Propagation of Uncertainties in Measurements: Generalized/ Fiducial Inference

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Frameworks for Quantifying Uncertainty

- Frequentist Inference
  - Likelihood Function
  - Pivotal Quantities
  - Confidence Intervals

- Bayesian Inference
  - Priors on Parameters
  - Posterior Distributions
  - Probabilty intervals

- Use of the Maximum Entropy Principle
A Simple Case

- Unknown value of measurand is $\mu$.
- Reported value is $x$.
- Reported standard uncertainty is $u$.
- Reported degrees of freedom is $k$. 
Frequentist Approach

- $\mu$ is regarded as a fixed but unknown constant.
- Typical working model:

$$X \sim N(\mu, \sigma^2)$$

$$kU^2/\sigma^2 = W \sim \chi^2_k.$$

$$\frac{X - \mu}{U} \sim \text{Student’s } t \text{ with } k \text{ df.}$$

- Obtain an interval of plausible values of $\mu$.

$$x \pm t_{1-\frac{\alpha}{2}:k} u.$$
Bayesian Approach

- Describe our prior knowledge about plausible values of $\mu$ and $\sigma$ by specifying prior distributions.
- Use Bayes’ theorem from probability theory to deduce the distribution of $\mu$ conditional on the observed data.
- In the absence of any detailed prior knowledge one could assume that $\mu$ has a distribution over some very wide interval.
- The wider this interval is the closer the Bayes solution is to the classical frequentist solution.
Who Is This?
Sir Ronald Fisher
Fisher’s Fiducial Method

\[
\frac{X - \mu}{\sigma} = Z, \quad Z \sim N(0, 1)
\]

\[
k U^2 \frac{1}{\sigma^2} = W, \quad W \sim \chi^2_k.
\]

\[
\mu = X - \sigma Z = X - U \frac{Z}{\sqrt{W/k}} = X - U T_k.
\]

Fiducial distribution:

\[
\tilde{\mu} = x - u T_k.
\]
A More Realistic Case

- Value of a measurand is $\mu$ units.
- Measured value is $x$ and reported uncertainty is $u$ with $k$ df.
- Measurement equation: $X = \mu + \delta + \sigma Z$
  $\delta$ is an unknown offset.
- \[ \frac{k U^2}{\sigma^2} = W \sim \chi_k^2. \]
- Information about $\delta$ is expressed in terms of an appropriate probability distribution.
- How to characterize our state of knowledge of $\mu$?
Fisher’s Fiducial Method

\[ X = \mu + \delta + \sigma Z, \quad Z \sim N(0, 1) \]

\[ \frac{k \ U^2}{\sigma^2} = W, \quad W \sim \chi^2_k. \]

\[ \mu = X - \delta - \sigma Z \]
\[ = X - \delta - U \frac{Z}{\sqrt{W/k}} \]
\[ = X - \delta - U \ T_k. \]

Fiducial distribution: \[ \tilde{\mu} = x - \delta - u \ T_k. \]
A Numerical Illustration

- Measurand $\mu$
- $X = \mu + \delta + \sigma Z, \quad Z \sim N(0, 1)$
- $k: \frac{U^2}{\sigma^2} = W, \quad W \sim \chi^2_k$.
- Measurement result = 10.031 mm;
- Standard Uncertainty = 0.011 mm;
- df = 4.
- Fiducial distribution:

$$\tilde{\mu} = 10.031 - \delta - 0.011 T_4.$$
Scenario 1

\[ \delta \sim N(0, sd = 0.017) \]

- Mean = 10.031
- 2.5 percentile = 9.994
- 97.5 percentile = 10.068
Scenario 2

\( \delta \) skewed with mean = 0, sd = 0.017

Mean = 10.031
2.5 pctl = 9.979
97.5 pctl = 10.071
About Fiducial Inference

- Introduced by R. A. Fisher in the 1930s.
- Fell into disrepute because of an apparent lack of a mathematical framework for its justification.
- Re-invented in the 1990’s by Weerahandi under the name Generalized Inference.
- Recent research results show Fiducial Inference is a valid method, justifiable within the frequentist setting.
- Fiducial Inference leads to POSTERIOR distributions on parameters without having to declare prior distributions.
Bayesian Omelette

Fiducial inference is an attempt to make the Bayesian Omelette without breaking the Bayesian eggs

– L. J. Savage.
Fiducial Method in the Published Literature


Example Annex H.1 of the GUM: Gauge Block Calibration
Gauge Blocks
Gauge Block Calibration
Gauge Block Calibration

\[ d = l[1 + (\delta_\alpha + \alpha_s)\theta] - l_s[1 + (\theta - \delta_\theta)\alpha_s] \]

\[
l = \frac{l_s[1 + (\theta - \delta_\theta)\alpha_s] + d}{1 + (\delta_\alpha + \alpha_s)\theta}.
\]
Gauge Block Calibration

- $l$ is the length (at 20 °C) of the test gauge block
- $l_s$ is the length (at 20 °C) of the reference gauge block
- $\alpha$ is the coefficient of linear expansion of the test gauge block
- $\alpha_s$ is the coefficient of linear expansion of the reference gauge block
- $\theta$ is the deviation in temperature from 20 °C of the test gauge
- $\theta_s$ is the deviation in temperature from 20 °C of the reference gauge

$$d = l - l_s \quad \delta_\alpha = \alpha - \alpha_s \quad \delta_\theta = \theta - \theta_s.$$
Fiducial Distribution for $d$

- $d_i \sim N(\mu_d, \sigma_d)$, $i = 1, \ldots, 5$. Mean of $d_i$ is 215 nm.
- Estimate of $\sigma_d$ is $u_d = 13$ nm (24 df) based on an independent experiment.
- The value $d$ differs from $\mu_d$ due to errors associated with the comparator device. We have $d - \mu_d = d_R + d_S$
- $d_R \sim N(0, \sigma_{dR})$ and $d_S \sim N(0, \sigma_{dS})$.
- $u_{dR} = 3.9$ nm (5 df) $u_{dS} = 6.7$ nm (8 df).
- $\tilde{d} = \tilde{\mu}_d + \tilde{d}_R + \tilde{d}_S$ (assume independence)
  
  $$= \left(215 - \frac{13T_{24}}{\sqrt{5}}\right) + 3.9T_5 + 6.7T_8$$
Fiducial Distribution for $l_s$

- The estimated value of $l_s$ (i.e., the value given in the calibration certificate), denoted by $\hat{l}_s$, is equal to 50000623 nm.

- Uncertainty stated in the calibration certificate is $u_{l_s} = 25$ nm (with 18 degrees of freedom).

- Regard $\hat{l}_s \sim N(l_s, \sigma^2_{l_s})$ and $\frac{18s^2_{l_s}}{\sigma^2_{l_s}} \sim \chi^2_{18}$.

- Fiducial distribution of $l_s$ is:

$$\tilde{l}_s = 50000623 - 25 T_{18}.$$
Subjective Distribution for $\theta$

- $\theta$ = Devn of test bed temp from 20 °C.
- $\mu_\theta$ = average devn from 20 °C. We write $\theta = \mu_\theta + \Delta$
- $\Delta$ represents the cyclic variation of the temperature of the test bed under a thermostat system.
- Estimate of $\mu_\theta = \bar{\theta} = -0.1$ °C with a standard uncertainty equal to 0.2 °C (df = $\infty$).
- Regard $\bar{\theta} \sim N(\mu_\theta, (0.2)^2)$. (units are °C)
- $\Delta$ is assumed to have a U-shaped density function
  \[ g(\Delta) = \frac{2}{\pi \sqrt{1-4\Delta^2}}, \quad -0.5 \, ^\circ \text{C} < \Delta < 0.5 \, ^\circ \text{C}. \]
  Mean = 0 and Standard deviation = $1/\sqrt{8} = 0.353553$. 
Subjective Distribution of $\Delta$

- Observation: If $U$ is a uniform $[0, 1]$, then $\Delta = -\cos(\pi U)/2$ has the arcsine distribution.

- Fiducial distribution for $\theta$ is given by
  $$\tilde{\theta} = -0.1 - 0.2Z - \cos(\pi U)/2$$
Distributions for $\delta_\alpha, \delta_\theta, \alpha_s$

1. Assume $\delta_\alpha$ is a uniform random variable over the interval $\pm 1 \times 10^{-6} \degree C^{-1}$.
2. Assume $\delta_\theta$ is uniform over the interval $\pm 0.05 \degree C$.
3. Assume $\alpha_s$ is uniform over the interval $11.5 \times 10^{-6} \pm 2 \times 10^{-6} \degree C^{-1}$.

\[
l = \frac{l_s[1 + (\theta - \delta_\theta)\alpha_s] + d}{1 + (\delta_\alpha + \alpha_s)\theta}.
\]
\[ l = \frac{l_s[1 + (\mu_\theta + \Delta - \delta_\theta)\alpha_s] + \mu_d + d_R + d_S}{1 + (\delta_\alpha + \alpha_s)\theta}. \]

\[ l = \frac{l_s[1 + (\theta - \delta_\theta)\alpha_s] + d}{1 + (\delta_\alpha + \alpha_s)\theta}. \]
Fiducial Representation of $l$

\[
l = \begin{cases} 
(50000623 - 25T_{18})[1 - (0.1 + 0.2Z + U_3) \\
+ \cos(\pi U_1)/2)(11.5 \times 10^{-6} + U_4)] \\
+ 215 - 13 T_{24}/\sqrt{5} - 3.9 T_{5} - 6.7 T_{8} \\
\end{cases} \begin{cases} 
1 - (U_2 + 11.5 \times 10^{-6} + U_4)(0.1 + 0.2Z + \\
\cos(\pi U_1)/2) \\
\end{cases}
\]

- Using simulation, the mean and the standard deviation of $l$ are 50000838 nm and 35 nm, respectively.
- A 99 % confidence interval for $l$ can be obtained by using the interval between the 0.005 and 0.995 quantiles of the simulated distribution.
A 99% CI is given by $(50000746, 50000930)$ nm.

For comparison, the result from GUM is $(50000745, 50000931)$ nm.

Histogram of the Fiducial distribution of the true length $l$ based on 500000 Monte Carlo samples.
Final Remarks

- Fiducial Inference/Generalized Inference is a valid statistical method with good operating characteristics.
- Fiducial/Generalized intervals have satisfactory coverage.
- Similarity with Bayesian approaches – get “posterior distribution” for measurand without assuming prior distributions for unknown parameters.
- The method allows the combination of type-B and type-A uncertainty information in a logical and straightforward manner. Principle of Maximum Entropy can be used when deemed appropriate.
- Generally requires a Monte-carlo approach for computing uncertainty intervals. Computations are straight-forward.
Thank You