Some observations on the reversion of series

by Jörg W. Müller

Bureau International des Poids et Mesures, F-92310 Sèvres

Abstract

A convergent power series $y = f(x)$ can be reversed to an equivalent series of the form $x = g(y)$ by the use of well-established formulae which connect the coefficients of the corresponding developments. Some simple observations on symmetries are discussed which appear when the formulae are expressed in the form of recursions for the coefficients. Similarly, the respective effects of sign changes are described and illustrated by examples.

1. Introduction

Let us consider a convergent power series of the type

$$Y = \alpha + a\,x + b\,x^2 + c\,x^3 + d\,x^4 + e\,x^5 + \ldots \quad (1)$$

The corresponding reversed series, with x expressed in powers of $y = Y - \alpha$, can be written as

$$x = A\,y + B\,y^2 + C\,y^3 + D\,y^4 + E\,y^5 + \ldots , \quad (2)$$

where the problem consists in finding the new coefficients in terms of the old ones. Elementary, but tedious, rearrangements yield for the first few of them the expressions (for $a \neq 0$)

$$\begin{aligned}
A &= 1/a , \\
B &= -b/a^3 , \\
C &= (2b^2 - ac)/a^5 , \\
D &= (5abc - a^2b - 5b^3)/a^7 , \\
E &= (6a^2bd + 3a^2c^2 + 14b^4 - a^3e - 21ab^2c)/a^9 , \\
&\ldots
\end{aligned} \quad (3)$$

The first seven coefficients (written in this notation) can be readily found in tables of mathematical formulae (e.g. in [1] or [2]).

By an appropriate change of variables it is always possible to achieve $a = 1$ and $\alpha = 0$, which simplifies the expressions. In addition, the problem of the changing signs can be avoided by writing the original formulae (1) and (2) in a slightly different (but equivalent) way, namely by putting

$$y = x\,(1 - \sum_{n=1}^{\infty} b_n\, x^n)\;, \tag{4}$$

and for the reversed series

$$x = y\,(1 - \sum_{n=1}^{\infty} c_n\, y^n)\;. \tag{5}$$

The previous relations (3) then appear in the simpler form

$$\begin{aligned} -c_1 &= b_1\;, \\ -c_2 &= b_2 + 2b_1^2\;, \\ -c_3 &= b_3 + 5b_1b_2 + 5b_1^3\;, \\ -c_4 &= b_4 + 6b_1b_3 + 3b_2^2 + 21b_1^2b_2 + 14b_1^4\;, \\ &\dots \end{aligned} \tag{6}$$

Expressions for the coefficients c_n up to order 12 have been published a long time ago by Van Orstrand [3]. The only (very minor) difference lies in our choice of the sign of c_n (which is opposite to the convention used in [3]); the reason for this apparently unmotivated change will become obvious at a later stage (section 2).

It is possible to give a general expression for the coefficients in question, although in an operational rather than in a very explicit form. According to [4] we have

$$-c_n = \frac{1}{n+1} \sum_{\langle k_j \rangle} \binom{n+k}{k} \frac{k!}{\prod_j k_j!} \prod_{j=1}^{n} b_j^{k_j}\;, \tag{7}$$

with the conditions that

$$\begin{aligned} \sum k_j &= k \qquad \text{and} \\ \sum j\, k_j &= n\;. \end{aligned} \tag{8}$$

The numbers k_j are non-negative integers and the range of the summation, denoted symbolically by $\langle k_j \rangle$, extends over all p(n) partitions of n (see for instance [2]). Hence c_4, for example, includes only 5 terms, whereas for c_{10} we have already p(10) = 42. An explicit tabulation of the full expressions for these coefficients therefore rapidly becomes a rather cumbersome task.

We may mention that P. Carré (BIPM) has recently set up a computer program which evaluates and prints out the explicit form of (7) for a given order n, i.e. the complete sequence of products of coefficients b_j, with the appropriate factors and powers. In Table 1 we reproduce for illustration the set corresponding to $-c_{10}$.

Table 1 - Computer outprint of the formulae for the coefficients of a reversed power series. The example chosen gives the expression for $-c_{10}$ (by courtesy of the author).

```
         B(10)
    12 * B(1) * B(9)
    12 * B(2) * B(8)
    12 * B(3) * B(7)
    12 * B(4) * B(6)
     6 * B(5)**2
    78 * B(1)**2 * B(8)
   156 * B(1) * B(2) * B(7)
   156 * B(1) * B(3) * B(6)
   156 * B(1) * B(4) * B(5)
    78 * B(2)**2 * B(6)
   156 * B(2) * B(3) * B(5)
    78 * B(2) * B(4)**2
    78 * B(3)**2 * B(4)
   364 * B(1)**3 * B(7)
  1092 * B(1)**2 * B(2) * B(6)
  1092 * B(1)**2 * B(3) * B(5)
   546 * B(1)**2 * B(4)**2
  1092 * B(1) * B(2)**2 * B(5)
  2184 * B(1) * B(2) * B(3) * B(4)
   364 * B(1) * B(3)**3
   364 * B(2)**3 * B(4)
   546 * B(2)**2 * B(3)**2
  1365 * B(1)**4 * B(6)
  5460 * B(1)**3 * B(2) * B(5)
  5460 * B(1)**3 * B(3) * B(4)
  8190 * B(1)**2 * B(2)**2 * B(4)
  8190 * B(1)**2 * B(2) * B(3)**2
  5460 * B(1) * B(2)**3 * B(3)
   273 * B(2)**5
  4368 * B(1)**5 * B(5)
 21840 * B(1)**4 * B(2) * B(4)
 10920 * B(1)**4 * B(3)**2
 43680 * B(1)**3 * B(2)**2 * B(3)
 10920 * B(1)**2 * B(2)**4
 12376 * B(1)**6 * B(4)
 74256 * B(1)**5 * B(2) * B(3)
 61880 * B(1)**4 * B(2)**3
 31824 * B(1)**7 * B(3)
111384 * B(1)**6 * B(2)**2
 75582 * B(1)**8 * B(2)
 16796 * B(1)**10
```

It will be obvious that such an approach (which is practically only limited by the length of the paper) is a most efficient method of obtaining reliable formulae for coefficients beyond those given in Van Orstrand's table (which was found to be free of misprints). Otherwise, the danger of errors creeping into a traditional evaluation "by hand" would no doubt prevent the safe use of higher coefficients.

After these introductory remarks we should like to make some simple observations.

2. Emergence of a strange symmetry

For evaluating numerically the new set of coefficients c_n, the relations given in (6) or (for c_5 to c_{12}) those listed in [3] can be used. However, this is clearly not the only possible approach, and perhaps not the simplest either, for this depends on the explicit form of the coefficients (which may be available in the form of fractions). In this context it will be useful to remember the similar situation which we have studied recently for the square root of a power series [5], where it turned out that an explicit expression in terms of the original coefficients is rather complicated (eq. 12), whereas a general recursion formula which makes use of all the new coefficients already determined (eq. 10) is much simpler. It therefore seems worthwhile to check this possibility.

If we modify (6) by replacing successively on the right-hand side of a relation for $-c_n$ all the coefficients b_j by their equivalents expressed in terms of c_j, except for b_n, we arrive, after some elementary rearrangements, at the expressions

$$\begin{aligned} -c_1 &= b_1 , \\ -c_2 &= b_2 + 2c_1^2 , \\ -c_3 &= b_3 + 5c_1c_2 + 5c_1^3 , \\ -c_4 &= b_4 + 6c_1c_3 + 3c_2^2 + 21c_1^2c_2 + 14c_1^4 , \\ &\dots \end{aligned} \tag{9}$$

This set of equations looks strikingly similar to (6), and by forming $b_n + c_n$ we find the surprising relations

$$\begin{aligned} -(b_1+c_1) &= 0 &&= 0, \\ -(b_2+c_2) &= 2b_1^2 &&= 2c_1^2 , \\ -(b_3+c_3) &= 5b_1(b_1^2+b_2) &&= 5c_1(c_1^2+c_2) , \\ -(b_4+c_4) &= 3(2b_1b_3+b_2^2)+7b_1^2(2b_1^2+3b_2) &&= 3(2c_1c_3+c_2^2)+7c_1^2(2c_1^2+3c_2) , \\ &\dots \end{aligned} \tag{10}$$

It therefore seems that in any expression of c_n, as given either by (6) or by (9), we can simply replace all the coefficients b_j by c_j (for $j < n$) and obtain thereby new valid relations, which are just the required recursion formulae for c_n. This is a rather surprising observation - at least at first sight.

At second sight, however, the mystery is quickly resolved. In view of the complete symmetry between the forms expressed by (4) and (5) it is obvious that a reversion of (5) into (4) would give relations identical with those in (6), but with b_j everywhere replaced by c_j. Thus (after changing sides of c_n and b_n) we find

$$-b_1 = c_1 ,$$

$$-b_2 = c_2 + 2c_1^2 ,$$

$$-b_3 = c_3 + 5c_1c_2 + 5c_1^3 ,$$

$$-b_4 = c_4 + 6c_1c_3 + 3c_2^2 + 21c_1^2c_2 + 14c_1^4 ,$$

$$\ldots$$

but this set is clearly identical* with (9). Hence, the surmised rule for replacing the coefficients in (6) is generally valid, and so are the identities given in (10).

Although the new set of equations (9) is not of a simpler structure than the original one (6), it allows additional checks to be performed, and this may be very useful in lengthy numerical evaluations.

3. A similar situation

In retrospect, it can be easily seen that the above considerations could have been applied to the inversion of a power series as well. Indeed, if we start with a series of the form

$$S = 1 - \sum_{n=1}^{\infty} a_n x^n , \qquad (11)$$

the first coefficients b_n of the inverted series

$$1/S = 1 - \sum_{n=1}^{\infty} b_n x^n \qquad (12)$$

---

* We may note that this is only so because of the sign convention chosen in (5).

are known [5] to be given by

$$
\begin{aligned}
-b_1 &= a_1 , \\
-b_2 &= a_2 + a_1^2 , \\
-b_3 &= a_3 + 2a_1a_2 + a_1^3 , \\
-b_4 &= a_4 + 2a_1a_3 + a_2^2 + 3a_1^2a_2 + a_1^4 , \\
&\dots
\end{aligned}
\tag{13}
$$

Since a double inversion leaves S invariant, we can simply exchange in (13) all a_j by b_j, and this then leads (after changing sides of a_n and b_n) to the relations

$$
\begin{aligned}
-b_1 &= a_1 , \\
-b_2 &= a_2 + b_1^2 , \\
-b_3 &= a_3 + 2b_1b_2 + b_1^3 , \\
-b_4 &= a_4 + 2b_1b_3 + b_2^2 + 3b_1^2b_2 + b_1^4 , \\
&\dots
\end{aligned}
\tag{14}
$$

These recursion formulae for the coefficients b_j exhibit the same "symmetries" (compared to (13)) as those we have observed above for the reversion problem, and they are obviously there for the same reason.

4. The case of a geometric series

Let us come back to the reversion problem described in the introduction. A particularly simple situation arises for the special case where $b_n = -1$, for any value of n. It will be interesting to see if and how an application of the general relation (7) can verify our expectation.

As (4) now leads to (for $x^2 < 1$)

$$
y = x\,(1 + \sum_{n=1}^{\infty} x^n) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x} , \tag{15a}
$$

we simply have the reversion

$$
x = \frac{y}{1+y} = y\,[1 + \sum_{n=1}^{\infty} (-y)^n] , \tag{15b}
$$

hence, in the notation of (5),

$$- c_n = (-1)^n . \tag{15c}$$

We now look at the general formula (7) and try to perform the summation. For a fixed value of k the factor $\binom{n+k}{k}$ can be kept apart and we obviously have

$$\prod_j b_j^{k_j} = (-1)^k . \tag{16}$$

The evaluation of the sum

$$\sum_{\langle k_j \rangle} \frac{k!}{\prod_{j=1}^{n} k_j!} , \qquad \text{where} \quad k = \sum_{j=1}^{n} k_j , \tag{17}$$

is complicated by the fact that for the summation we have also to take into account the condition

$$\sum_{j=1}^{n} j\, k_j = n . \tag{18}$$

In the traditional language of combinatorics, the expression $k!/(\prod k_j!)$ corresponds to the number of partitions of k "balls" into n "cells", if these are to contain $k_1, k_2, \ldots, k_n$ "balls" (with $0 \leqslant k_j \leqslant k$).

The problem can be simplified by looking at it in a different way. As may be seen from (7), k is in our case the number of coefficients b_j which appear as factors in a term. For instance, for n = 5 and k = 3 these are $b_1 b_2 b_2$ and $b_1 b_1 b_3$. They are thus all of the form

$$\prod_{r=1}^{k} b_{j_r} , \qquad \text{with} \quad \sum_{r=1}^{k} j_r = n . \tag{19}$$

Since $j_r > 0$ for all k factors, we may now consider this as an occupancy problem, where we look for the number of different combinations in which n "balls" can be put into k non-empty "cells". For the original formula (7), this approach corresponds to allocating the "indices" j to the k "factors" b_j, where the constraints are now automatically taken into account. As is well known, the solution to this problem is given by the simple binomial coefficient $\binom{n-1}{k-1}$. Therefore, the summation (17) can now be performed and it turns out to be

$$\sum_{\langle k_j \rangle} \frac{k!}{\prod_j k_j!} = \binom{n-1}{k-1} . \tag{20}$$

For an instructive discussion of occupancy problems see for instance [6].

A subdivision of the sum appearing in (7) for constant values of k therefore leads to

$$- c_n = \frac{1}{n+1} \sum_{k=1}^{n} c_{n,k} , \tag{21a}$$

where $c_{n,k}$, according to (16) and (20), is given by

$$c_{n,k} = \binom{n+k}{k} \binom{n-1}{k-1} (-1)^k . \tag{21b}$$

Combining this with (15c) leads us to the rather surprising relation

$$(-1)^n = \frac{1}{n+1} \sum_{k=1}^{n} (-1)^k \binom{n+k}{k} \binom{n-1}{k-1} . \tag{22}$$

In the tabulations of formulae to which we have ready access we could not find (22), although it must be strongly suspected to be an identity. Indeed, a proof seems possible along the following lines.

We start with the elementary recurrence relation

$$\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1} \tag{23}$$

and use for the first term on the right-hand side the decomposition [7]

$$\binom{n}{m} = \sum_{k=0}^{m} (-1)^{m+k} \binom{n+k}{n} \binom{m}{k} . \tag{24a}$$

Hence, the second term yields likewise $\left(\text{since } \binom{m-1}{m} = 0\right)$

$$\binom{n}{m-1} = \sum_{k=0}^{m} (-1)^{m+k-1} \binom{n+k}{n} \binom{m-1}{k} . \tag{24b}$$

Therefore (23) now becomes

$$\binom{n+1}{m} = \sum_{k=0}^{m} (-1)^{m+k} \left[\binom{m}{k} - \binom{m-1}{k}\right] \binom{n+k}{n}$$

$$= \sum_{k=1}^{m} (-1)^{m+k} \binom{m-1}{k-1} \binom{n+k}{k} . \tag{25}$$

By putting m = n we finally obtain

$$\binom{n+1}{n} = n+1 = (-1)^n \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \binom{n+k}{k} ,$$

or also

$$\frac{1}{n+1} \sum_{k=1}^{n} (-1)^k \binom{n+k}{k} \binom{n-1}{k-1} = (-1)^n ,$$

which proves (22) and thereby confirms (15).

5. Possible further applications

The complicated structure of (7) severely restricts the number of potential direct applications; the apparently innocent case of a geometric series, treated in the previous section, well illustrates the situation and provides little encouragement for tackling more complicated examples. As possible candidates one might think of some trigonometric functions. Thus, for instance, if sin x is put into the form (4), i.e.

$$y = \sin x = x\,(1 - \sum_{k=1}^{\infty} b_n x^n) , \quad \text{we have } b_{2n} = \frac{(-1)^{n-1}}{(2n+1)!} , \qquad (26a)$$

whereas all coefficients with odd index vanish.

Its reversion is known to be

$$x = \text{arc sin } y = y\,(1 - \sum_{n=1}^{\infty} c_n y^n) ,$$

where now

$$c_{2n} = \frac{(2n-1)!!}{(2n+1)\,(2n)!!} . \qquad (26b)$$

Another pair is given by (for $x^2 < \pi^2/4$)

$$y = \text{tg } x \quad \text{and} \quad x = \text{arc tg } y ,$$

with the respective coefficients

$$b_{2n} = - \frac{2^{2n}(2^{2n}-1)}{(2n)!} \left| B_{2n} \right| \qquad (27a)$$

and

$$c_{2n} = \frac{(-1)^{n-1}}{2n+1} , \qquad (27b)$$

with B_n = Bernoulli numbers (see e.g. [2]).

It is easy to see from (7) that if $b_n = 0$ for odd indices n, then this also holds for the coefficients c_n. On the other hand, $b_n = 0$ for even indices would not imply such a consequence for c_n. Some general conclusions are also possible concerning the signs of the coefficients. Thus, if for example in a series we replace b_n by $b'_n = (-1)^n b_n$, then this results in the analogous change $c'_n = (-1)^n c_n$ for the reversed series. The proof is simple, for the relation only reflects the fact that in (19) an odd index n also requires in the sum an odd number of odd indices j_r, whereas for n even their number is always even (including zero), independent of the number of terms k. As an illustration, we may take for the first pair (for $y^2 < 1$)

$$y = e^x - 1 , \qquad x = \ln (1 + y) ,$$

with the respective coefficients

$$b_n = \frac{-1}{(n+1)!} \quad \text{and} \quad c_n = \frac{(-1)^{n-1}}{n+1} . \tag{28a}$$

Then, in the second pair

$$y = 1 - e^{-x} , \qquad x = - \ln(1 - y)$$

we have

$$\begin{aligned} b'_n &= \frac{(-1)^{n-1}}{(n+1)!} = (-1)^n b_n \quad \text{and} \\ c'_n &= \frac{-1}{n+1} = (-1)^n c_n , \end{aligned} \tag{28b}$$

in agreement with the above statement.

For the practically important case where $b_{2n+1} = 0$, the explicit relations (9) can be simplified and there then only remain

$$\begin{aligned} - c_2 &= b_2 , \\ - c_4 &= b_4 + 3b_2^2 , \\ - c_6 &= b_6 + 8b_2b_4 + 12 b_2^3 , \\ - c_8 &= b_8 + 10b_2b_6 + 5b_4^2 + 55b_2^2b_4 + 55b_2^4 , \\ &\ldots \end{aligned} \tag{29}$$

Note that a coefficient c_{2n} now consists of exactly p(n) terms.

In this case, too, one can readily see that a change from b_{2n} to $b'_{2n} = (-1)^n\ b_{2n}$ results, for the reversed series, in $c'_{2n} = (-1)^n\ c_{2n}$. A simple illustration of this relation is, for instance, provided by the functions sin x, developed in (26), and sinh x. The coefficients of their respective reversions follow the relation indicated above.

The examples given in (26) and (27) show that even in apparently quite simple cases rather complicated expressions (e.g. with double factorials and Bernoulli numbers) may arise as coefficients of the reversed series. This fact could therefore suggest to make use of known functional pairs and to derive from them, by means of (7), new identities which will involve such mathematical expressions.

I thank P. Carré for his long-standing and active interest in these matters and G. Ratel for some useful discussions on problems treated above.

References

[1] H.B. Dwight: "Tables of Integrals and other Mathematical Data" (Macmillan, New York, 1957³)

[2] "Handbook of Mathematical Functions" (ed. by M. Abramowitz and I.A. Stegun) NBS, AMS 55 (GPO, Washington, 1965⁴)

[3] C.E. Van Orstrand: "Reversion of power series", Phil. Mag. (Series 6) 19, 366-376 (1910)

[4] J. Riordan: "Combinatorial Identities" (Wiley, New York, 1968), p. 149

[5] J.W. Müller: "Some simple operations with series", Rapport BIPM-84/1 (1984)

[6] W. Feller: "An Introduction to Probability Theory and its Applications, Volume I" (Wiley, New York, 1968³), p. 38 ff.

[7] C. Jordan: "Calculus of Finite Differences" (Chelsea, New York, 1965³), p. 93

(January 1985)