Monte Carlo Uncertainties: Choosing Parent Distributions

Alan Steele, Barry Wood and Rob Douglas
Institute for National Measurement Standards
National Research Council Canada

Disclaimer on Original Abstract

- We originally thought that talking about the *mechanics* of MC resampling was appropriate
  - some people are just *getting started*
  - easy *tools* becoming available
  - engines can handle *complex models*

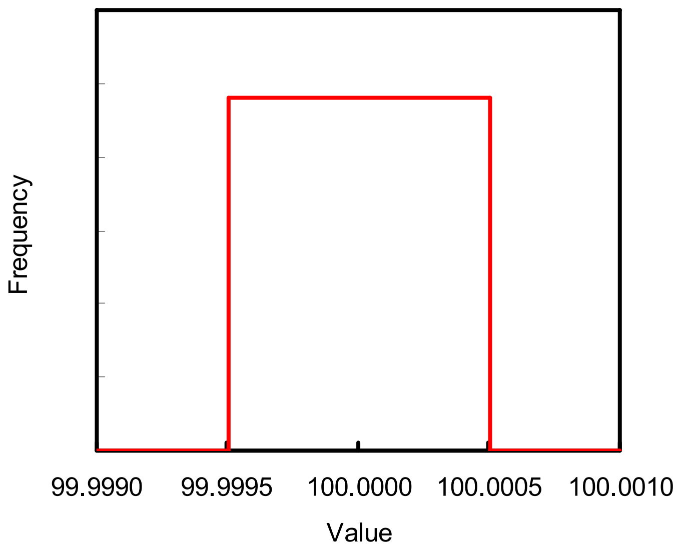
- We now think that talking about the *beliefs* underlying MC simulation is essential
  - different voices have *different opinions* on what is and what is not useful, or even permitted

A Word about "Resampling"

- The *re-sampling* that we want to discuss is *numerical* only, based on the uncertainty *claims* summarized in an *experimentally determined* uncertainty budget
  - There are *no new measurements*
  - Monte Carlo simulation is *one technique* to avoid other, perhaps more difficult, analysis
  - Our emphasis is on the *claim* made by the metrologist about nature of the measurements

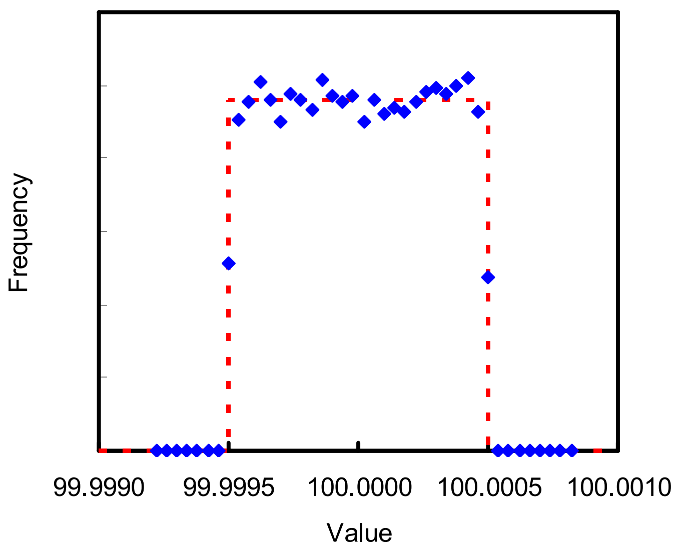
Resolution Limits and Monte Carlo

- The typical *Type B* component with a *uniform distribution* is a *resolution limited* DVM



- Monte Carlo *resampling* from this distribution is straightforward

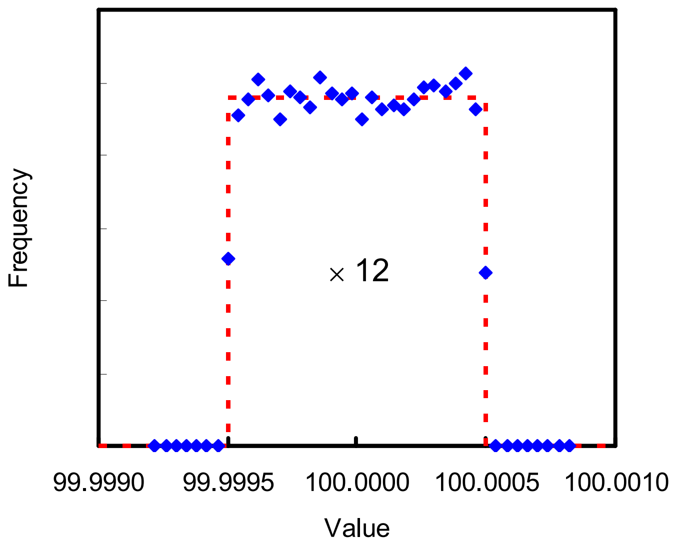
- Simulation of *measurand* or "*virtual*" measurements

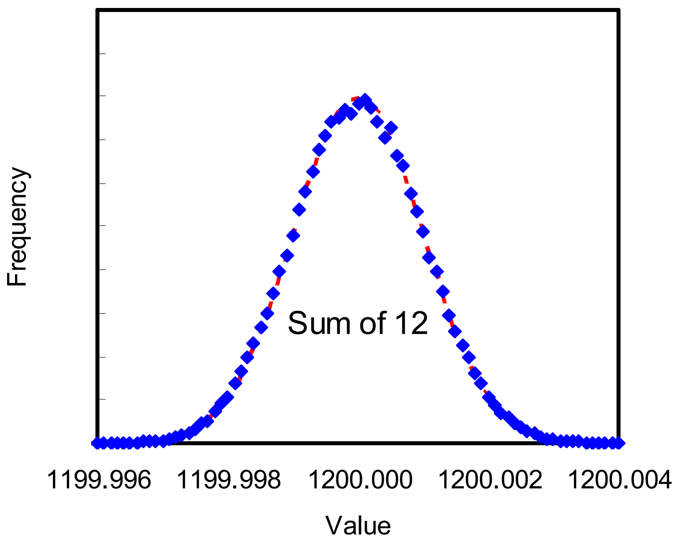


12 DVMs : Resolution Limited (Type B)

- The typical *Type B* component with a *uniform distribution* is a *resolution limited* DVM
- Suppose we use a divider network to make voltage measurements *beyond the range* of the DVM
  - 12 *uniform* terms
  - sum is *very nearly* Gaussian
  - "shape" *central limit* theorem

- *Which is the parent ?*

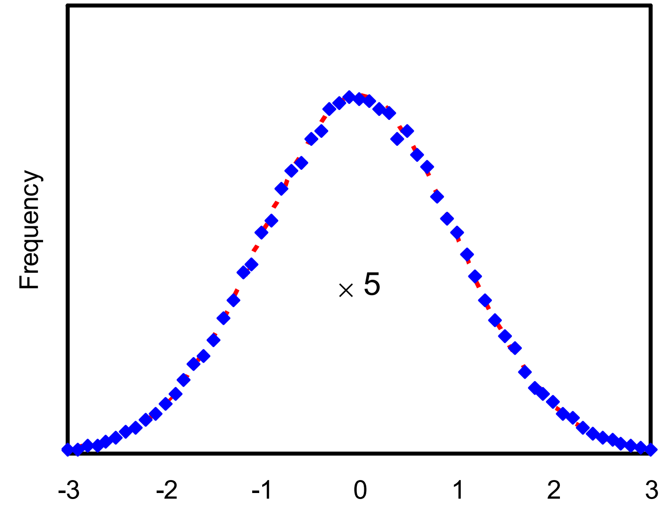


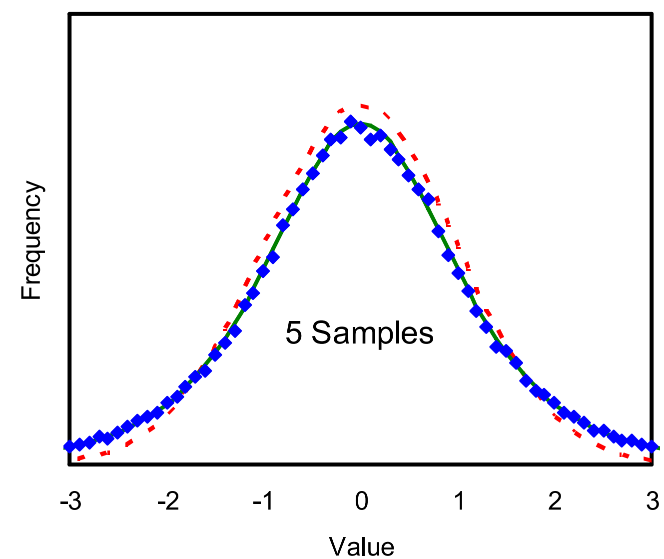


5 Measurements : Repeats (Type A)

- The typical *Type A* component with a *Student distribution* arises from a *limited number* of repeat measurements
- Suppose we take only *5 measurements* of a Gaussian process
  - uncertainty in σ gets captured
  - shape "*de-central limit*"
  - 1 sample from the ν = 4 *Student-t*

- *Which is the parent ?*

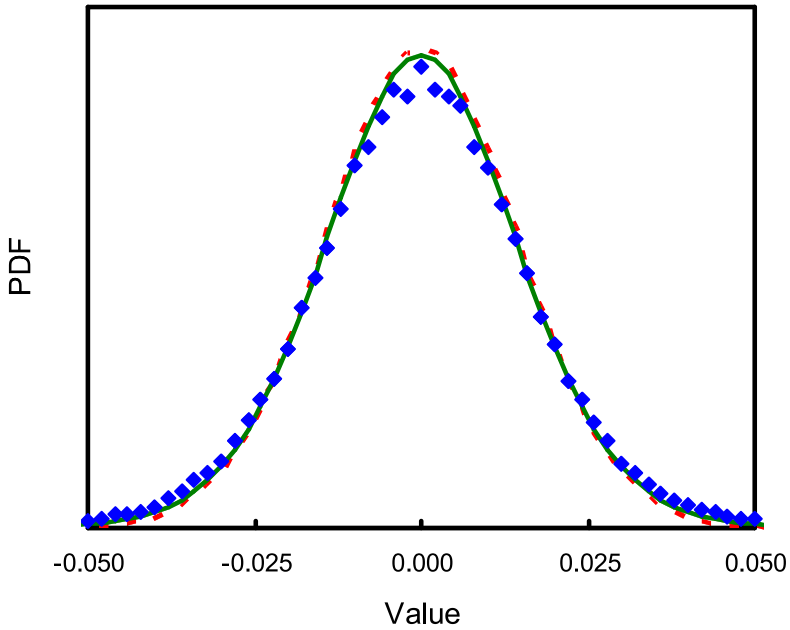




Real Uncertainty Budgets

- These *simple schemes* never completely describe what happens in ‘real life’
- Combination of many individual *components*
- Dominant terms often inferred using ‘expert opinion’ of *metrologist*

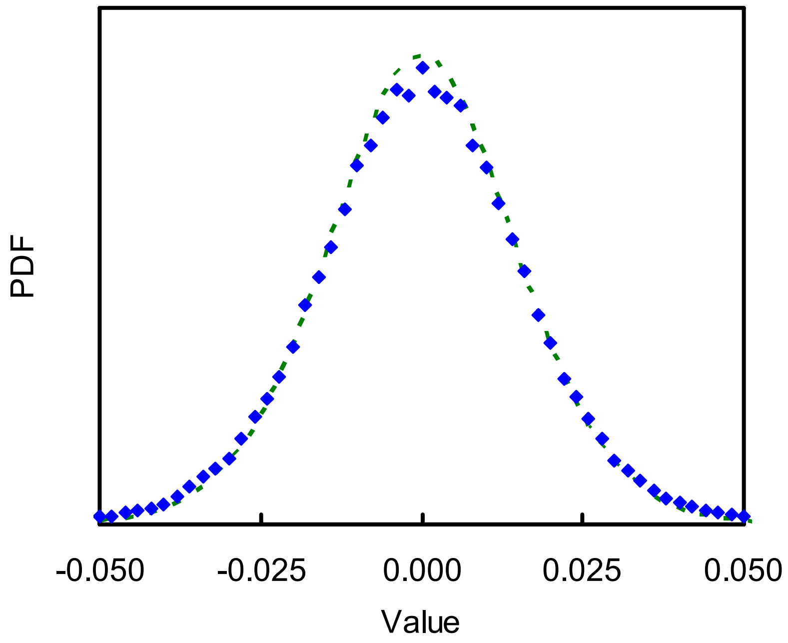
	u_i	ν_i	Shape	Type
Effect 1	0.01/2√3	∞	Rect	B
Effect 2	0.01	∞	Gauss	B
Effect 3	0.01	4	Stud	A
u_c	0.014	17		



Uncertainty in Uncertainty Perspective

- Finite *degrees of freedom* are generally "*effective*" numbers

- Physical interpretation as *uncertainty* in *uncertainty*



- Relevant in many *real* uncertainty budgets

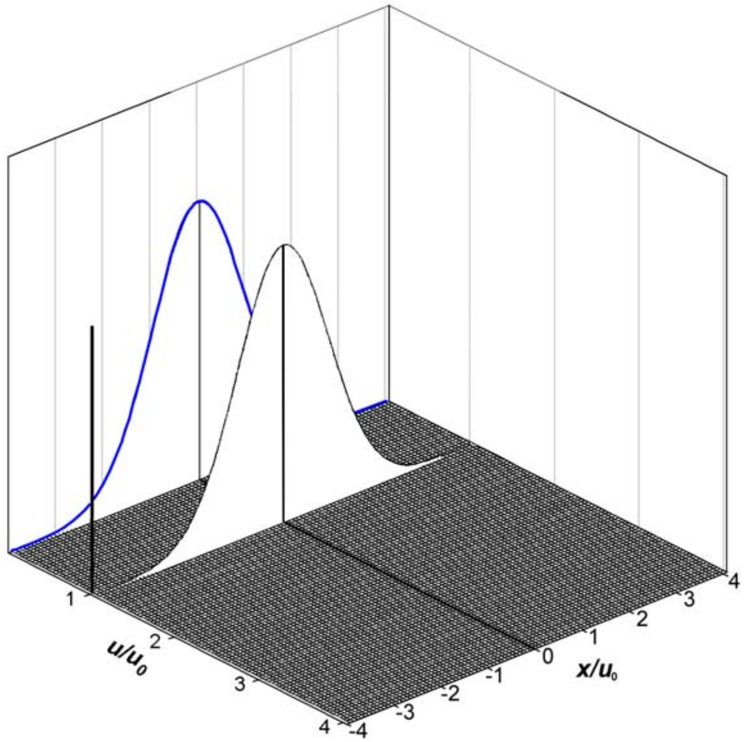
The "AND" versus "OR" question

- ISO Guide mainly discusses *combining effects* that manifest themselves *independently* and *simultaneously*
  - this is a logical AND operation

- Sometimes we have effects (or lab results) that are best described as *mutually exclusive*
  - this is a logical OR operation

A Normal Uncertainty Budget

- Here, infinite degrees of freedom implies *Gaussian*
- u_0 is *standard deviation* of a Gaussian population
- u/u_0 is distribution of *possible* values of u_0
- Integration runs over u/u_0
- Consider a δ*-function* for u/u_0
- Integral is just *original* Gaussian

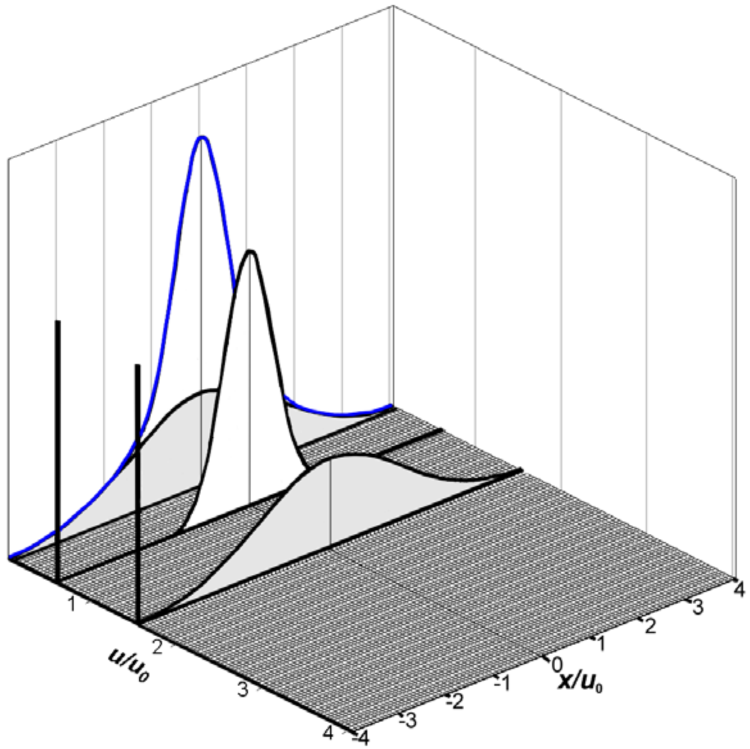


- *Which is the parent ?*

Two component mixture

- Here, u_0 is the *reported average* standard deviation
- u/u_0 has *two possible values* (0.5 and 1.5)
- larger u gives *wider* x
- Integration runs over u/u_0

- Consider *two* δ-functions
- Integral is *non-Gaussian* PDF(x)



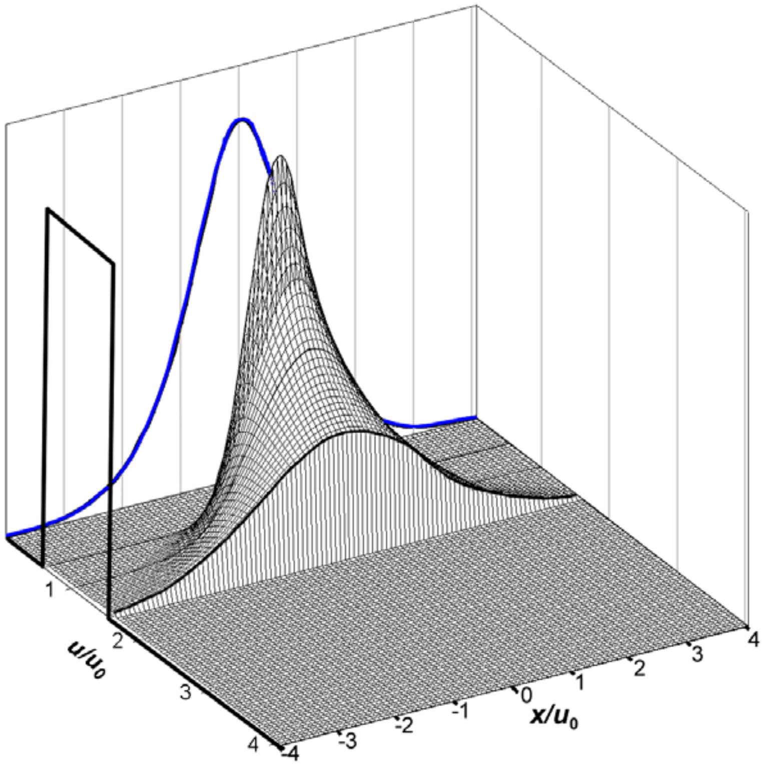
- *Which is the parent ?*

Uniform uncertainty in u

- Here, u_0 is the *reported average* standard deviation
- u/u_0 ranges continuously from 0.5 to 1.5
- larger u gives *wider* x
- Integration runs over u/u_0

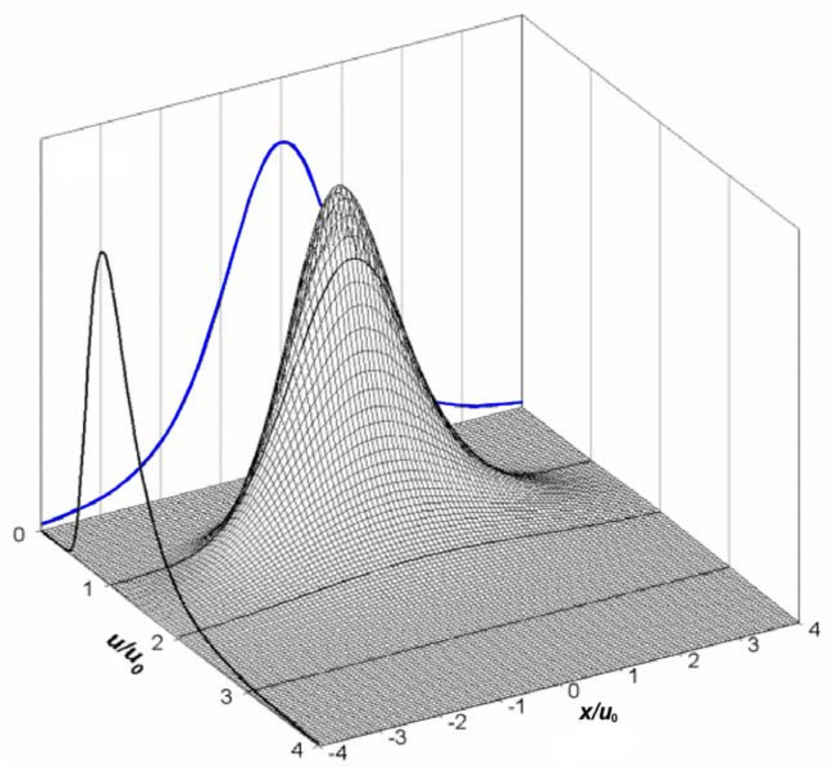
- Consider *uniform* u/u_0
- Integral is *non-Gaussian* PDF(x)
  that is almost, but not quite, entirely unlike t...

- *Which is the parent ?*



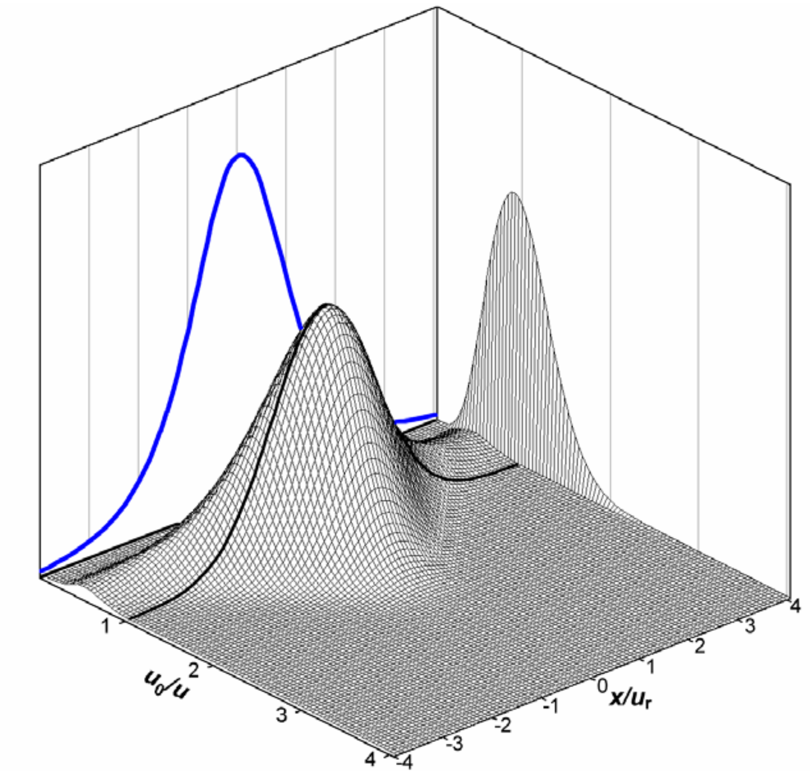
The Student Distribution

- Here, u_0 is the *reported standard uncertainty*
- u/u_0 is distributed as a reduced *inverse-chi* with $\nu=4$
- $(u_0/u)^2$ is distributed as a reduced chi-squared
- Integration runs over u/u_0
- Integral is $\nu=4$ *Student-t* PDF(*x*)
- *Which is the parent ?*



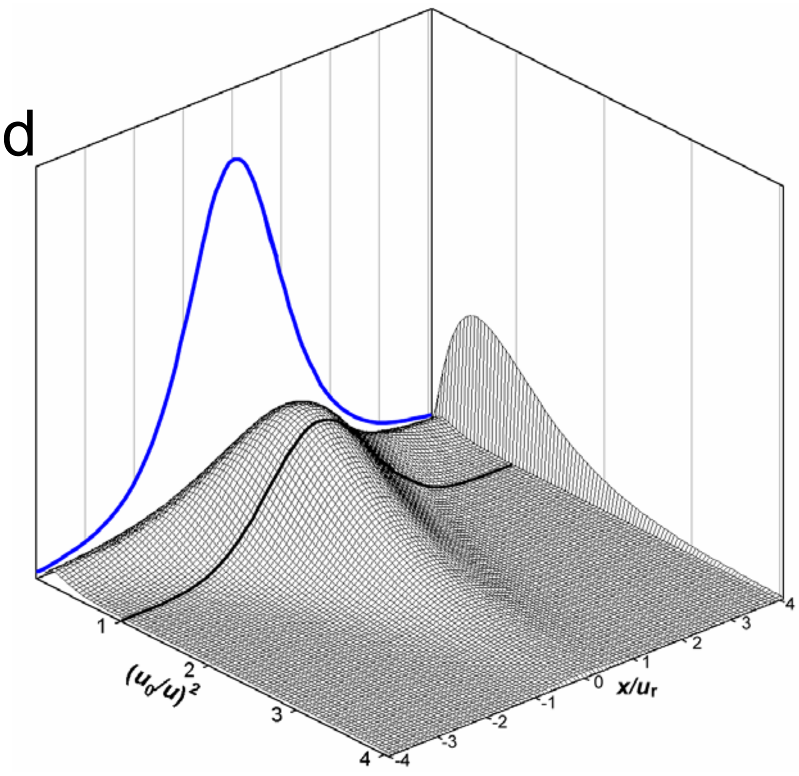
The Student Distribution

- Here, $u = u_r$ is the *reported standard uncertainty*
- u_0/u is distributed as a reduced *chi* with $\nu=4$
- $(u_0/u)^2$ is distributed as a reduced chi-squared
- Integration runs over u_0/u
- Integral is $\nu=4$ *Student-t* PDF(x)
- *Which is the parent ?*



The Student Distribution

- Here, $u = u_r$ is the *reported standard uncertainty*
- $(u_0/u)^2$ is distributed as a reduced *chi-squared* with $\nu=4$
- Integration runs over $(u_0/u)^2$
- Integral is $\nu=4$ *Student-t* PDF(x)
- *Which is the parent ?*



Conclusions

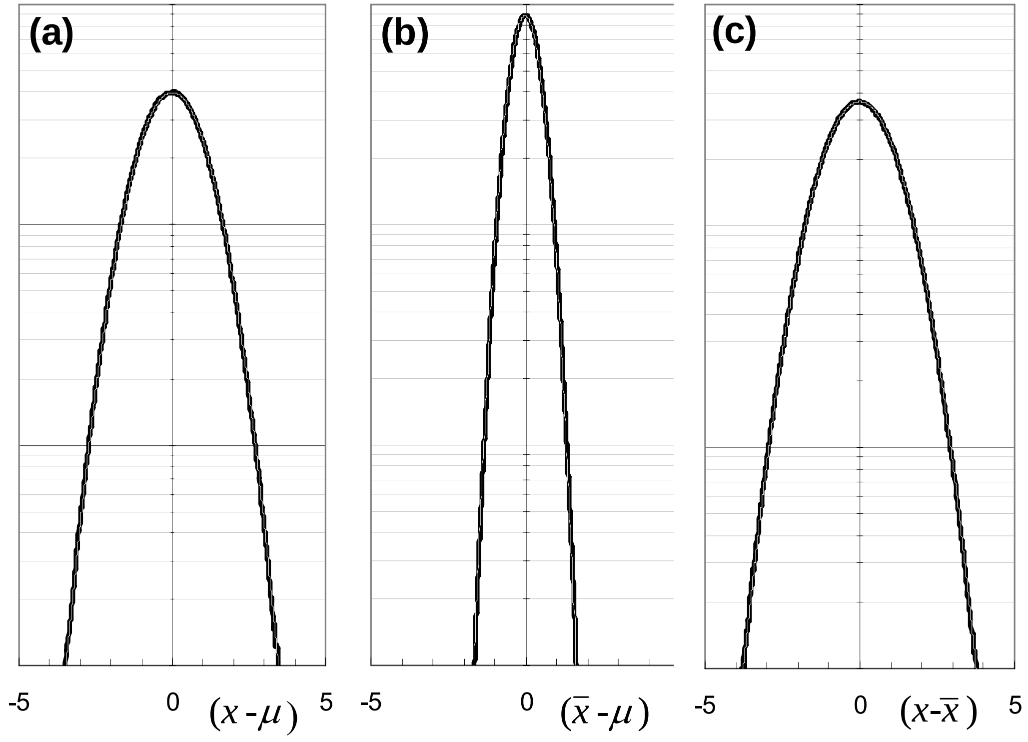
- Monte Carlo resampling from *Gaussian parents* can be useful to get the *right answer* without requiring much deep thought
- With *thoughtful interpretation*, scaled *Student distributions* can be used *directly* for resampling
- Rigid rules about how resampling *must* be done may be *too restrictive* for metrology

Conclusions

- Monte Carlo resampling *must respect* the *uncertainty claim* made by the metrologist

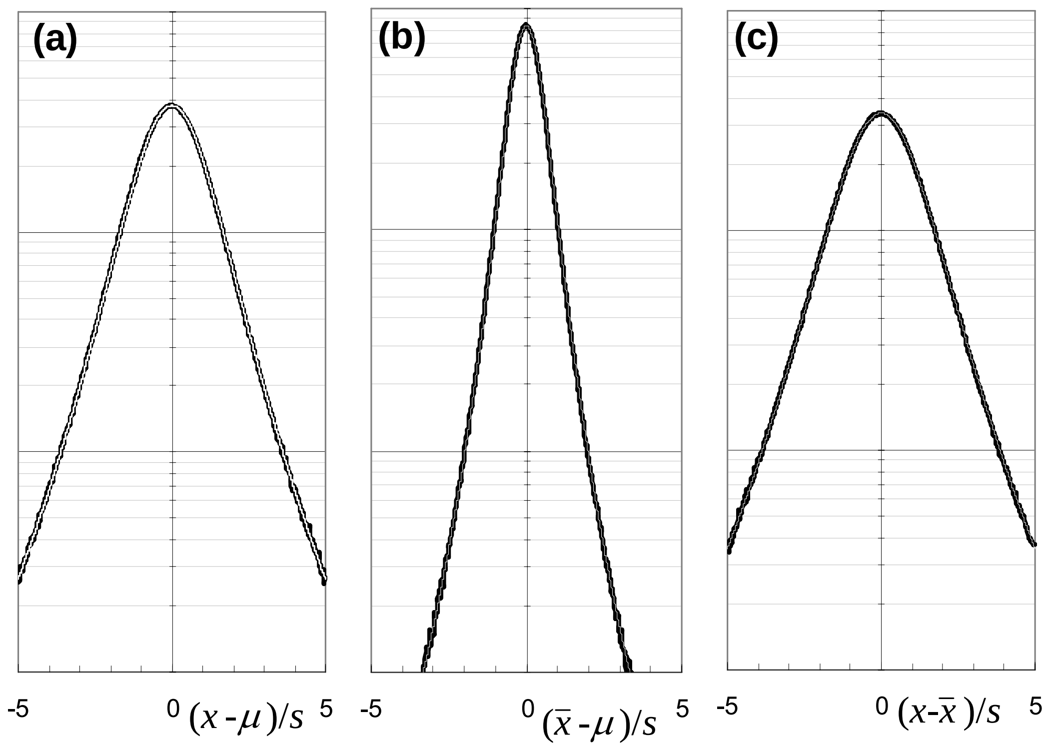
Predictions: Known Gaussian Parent

- for prediction of the next measurement when the standard deviation is known (5 samples for from Gaussian parent)



Unknown Gaussian Parent to Student-t's

- when the standard deviation has to be approximated by *s*, the sample standard deviation of the 5 samples, each curve is a ν=4 Student



Comparing Predicted Coverage Factors

- For prediction, using *5 samples* to predict a *6th*, the Student distributions are *not independent*

Examine coverage factors…

- *Exact* solution: ν=4 Student
- *Convolution*: quite close
- *Welch-Satterthwaite*: still a useful approximation