tag{19}$$

Analogous developments for q_j, given by (17), yield (again up to fourth order), after some lengthy rearrangements,

$$q_1 \cong -\frac{1}{2}\theta(1-\alpha)^2\left[x^2 - \frac{1}{3}(1-\alpha+3\alpha\theta)\,x^3 + \frac{1}{12}(1-2\alpha+4\alpha\theta+\alpha^2-4\alpha^2\theta+6\alpha^2\theta^2)\,x^4\right],$$
$$q_2 \cong \frac{1}{6}\theta^2(1-2\alpha)^3\left[x^3 - \frac{1}{4}(1-2\alpha+8\alpha\theta)\,x^4\right], \tag{20}$$
$$q_3 \cong -\frac{1}{24}\theta^3(1-3\alpha)^4\,x^4\ .$$

For their sum one finally arrives at the result

$$\sum_j q_j \cong -\frac{1}{2}\theta(1-\alpha)^2x^2$$
$$+ \frac{1}{6}\theta\left[1+\theta - 3\alpha(1+\theta) + 3\alpha^2(1+2\theta) - \alpha^3(1+5\theta)\right]x^3$$
$$- \frac{1}{24}\theta\left[1+\theta+\theta^2 - 4\alpha(1+\theta+\theta^2) + 6\alpha^2(1+2\theta+2\theta^2) - 4\alpha^3(1+5\theta+6\theta^2) + \alpha^4(1+12\theta+23\theta^2)\right]x^4\ . \tag{21}$$

By substitution of (19) and (21) in (13), the loss L can be seen to be given (up to fourth order in x) by

$$
\begin{aligned}
L \;\cong\; & (1-\alpha)\,x - \frac{1}{2}\,\theta\,(1-2\alpha+2\alpha^2)\,x^2 + \frac{1}{6}\,\theta^2\,(1-3\alpha+6\alpha^2-6\alpha^3)\,x^3 \\
& \quad - \frac{1}{24}\,\theta^3\,(1-4\alpha+12\alpha^2-24\alpha^3+24\alpha^4)\,x^4 \\
& + (1-\theta)\Big\{-\frac{1}{2}\,(1-\alpha)^2\,x^2 + \frac{1}{6}\,\big[(1+\theta - 3\alpha(1+\theta) + 3\alpha^2(1+2\theta) - \alpha^3(1+5\theta)\big]\,x^3 \\
& \quad - \frac{1}{24}\,\big[1+\theta+\theta^2 - 4\alpha(1+\theta+\theta^2) + 6\alpha^2(1+2\theta+2\theta^2) \\
& \qquad - 4\alpha^3(1+5\theta+6\theta^2) + \alpha^4(1+12\theta+23\theta^2)\big]\,x^4\Big\}\;.
\end{aligned}
$$

This can be rearranged to yield

$$
\begin{aligned}
L \;\cong\; & (1-\alpha)\,x - \frac{1}{2}\,\big[(1-\alpha)^2 + \alpha^2\theta\big]\,x^2 + \frac{1}{6}\,\big[(1-\alpha)^3 + \alpha^2\theta(3-4\alpha) - \alpha^3\theta^2\big]\,x^3 \\
& - \frac{1}{24}\,\big[(1-\alpha)^4 + \alpha^2\theta\,(6-16\alpha+11\alpha^2) - \alpha^3\theta^2\,(4-11\alpha) + \alpha^4\theta^3\big]\,x^4\;.
\end{aligned} \tag{22}
$$

6. Application to T_1

We now have at hand all the elements needed for the evaluation of the transmission factor which, according to (2) and (5), is given by

$$
T_1(\theta,E) \;=\; \frac{R}{\rho\,T_2(E)} \;=\; \frac{r}{\rho}\,e^x\,(1-L)\;. \tag{23}
$$

The explicit multiplication of the respective series expansions is quite elementary, but rather cumbersome. As a first step we can use a series development of (6) given previously (eq. 8 in [3]). This leads to

$$
\begin{aligned}
\frac{r}{\rho}\,e^x \;\cong\; & \Big\{1 - \alpha x + (1-\frac{1}{2}\,\theta)\,\alpha^2x^2 - (1-\theta+\frac{1}{6}\,\theta^2)\,\alpha^3x^3 \\
& + (1-\frac{3}{2}\,\theta + \frac{7}{12}\,\theta^2 - \frac{1}{24}\,\theta^3)\,\alpha^4x^4\Big\}\cdot e^x \\
\cong\; & 1 + (1-\alpha)\,x + \frac{1}{2}\,\big[1 - 2\alpha + (2-\theta)\alpha^2\big]\,x^2 \\
& + \frac{1}{6}\,\big[1 - 3\alpha + (6-3\theta)\alpha^2 - (6-6\theta+\theta^2)\alpha^3\big]\,x^3 \\
& + \frac{1}{24}\,\big[1 - 4\alpha + (12-6\theta)\alpha^2 - (24-24\theta+4\theta^2)\alpha^3 \\
& \qquad + (24-36\theta+14\theta^2-\theta^3)\alpha^4\big]\,x^4\;.
\end{aligned} \tag{24}
$$

The final multiplication with 1-L, where L is taken from (22), is again somewhat tedious. Since many terms cancel, the final result is nevertheless quite simple. Thus, for the required transmission factor we end up with the expression (up to fourth order and for $\alpha \leqslant 1/4$)

$$T_1(\theta,E) \cong 1 + \frac{1}{2}(\alpha x)^2 - \frac{1}{3}(1-2\theta)(\alpha x)^3 + \frac{1}{24}(9 - 11\theta + 11\theta^2)(\alpha x)^4 , \tag{25}$$

which is the main result of the present study.

The two special cases $\theta = 0$ and 1 can serve as welcome checks. They lead to

$$T_1(N,E) \cong 1 + \frac{1}{2}(\alpha x)^2 - \frac{1}{3}(\alpha x)^3 + \frac{3}{8}(\alpha x)^4 \quad \text{and}$$
$$T_1(E,E) \cong 1 + \frac{1}{2}(\alpha x)^2 + \frac{1}{3}(\alpha x)^3 + \frac{3}{8}(\alpha x)^4 . \tag{26}$$

The two results given in (26) are in agreement with what we know from previous studies [6]; the second requires $\alpha \leqslant 1/4$ whereas the first is valid for any value of α.

More important is the fact that (25) fully agrees with the result given in [2], confirming thereby the correctness of the heuristic procedure applied previously. The new contribution of fourth order is particularly instructive since it proves that the equality of the corresponding terms in (26) is only accidental and must not be interpreted as indicating an independence of θ.

Unfortunately, we know of no similar general method which would also allow us to derive $T_1(\theta,N)$ to higher order.

We should note that for larger dead-time ratios α, the coefficients appearing in a series expansion of the form given in (25) are in general more complicated as they depend on the location of α (e.g. between 1/2 and 1/3). They will be evaluated in a subsequent report.

APPENDIX

A closer look at T_1 for $\alpha = 1$

The special situation where the two dead times in a series arrangement are of equal length τ, but not necessarily of the same type (described by θ), can be used for a simple reasoning which we shall describe below.

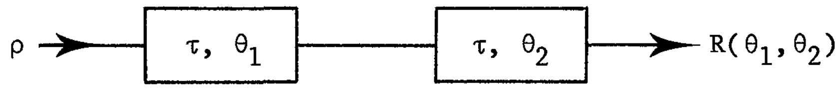

Fig. A1 - Schematic arrangement of two generalized dead times of equal length.

Let us first recall that the second dead time can have no effect at all on a sequence of pulses if in length it is smaller than (or equal to) a preceding one: the loss, and hence the output count rate R, is uniquely determined by the first dead time. Fig. A1 illustrates the special case where $\alpha = \tau_1/\tau_2 = 1$.

If $T(\theta)$ is the transmission factor corresponding to a (single) generalized dead time (τ, θ) and if the transmission factor T_1, when only valid for $\alpha = 1$, is denoted by ${}_1T_1$, then it follows from Fig. A1 that the output count rate R is given by

$$R(\theta_1, \theta_2) = \rho\ {}_1T_1(\theta_1, \theta_2)\ T(\theta_2) = \rho\ T(\theta_1) . \tag{A1}$$

This leads to the relation

$${}_1T_1(\theta_1, \theta_2) = T(\theta_1)/T(\theta_2) . \tag{A2}$$

We know from previous studies - using, for instance, eqs. 8 and 7 in [3], with $T = z/x$ - that

$$\begin{aligned} T(\theta_1) \cong\ & 1 - x + (1 - \frac{1}{2}\theta_1)\, x^2 - (1 - \theta_1 + \frac{1}{6}\theta_1^2)\, x^3 \\ & + (1 - \frac{3}{2}\theta_1 + \frac{7}{12}\theta_1^2 - \frac{1}{24}\theta_1^3)\, x^4 , \end{aligned} \tag{A3}$$

and likewise for the reciprocal

$$T^{-1}(\theta_2) \cong 1 + x + \frac{1}{2}\theta_2\, x^2 + \frac{1}{6}\theta_2^2\, x^3 + \frac{1}{24}\theta_2^3\, x^4 . \tag{A4}$$

${}_1T_1$ is now simply obtained by multiplying (A3) with (A4), and this leads, after some rearrangements and up to fourth order in x, to

$$
\begin{aligned}
{}_1T_1(\theta_1,\theta_2) \;\cong\; & 1 - \frac{1}{2}\,(\theta_1-\theta_2)\,x^2 + \frac{1}{6}\,[3(\theta_1-\theta_2)-\theta_1^2+\theta_2^2]\,x^3 \\
& - \frac{1}{24}\,[12(\theta_1-\theta_2)-10\theta_1^2+4\theta_2^2+\theta_1^3-\theta_2^3+6\theta_1\theta_2]\,x^4 \; .
\end{aligned} \tag{A5}
$$

From this general formula a number of special cases can be readily obtained. Thus, for $\theta_2 = 0$ or 1 we find (with $\theta_1 = \theta$)

$$
{}_1T_1(\theta,N) \;\cong\; 1 - \frac{1}{2}\,\theta\,x^2 + \frac{1}{6}\,\theta\,(3-\theta)\,x^3 - \frac{1}{24}\,\theta\,(12-10\theta+\theta^2)\,x^4 \tag{A6}
$$

and

$$
\begin{aligned}
{}_1T_1(\theta,E) \;\cong\; & 1 + \frac{1}{2}\,(1-\theta)\,x^2 - \frac{1}{6}\,(2-3\theta+\theta^2)\,x^3 \\
& + \frac{1}{24}\,(9-18\theta+10\theta^2-\theta^3)\,x^4 \; .
\end{aligned} \tag{A7}
$$

Or, by putting

$$
\theta_1 - \theta_2 \;=\; \delta \; , \tag{A8}
$$

we can also write instead of (A5)

$$
\begin{aligned}
{}_1T_1 \;\cong\; & 1 - \frac{1}{2}\,\delta\,x^2 + \frac{1}{6}\,(3\delta - \theta_1^2 + \theta_2^2)\,x^3 \\
& - \frac{1}{24}\,[12\delta - 3\delta^2 - 7(\theta_1^2 - \theta_2^2) + \theta_1^3 - \theta_2^3]\,x^4 \; .
\end{aligned} \tag{A9}
$$

In particular, this leads

- for $\delta = 0$ to

$$
{}_1T_1(\theta,\theta) \;=\; 1 \quad \text{and} \tag{A10}
$$

- for $\delta = \pm 1$ to

$$
{}_1T_1 \;\cong\; 1 - \frac{1}{2}\,\delta\,x^2 + \frac{1}{3}\,\delta\,x^3 + \frac{1}{8}\,(1 - 2\delta)\,x^4 \; . \tag{A11}
$$

While (A10) is certainly correct, but rather trivial, (A11) is of some interest for checking purposes as it contains the expansions

$$ {}_1T_1(E,N) \cong 1 - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{8} x^4 \quad \text{and} \tag{A12} $$

$$ {}_1T_1(N,E) \cong 1 + \frac{1}{2} x^2 - \frac{1}{3} x^3 + \frac{3}{8} x^4 \, . \tag{A13} $$

It can also be readily seen from (A2) that

$$ {}_1T_1(\theta_1,\theta_2) \cdot {}_1T_1(\theta_2,\theta_1) = 1 \, . \tag{A14} $$

All this is clearly consistent with the known exact expressions [6] which are (for $\alpha = 1$)

$$ {}_1T_1(E,N) = {}_1T_1^{-1}(N,E) = (1+x)\, e^{-x} \, . \tag{A15} $$

One can also show that, for $\frac{1}{2} \leqslant \alpha \leqslant 1$, there is

$$ T_1(E,E) \cong T_1^{-1}(N,N) \cong 1 + \frac{1}{2} (1-\alpha)(3\alpha - 1)\, x^2 \, . \tag{A16} $$

In the limit $\alpha = 1$ we thus have indeed ${}_1T_1 = 1$, in agreement with (A10).

Unfortunately, (A5) cannot be readily used as a guideline for establishing the structure of a general formula for $T_1(\theta_1,\theta_2)$ since it will, as a rule, depend on the exact region in which α is located, a feature which is well known from the special cases $T_1(N,N)$ and $T_1(E,E)$.

References

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