



Basic of time and frequency analysis

P. Tavella

Time Department, BIPM



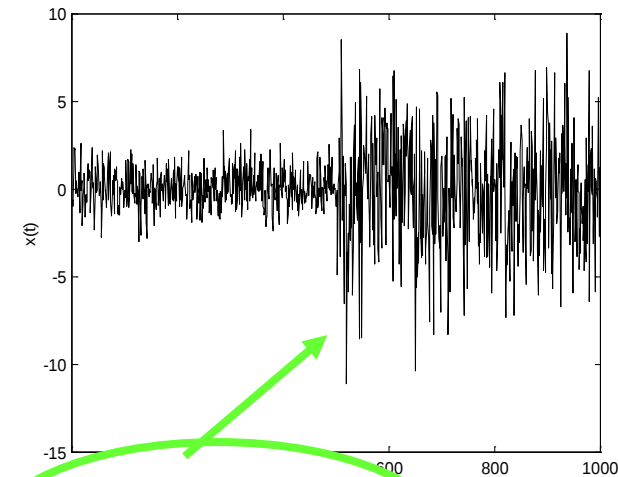
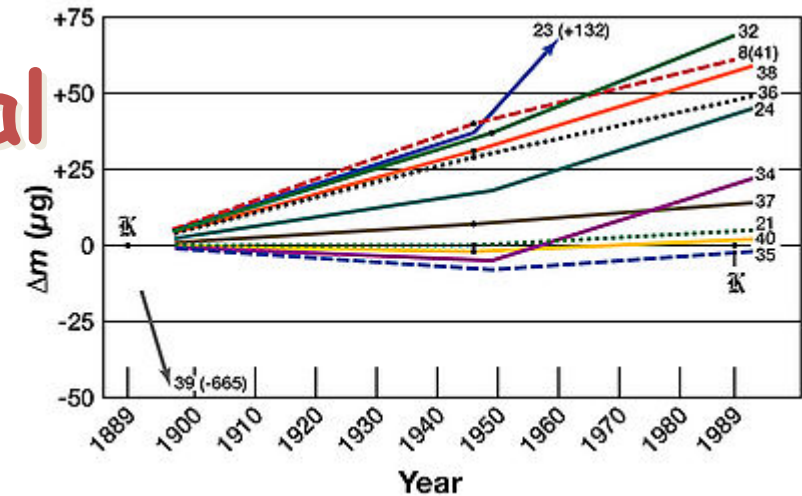
Sponsored by BIPM and UFFC



In time and frequency metrology we repeat the measurements and we look for their «stability»

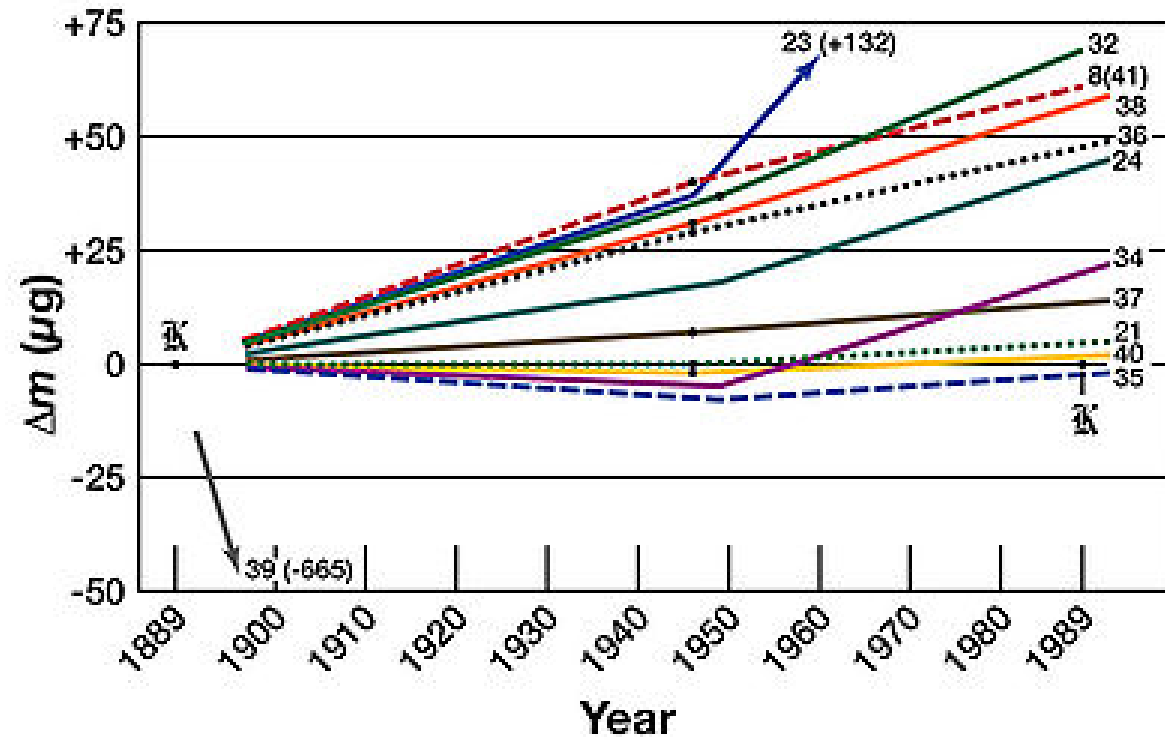
Repeated measurements reveal always something

- *stationarity*
- *aging*
- *slow drift*
- *abrupt change*
- *failure*



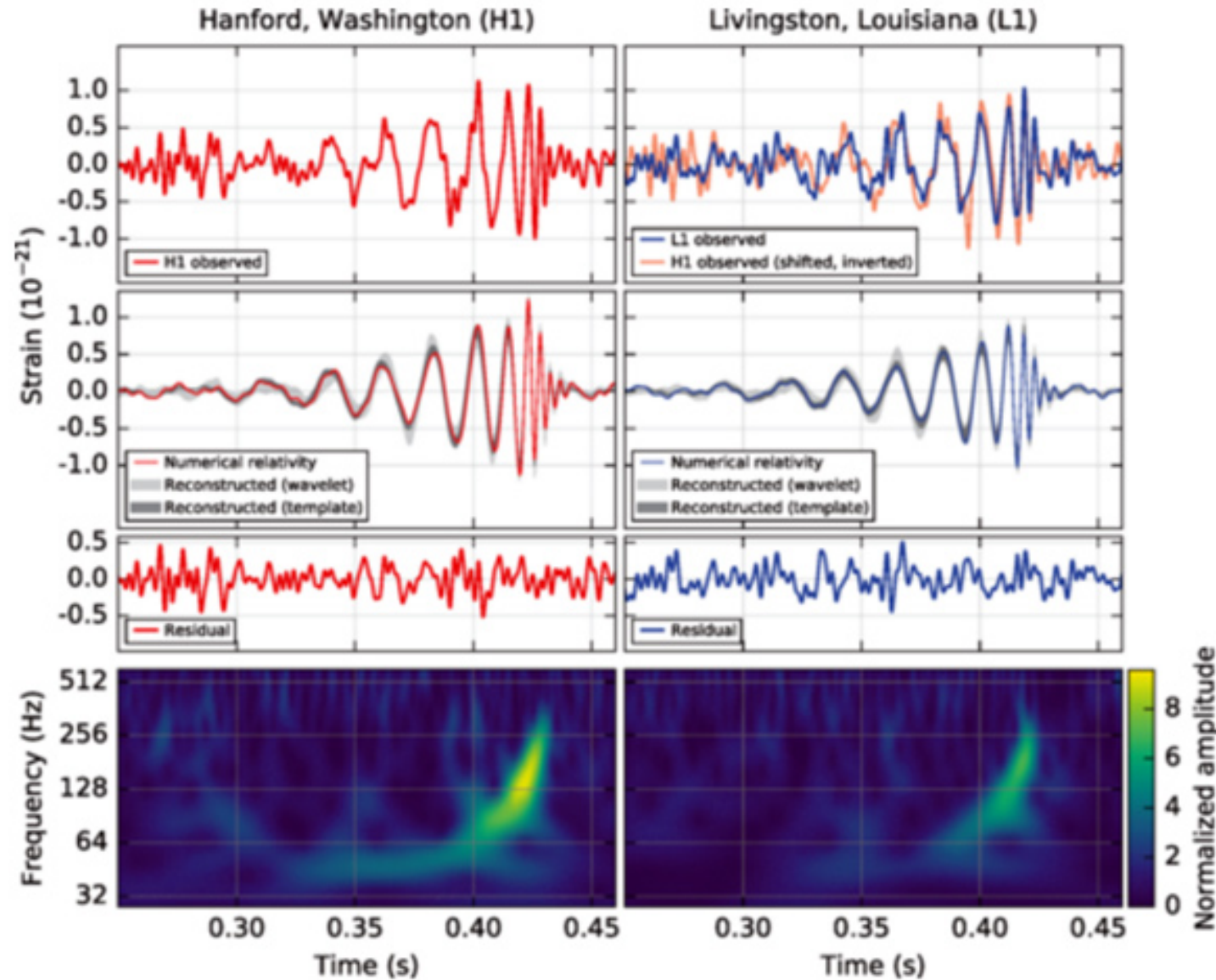
around $t = 500$
the noise
increases

Changes may cause disturbance: the mass of the International Prototype Kilogram



Masses of the ensemble of prototypes have been slowly diverging from each other.

Or may reveal new physics



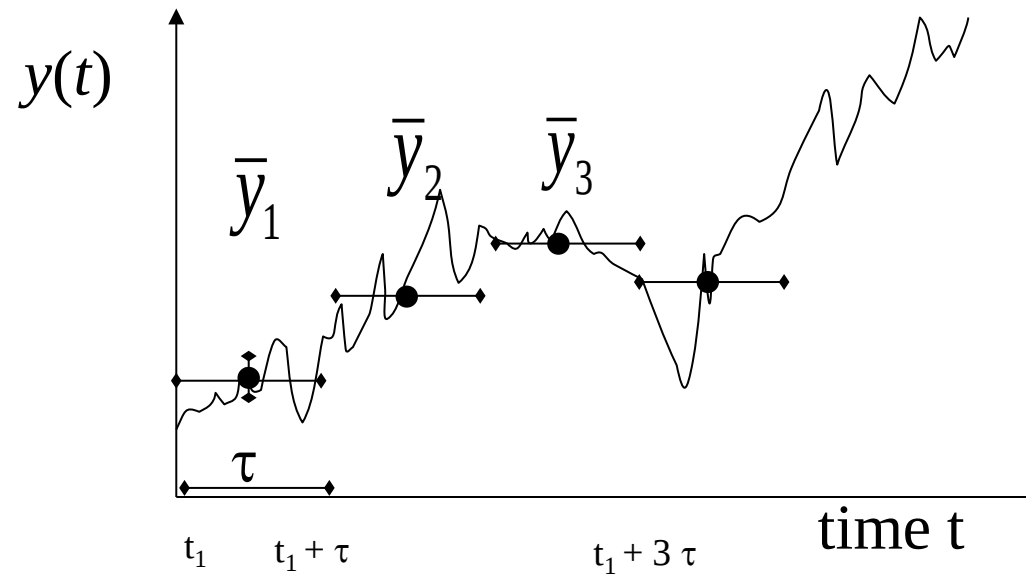
*LIGO Sees First
Gravitational
Waves*

What is changing?

- ◆ Sometime the physical quantity (the measurand) is varying
- ◆ Sometime it is the measuring system
- ◆ The value itself depends on time
- ◆ or just the noise
- ◆ The uncertainty may vary with time, even if it is perfectly estimated now, it may not be valid tomorrow

Repeating measurements

The uncertainty on the single measure can be evaluated according to the GUM: Guide for Uncertainty in Measurement



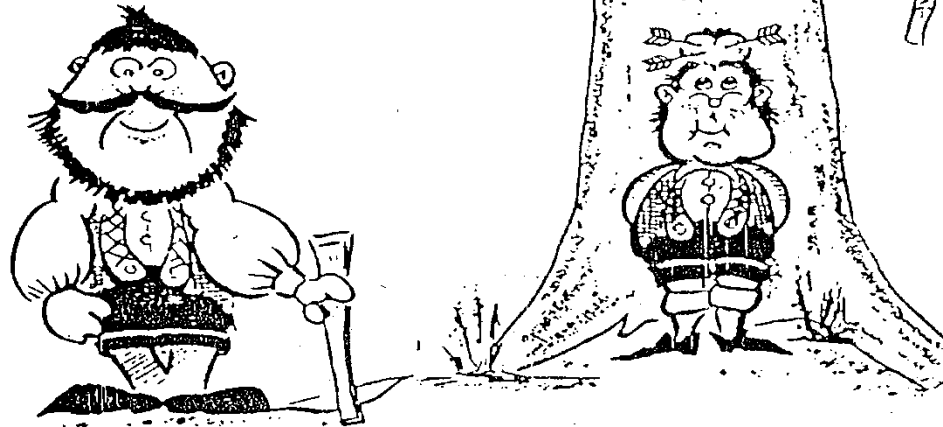
In same cases, instability is more impacting than single measure uncertainty

Stability Accuracy

Cons:

Stable or accurate?

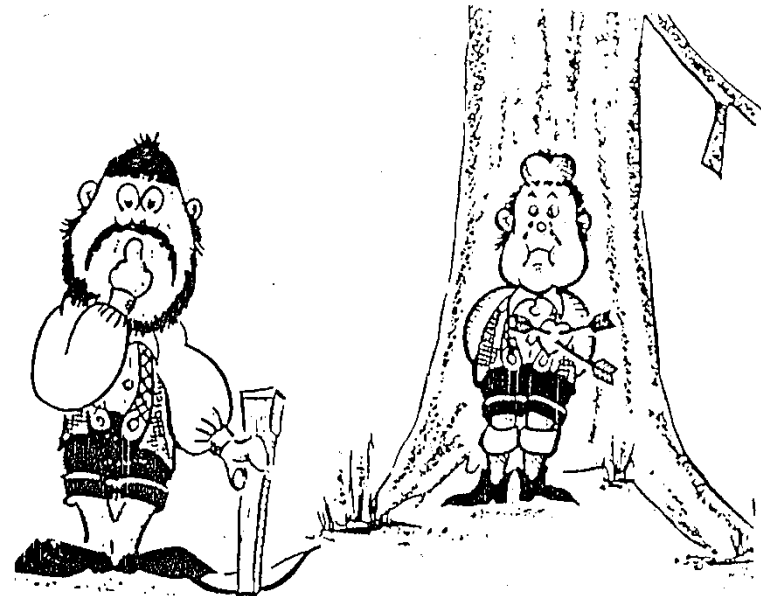
CI



DR. ALAN HOWARD, NE LABS, SCOTLAND

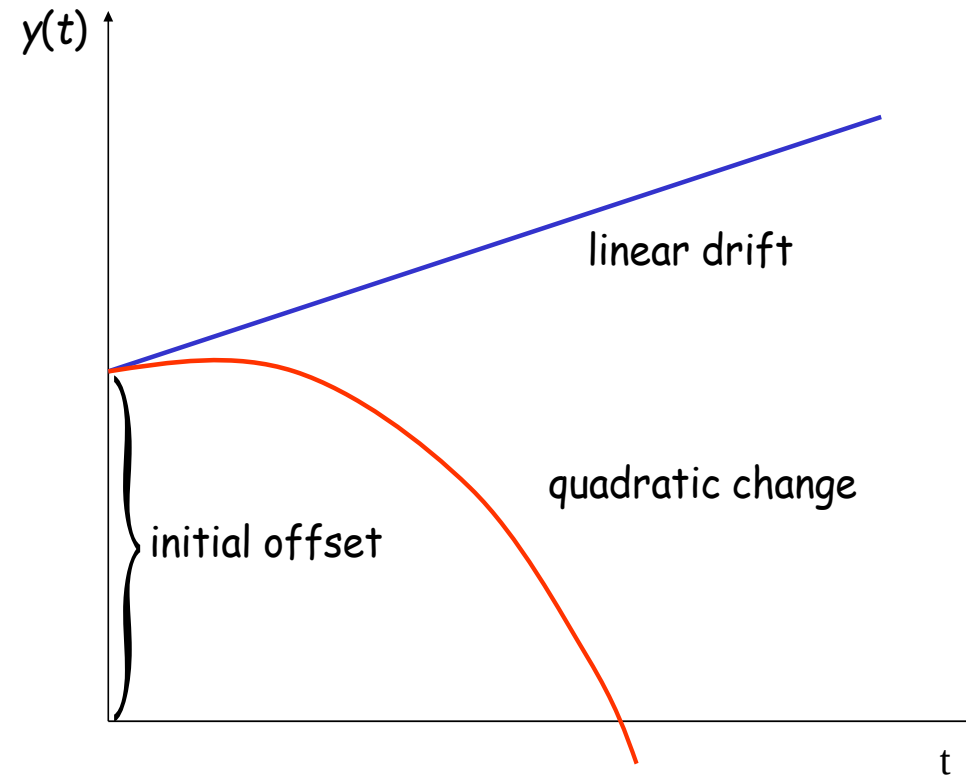


DR. ALAN HOWARD, NE LABS, SCOTLAND

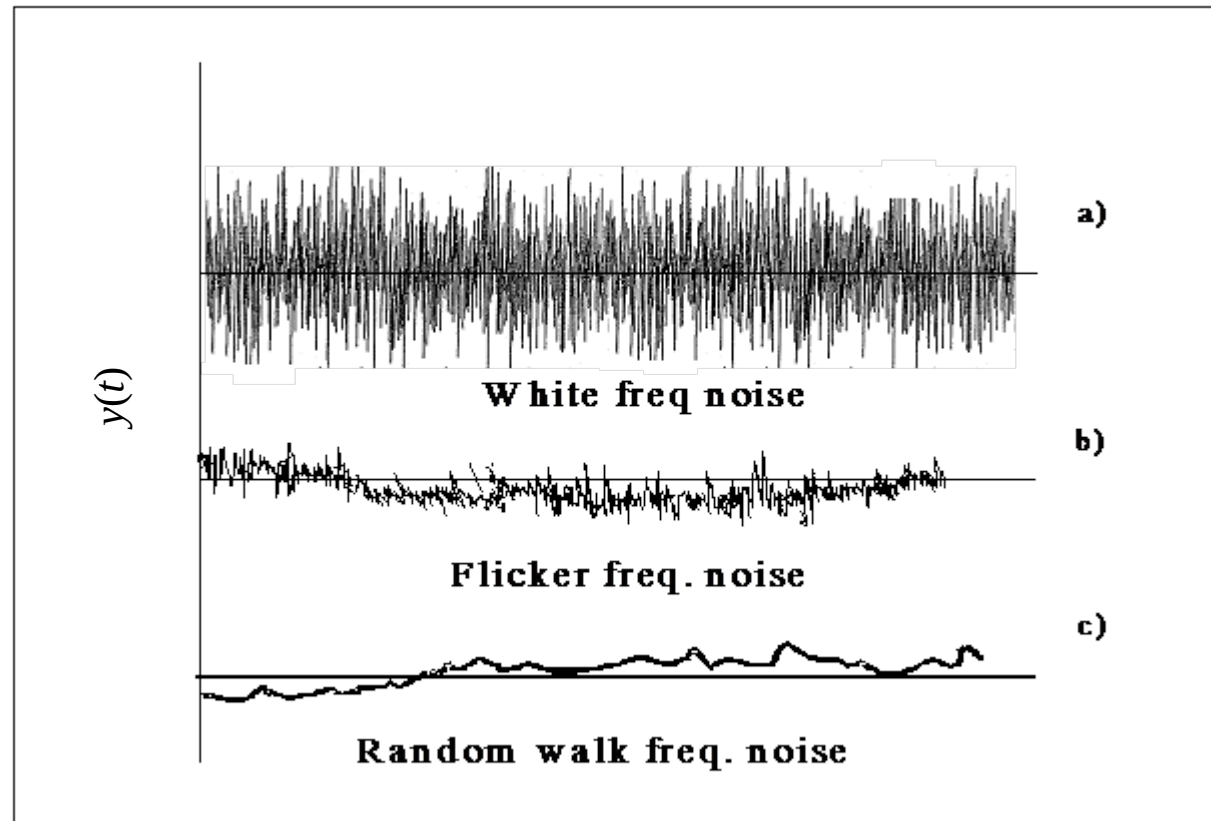


DR. ALAN HOWARD, NE LABS, SCOTLAND

Time series of measures may contain polynomial component to be treated by least square estimation techniques (batch, recursive...)

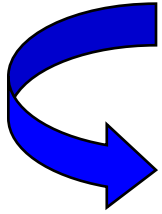


*but stochastic components are also present
and sometime dominant*

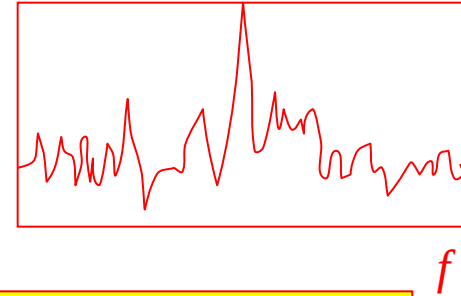


At each instant the measure $y(t)$ is a random variable

Appropriate mathematical tools for describing the stochastic noises:



- ◆ Spectral analysis of noise



decomposition in
elementary frequencies

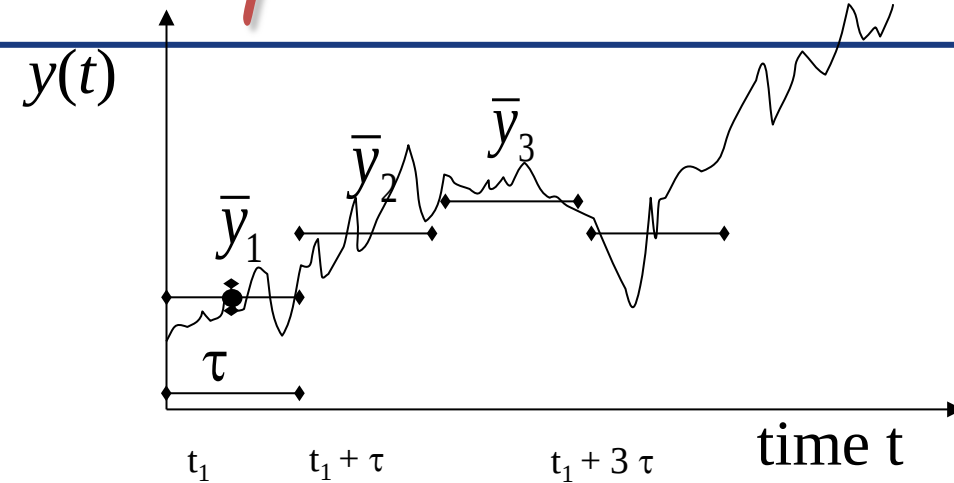
- ◆ Filtered variances such as the Allan variance

dispersion of the average
frequency values

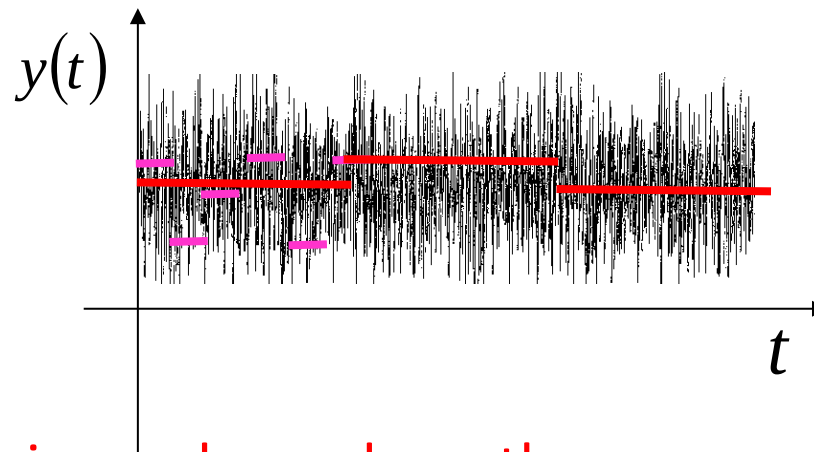
Instability - dispersion - variability

The variance is the measure of dispersion

$$\sigma^2(\tau) = \frac{\sum_{i=1}^N (\bar{y}_i - \mu)^2}{N}$$



First case of typical noisy behaviour (short term noise):



If the “noise” is stationary, the estimation of the variance converges as N grows

For longer observation interval τ , the variability diminishes

The “noise” is white and

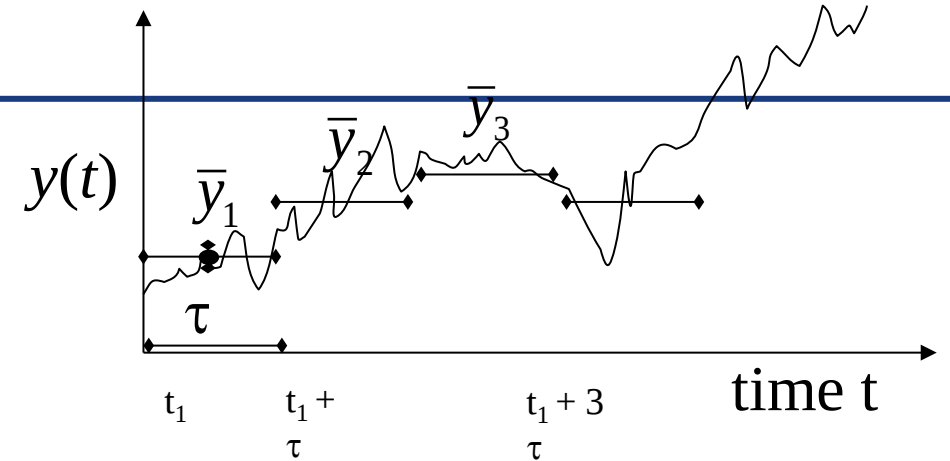
$$\sigma^2(\tau) \propto \frac{1}{\tau}$$

The variance depends on the observation interval τ

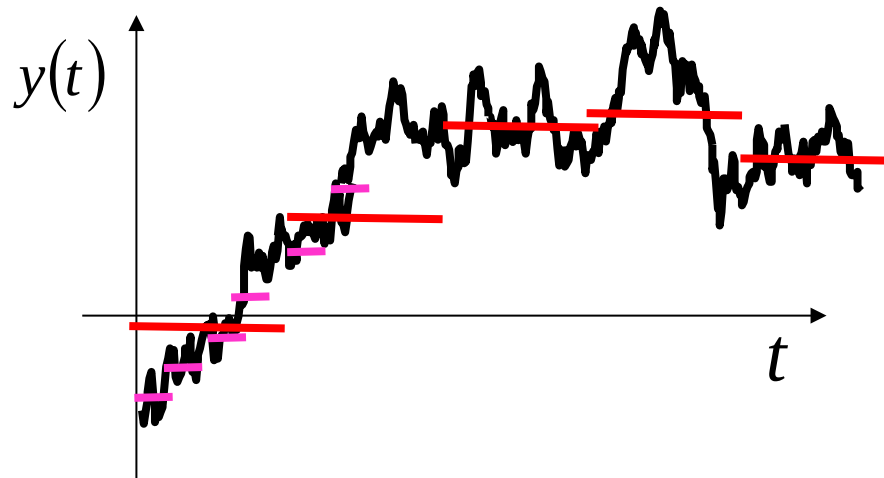
Instability - dispersion - variability

The variance is the measure of dispersion

$$\sigma^2(\tau) = \frac{\sum_{i=1}^N (\bar{y}_i - \mu)^2}{N}$$



Other typical noisy behaviour (long term noise)



If the “noise” is not strictly stationary, e.g. a random walk, the estimation of the variance do not converge and depends on N the number of samples (for any τ)

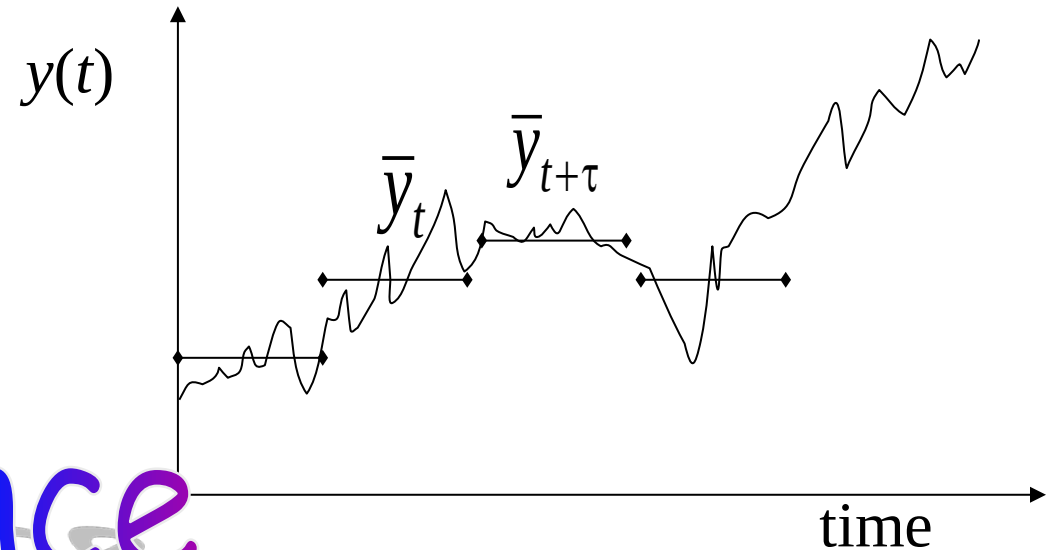
If the “noise” is not strictly stationary, e.g. a random walk, the estimation of the variance do not converge and depends on N the number of samples (for any τ)

IDEA of Allan and Barnes (1966) let's agree on the number of samples $N=2$

$$\sigma^2(N = 2, \tau) = \frac{\sum_{i=1}^2 (\bar{y}_i - \mu)^2}{2} = \frac{1}{2} (\bar{y}_1 - \bar{y}_2)^2$$

and let's average many 2-sample variances

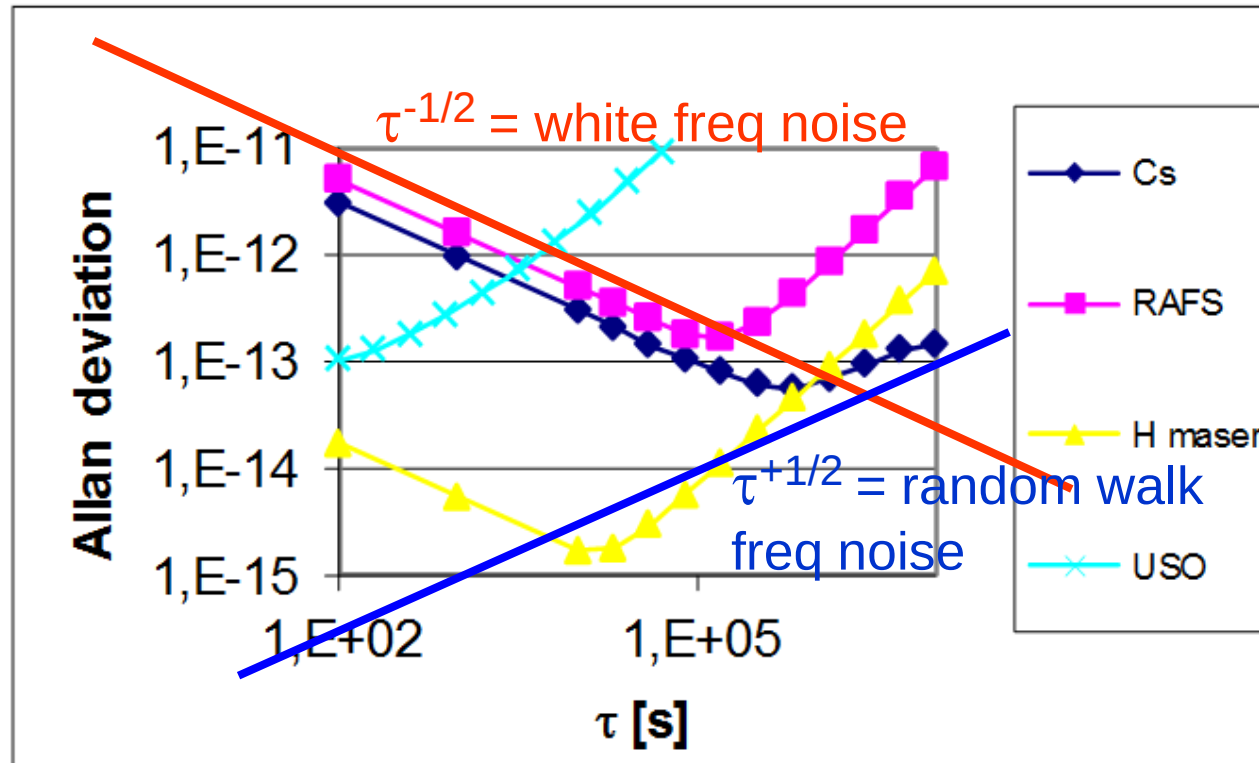
$$\sigma_y^2(\tau) = \frac{1}{2} \langle (\bar{y}_{t+\tau} - \bar{y}_t)^2 \rangle$$



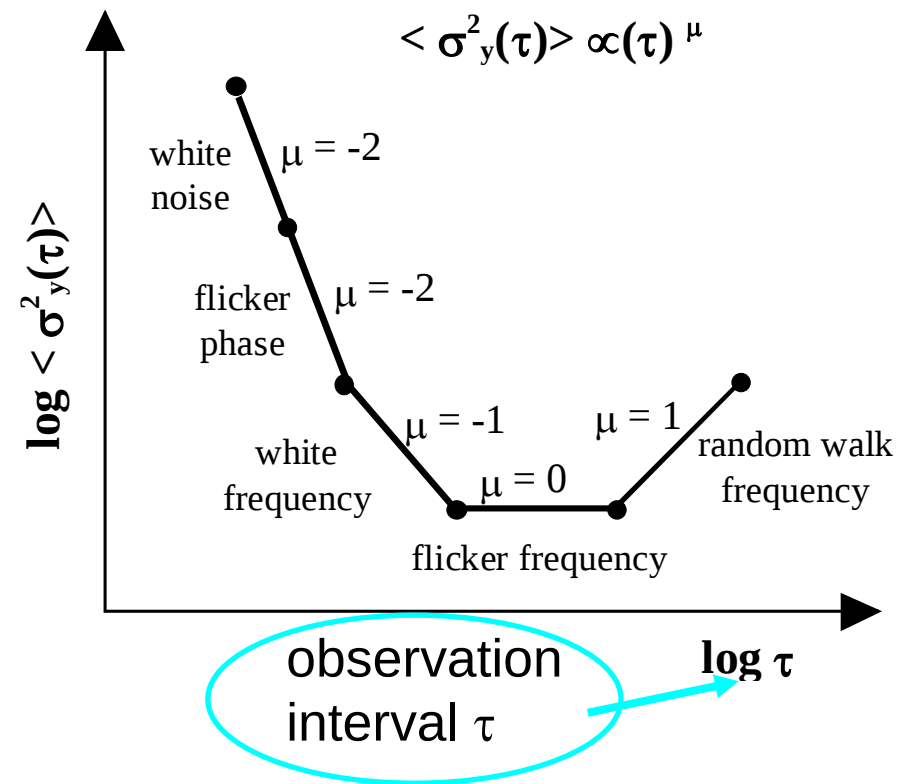
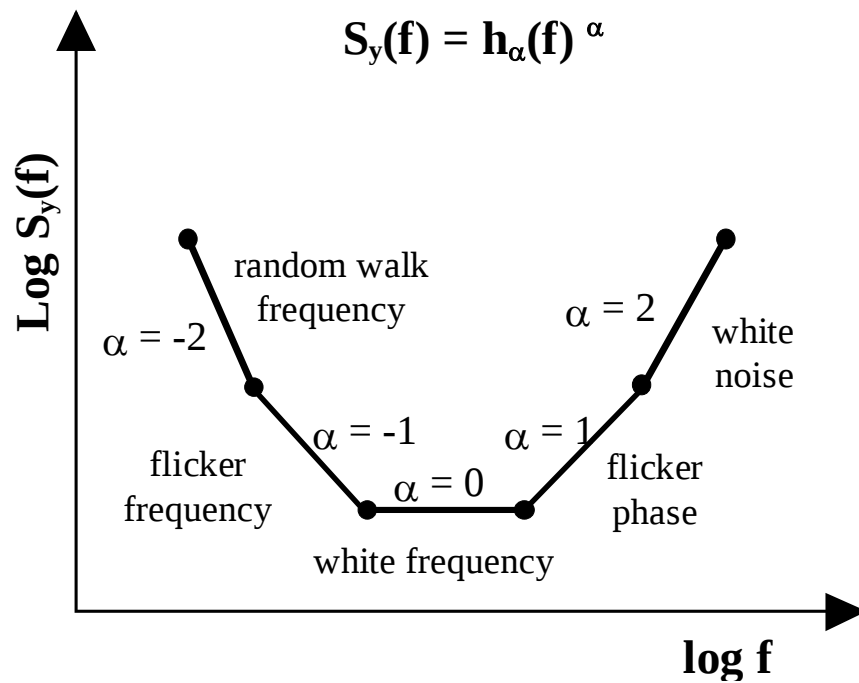
Allan variance

stability of the mean values (on τ intervals), actually variance of increments

With the Allan deviation we understand which type of noise is (mostly) affecting the measures, usually affecting the frequency of the atomic clocks



Allan variance $\leftrightarrow$ spectral density



What we learnt

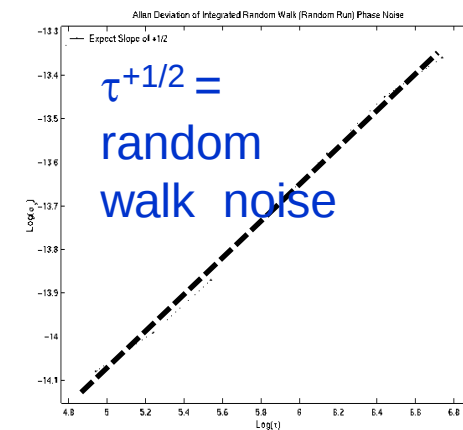
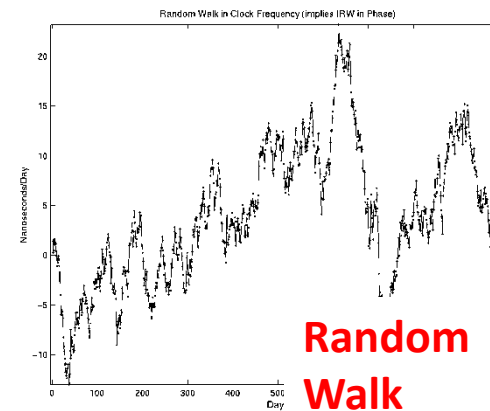
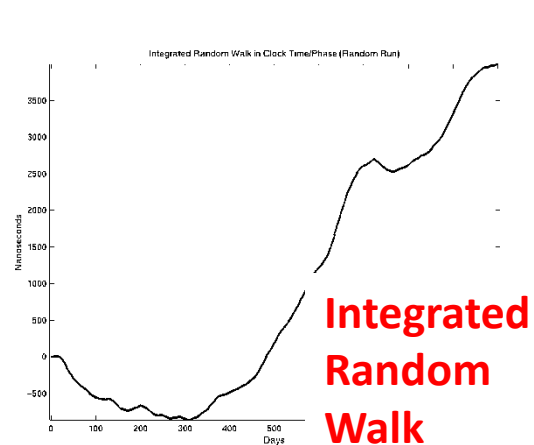
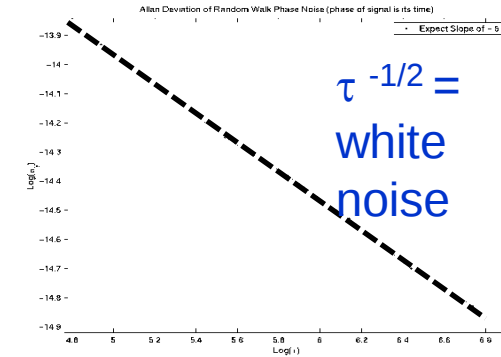
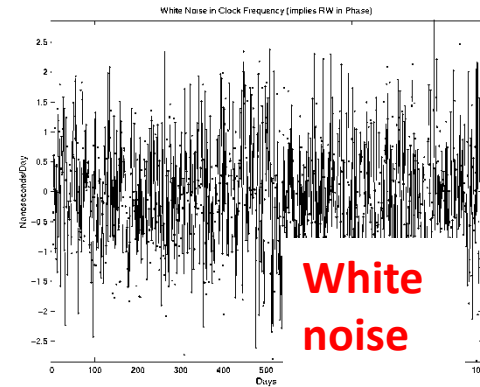
- 1- In some cases, the frequency instability is more impacting than single measure uncertainty
2. The instability variance depends on the observation interval τ
3. In some cases, the noise is non strictly stationary and the classical variance is not an appropriate tool. The Allan variance was proposed and important properties were found

Most common noise are white, random walk, and integrated random walk

Time offset = Integral of

Frequency offset

Allan Deviation



Often we want to estimate the accumulated time error due to stochastic components? Is there a method?

Allan, UFFC 1987

IEEE TRANSACTIONS ON ULTRASONICS, FERROELECTRICS, AND FREQUENCY CONTROL, VOL. UFFC-34, NO. 6, NOVEMBER 1987

TABLE III

α	Typical Noise Types Name	Optimum Prediction $x(\tau_p)$ ms ^a	Time Error: Asymptotic Form
2	white-noise PM	$\tau_p \cdot \sigma_y(\tau_p) / \sqrt{3}$	constant
1	flicker-noise PM	$\sim \tau_p \cdot \sigma_y(\tau_p) \sqrt{\ln \tau_p / 2 \ln \tau_0}$	$\sqrt{\ln \tau_p}$
0	white-noise FM	$\tau_p \cdot \sigma_y(\tau_p)$	$\tau_p^{1/2}$
-1	flicker-noise FM	$\tau_p \cdot \sigma_y(\tau_p) / \sqrt{\ln 2}$	τ_p
-2	random-walk FM	$\tau_p \cdot \sigma_y(\tau_p)$	$\tau_p^{3/2}$

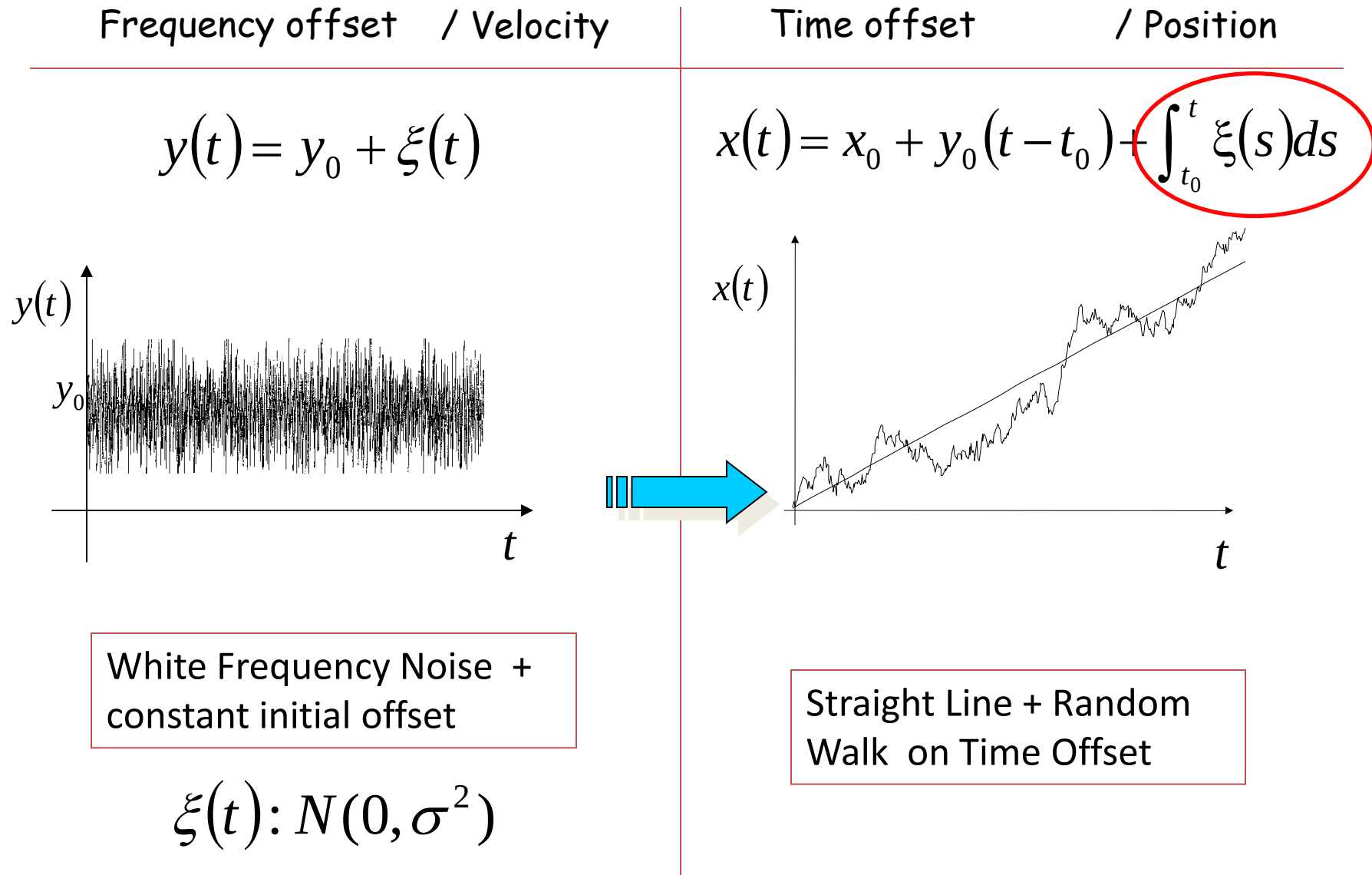
^a τ_p is the prediction interval.

let's derive that

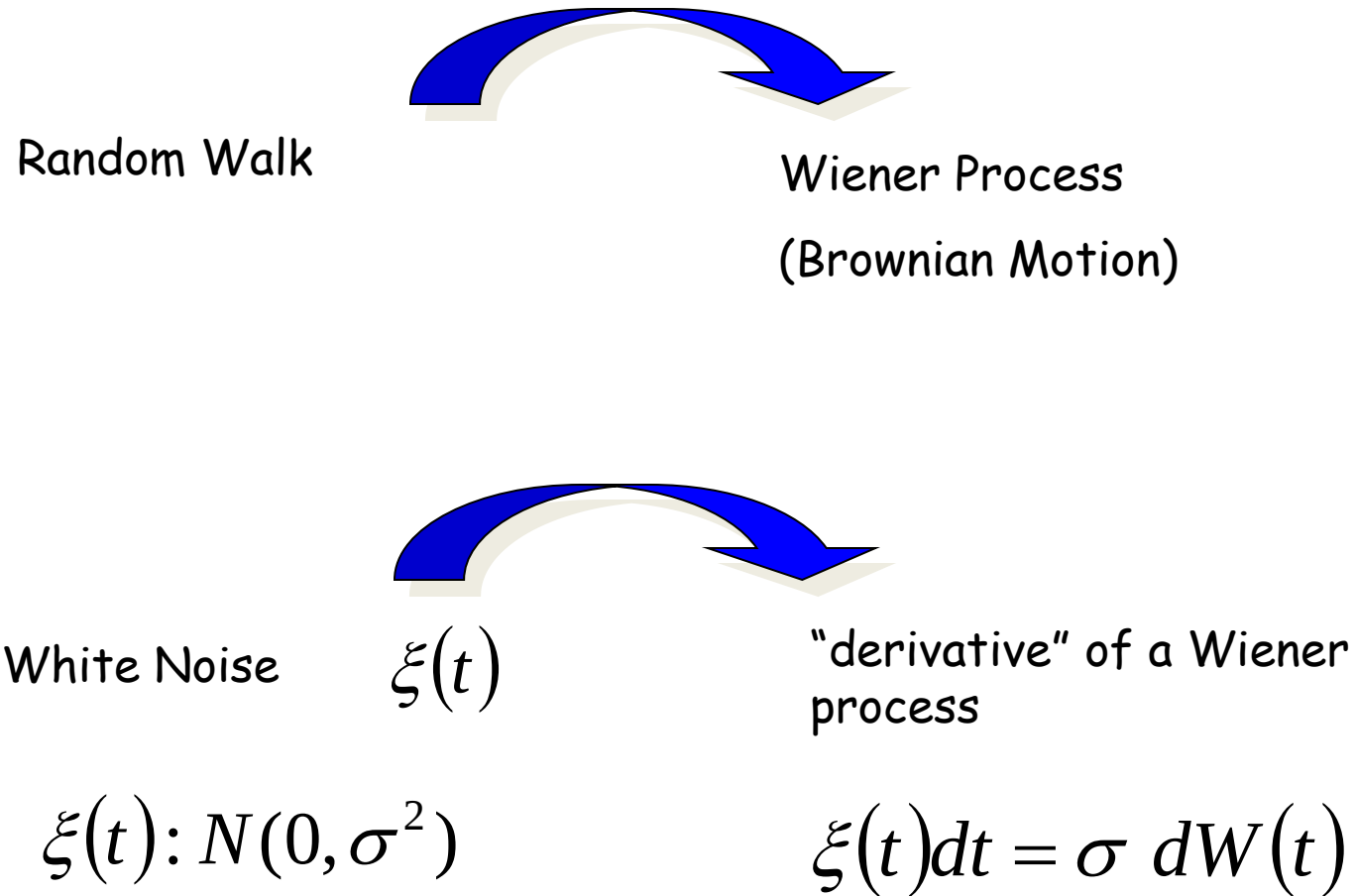
By the mathematical model for
deterministic and stochastic
behaviour

A mathematical model for determinist and stochastic behaviour

White noise plus constant offset and its integrated effect



Mathematical context: notations



Stochastic calculus

$$dx(t) = y_0 dt + \sigma dW(t)$$

Riemann integral

Stochastic integral

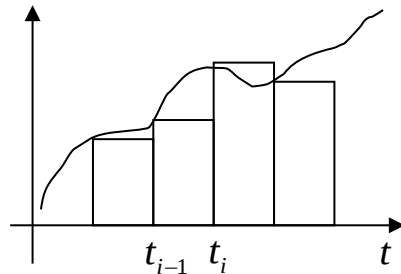
$$x(t) = \int_{t_0}^t y_0 dt + \sigma \int_{t_0}^t dW(t)$$

$$x(t) = x_0 + y_0(t - t_0) + \int_{t_0}^t \xi(s) ds$$

$$y(t) = \frac{dx(t)}{dt} = y_0 + \xi(t)$$

White noise is the derivative of Wiener process

$$\xi(t)dt = \sigma dW(t)$$



Stochastic integral

$$\approx \lim_{\delta \rightarrow 0} \sum_{i=1}^n (W(t_i) - W(t_{i-1}))$$

$$\delta = t_i - t_{i-1}$$

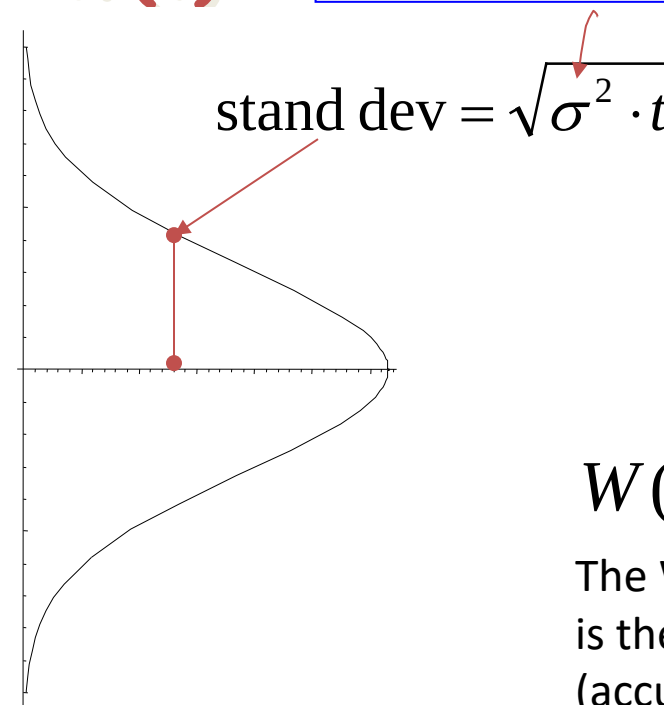
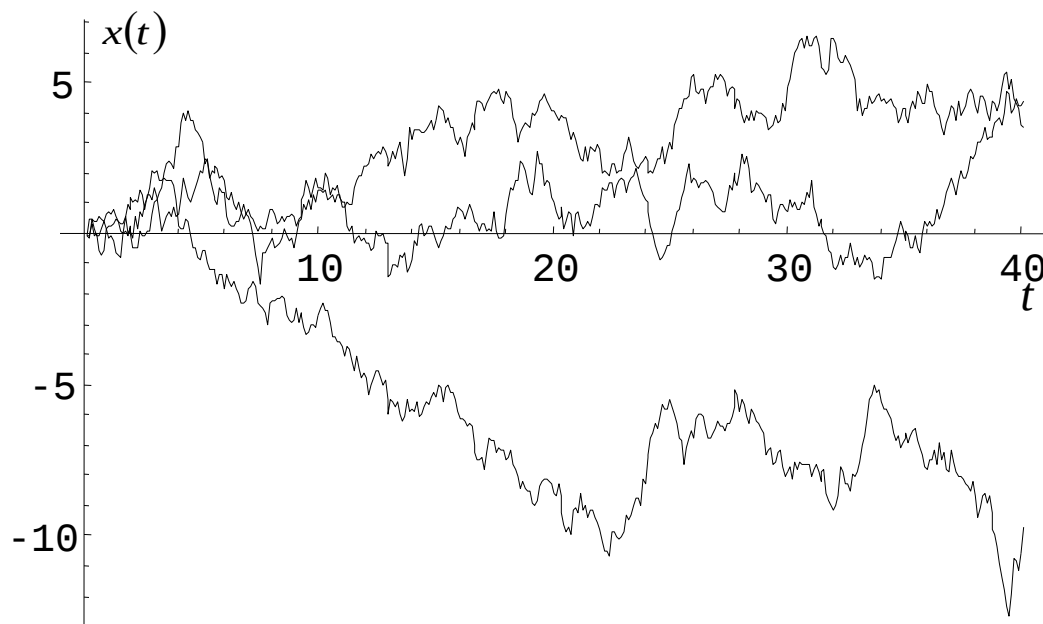
It depends on the choice of t_i

Itô stochastic integration

Random Walk position $x(t) = \sigma W(t)$

Diffusion coefficient
linked to Allan Deviation

$$x_0 = y_0 = 0$$



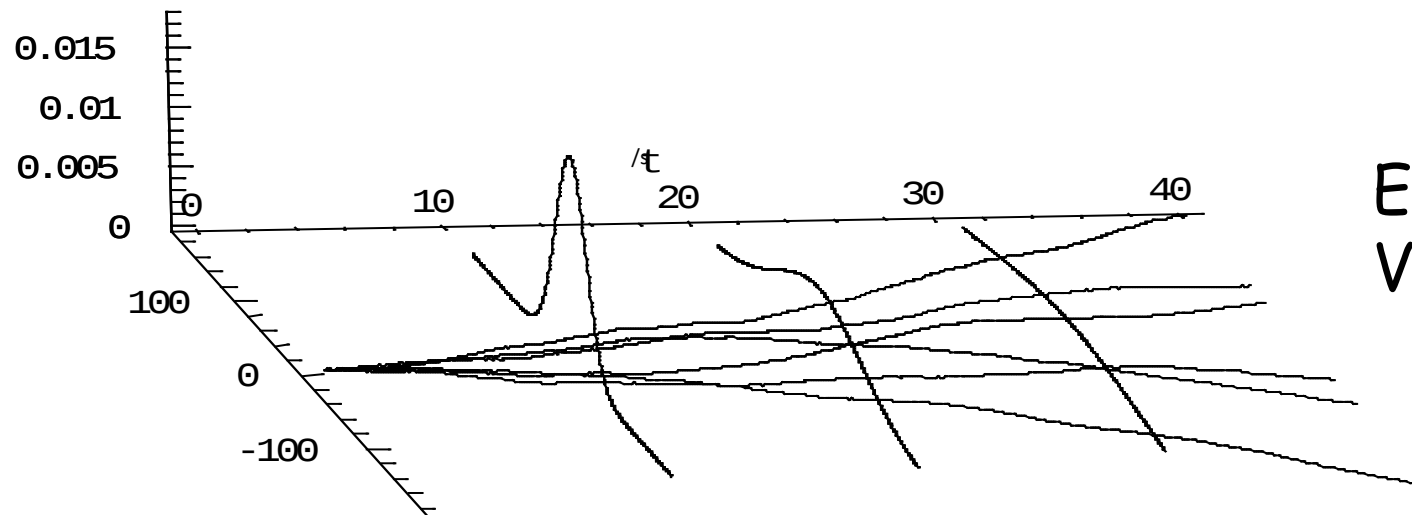
$$W(t) = \int_0^t \xi(s) ds$$

The Wiener process
is the integration
(accumulation) of
random steps

The probability of position at epoch t can be evaluated

FOR ANY NOISE

At time t after synchronisation, the time error $x(t)$ is described by a Gaussian probability density with



$$E[x(t)] = 0$$

$$\text{Variance}[x(t)] = \sigma^2 \cdot t$$

Diffusion coefficient
linked to Allan
Deviation
 $\sigma^2 t = \text{AVAR}(t) \cdot t^2$

IEEE TRANSACTIONS ON ULTRASONICS, FERROELECTRICS, AND FREQUENCY CONTROL, VOL. UFFC-34, NO. 6, NOVEMBER 1987

TABLE III

α	Typical Noise Types Name	Optimum Prediction $x(\tau_p)$ rms ^a	Time Error: Asymptotic Form
2	white-noise PM	$\tau_p \cdot \sigma_x(\tau_p) / \sqrt{3}$	constant
1	flicker-noise PM	$-\tau_p \cdot \sigma_x(\tau_p) \sqrt{\ln \tau_p / 2 \ln \tau_0}$	$\sqrt{\ln \tau_p}$
0	white-noise FM	$\tau_p \cdot \sigma_f(\tau_p)$	$\tau_p^{1/2}$
-1	flicker-noise FM	$\tau_p \cdot \sigma_f(\tau_p) / \sqrt{\ln 2}$	τ_p
-2	random-walk FM	$\tau_p \cdot \sigma_f(\tau_p)$	$\tau_p^{3/2}$

^a τ_p is the prediction interval.

We do not know the exact time error of an individual clock but we can estimate its statistical overall behaviour

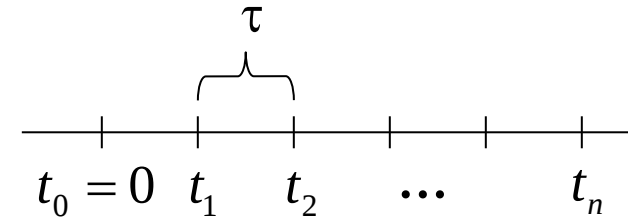
In case of more noises: integrated white and random walk the exact solution exists:

$$\begin{cases} x_1(t) = x_0 + y_0 t + a \frac{t^2}{2} + \sigma_1 W_1(t) + \sigma_2 \int_0^t W_2(s) ds \\ x_2(t) = y_0 + a t + \sigma_2 W_2(t) \end{cases}$$

White Freq. noise

RW Freq. noise

Iterative solution useful
for **simulations**, filter, ...



$$\begin{cases} X_1(t_{k+1}) = X_1(t_k) + X_2(t_k) \tau + a \frac{\tau^2}{2} + \sigma_1 W_{1,k}(\tau) + \sigma_2 \int_{t_k}^{t_{k+1}} W_2(s) ds \\ X_2(t_{k+1}) = X_2(t_k) + a \tau + \sigma_2 W_{2,k}(\tau) \end{cases}$$

correlated processes

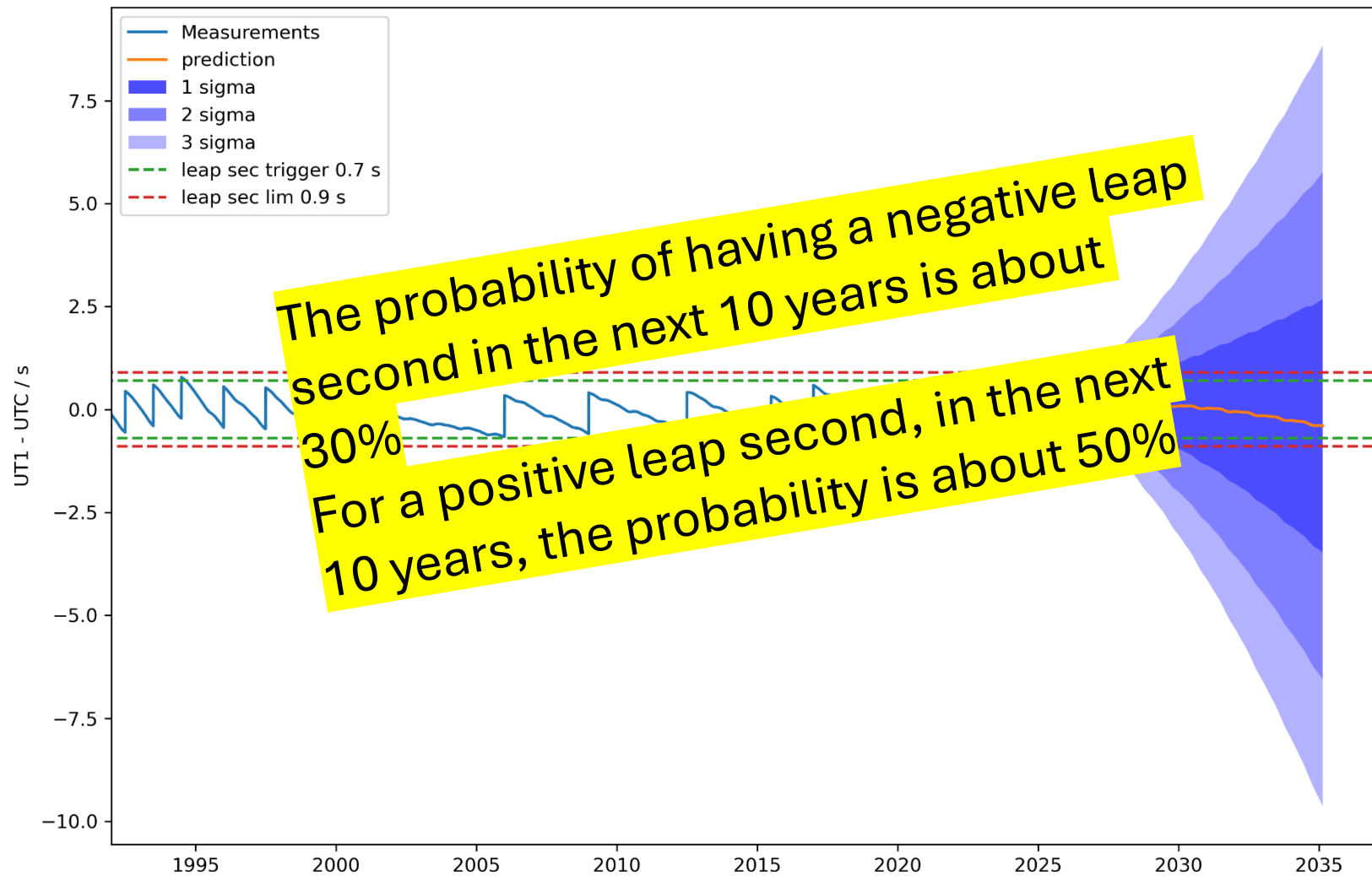
with initial conditions

$$\begin{cases} X_1(0) = x_0 \\ X_2(0) = y_0 \end{cases}$$

Prediction on UT1-UTC

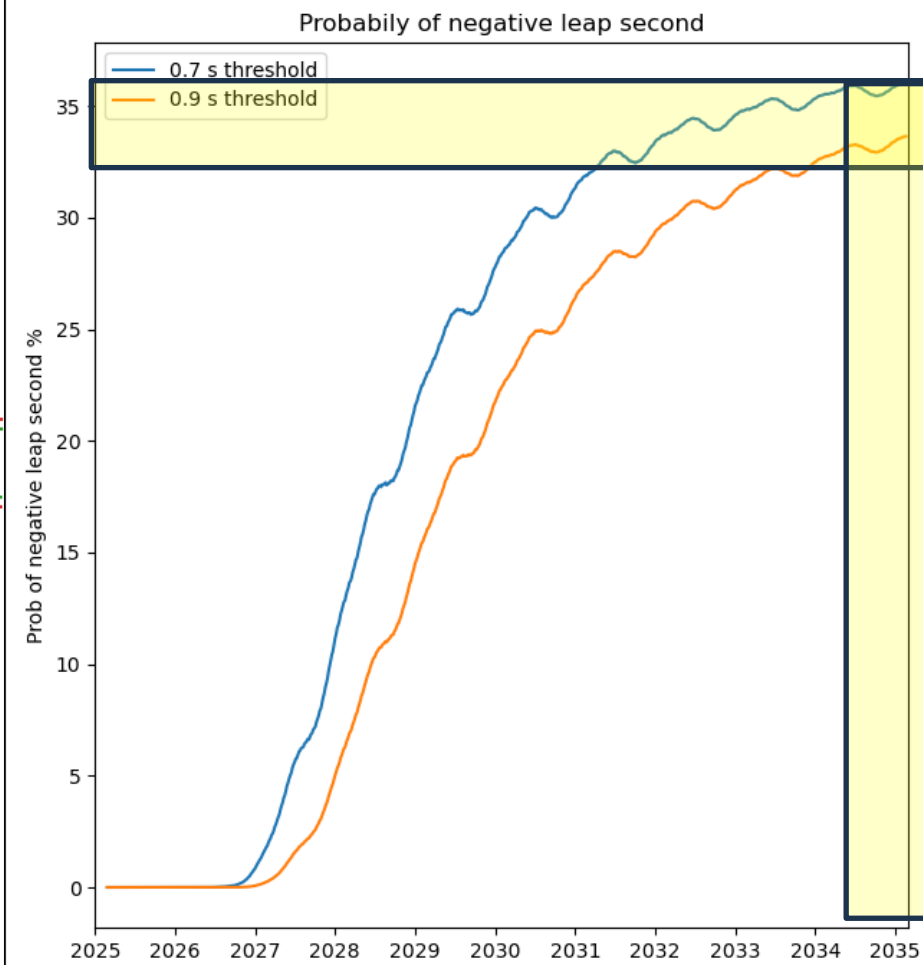
- current UT1-UTC ≈ 0
- current LOD excess ≈ 0
- millennial slowing down is slow
- periodical terms are limited
- the stochastic behavior has zero mean value and contributes, largely, to the uncertainty

Example of recent application



The probability of having a negative leap second in the next 10 years is about 30%

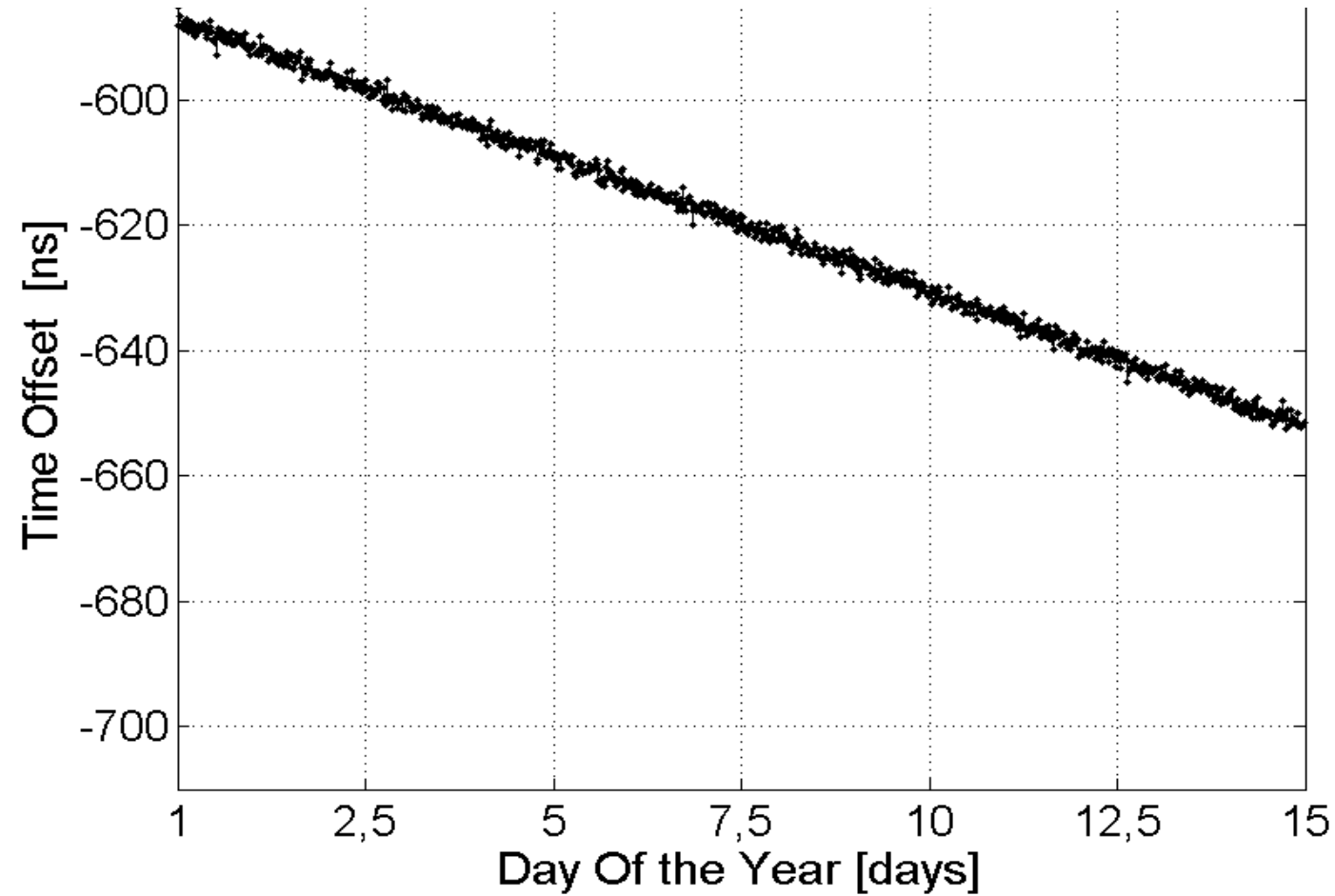
For a positive leap second, in the next 10 years, the probability is about 50%



What we learnt

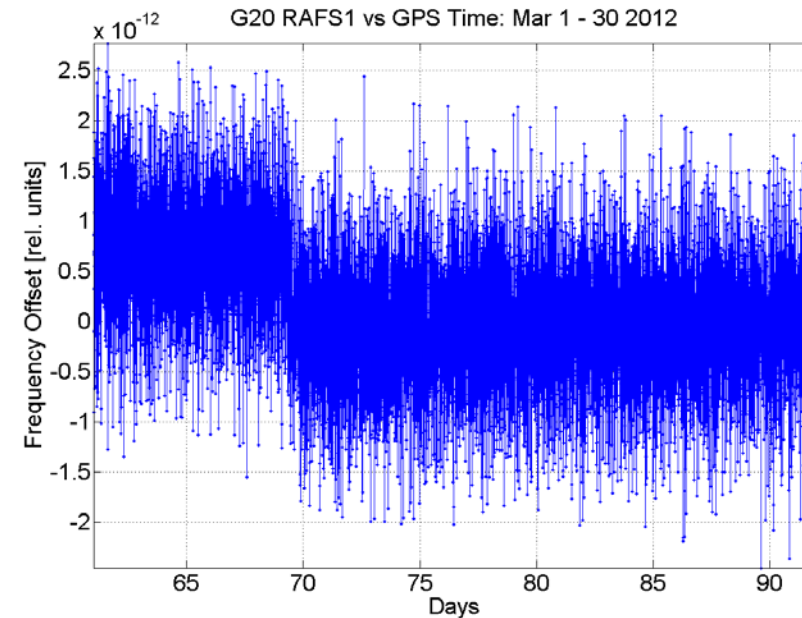
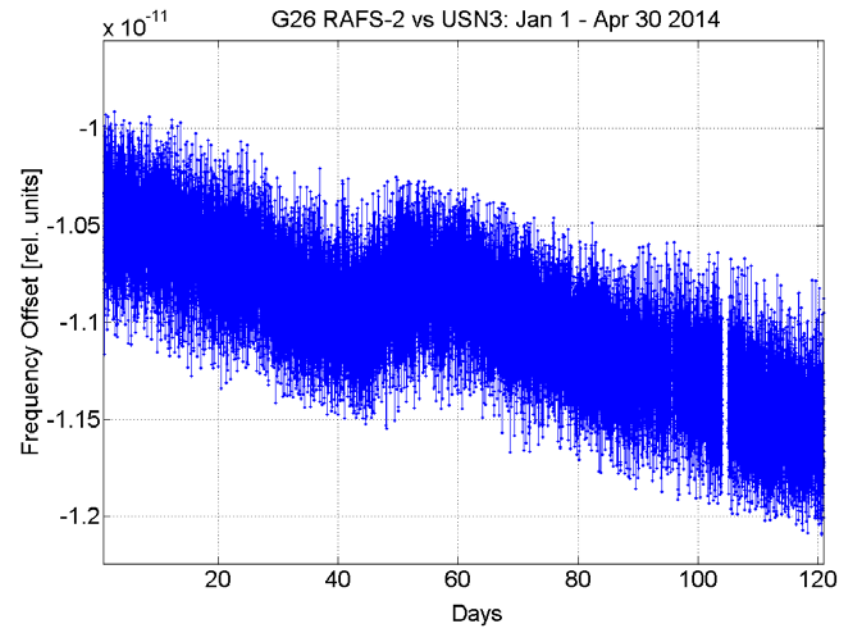
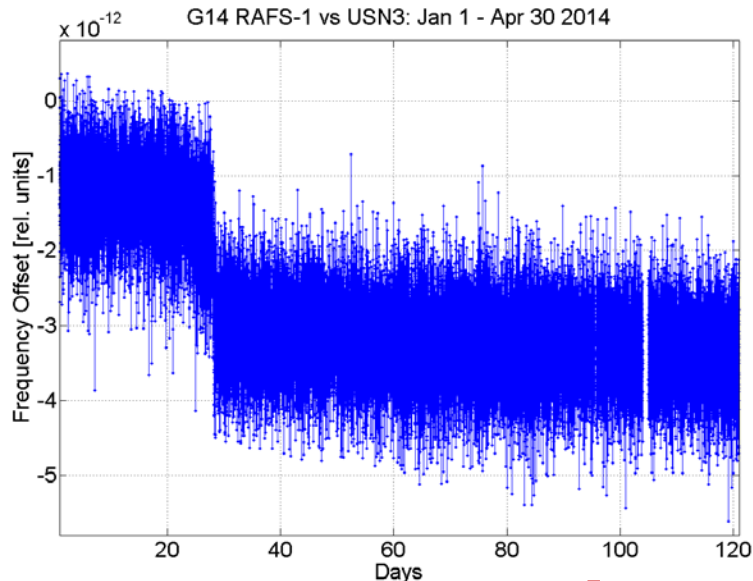
- 1- Deterministic and stochastic behaviour can be modeled by stochastic differential equations(SDE)
2. The exact (or approximated) solution of the SDE allows the dynamic behaviour estimation, simulation, and prediction
3. The noises imbedded in the model has usually zero mean value, they do not impact on the prediction but on the uncertainty of the predicted values
(therefore allowing to evaluate confidence intervals)

Will the well understood and modeled behaviour last forever?



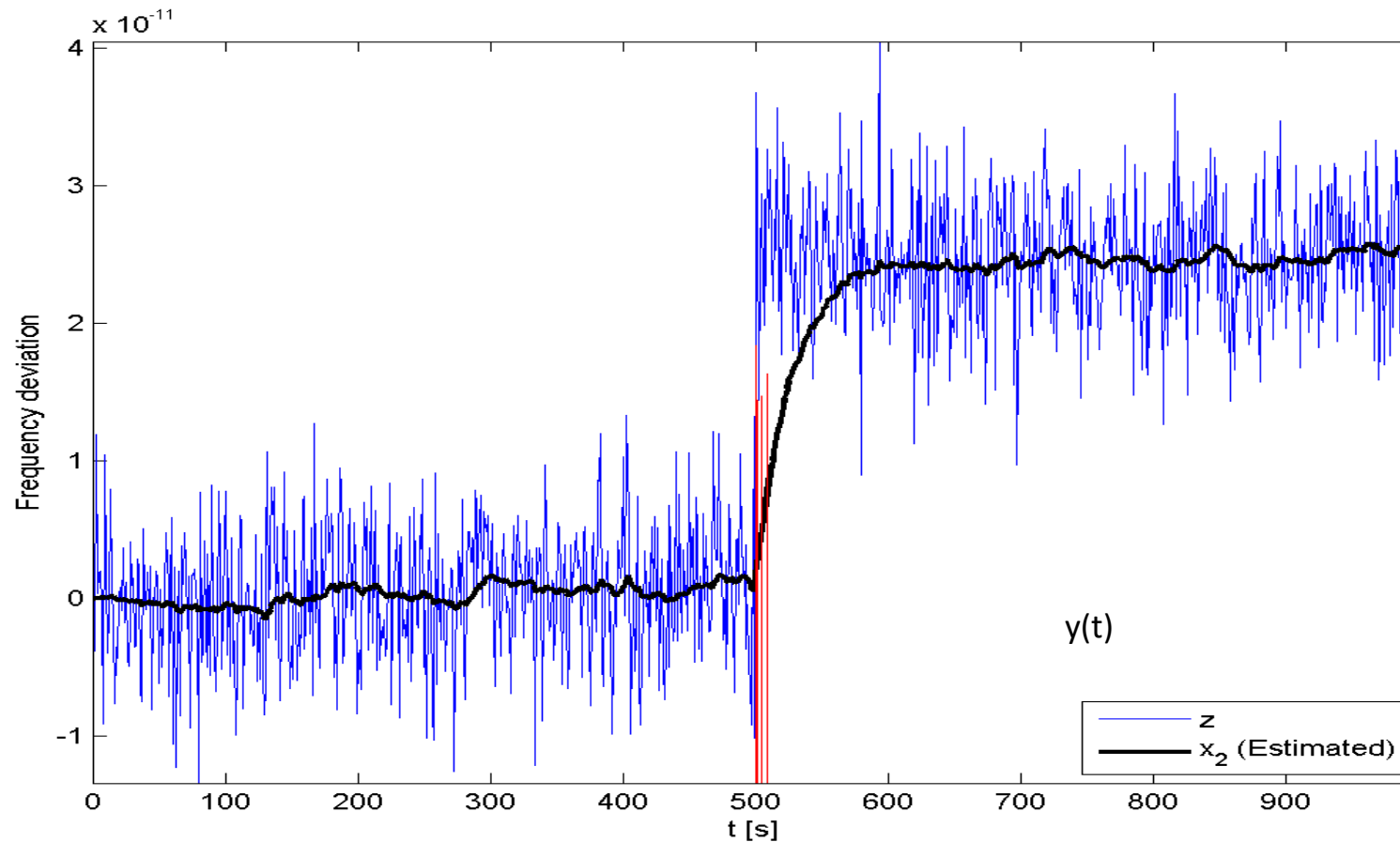
Example of GPS space clocks

Possible causes of nonstationarities on space clocks are manoeuvres and tests on board, environmental variations, eclipses, etc. In other cases, the nonstationarities may be due to the clock itself.



Frequency jumps are observed

WE NEED A FREQUENCY JUMP DETECTOR



Q. Wang, F.Droz, P. Rochat, "Robust Clock Ensemble for Time and Frequency Reference System", presented at the [EFTF/IFCS Denver 2015](#)

L.Galleani, P.Tavella, "Detection of Atomic Clock Frequency Jumps with the Kalman Filter," [IEEE Transactions on Ultrasonics, Ferroelectr, Freq Control](#) March 2012. vol. 59, no. 3, p. 504-509, March 2012

[Huang X](#), [Gong H](#), [Ou G](#), Detection of weak frequency jumps for GNSS onboard clocks, [IEEE Trans Ultrasonics, Ferroelectr Freq Control](#). 2014 May;61(5):747-55

Several detectors have been studied, the main idea is that from the past we predict the future and then we compare prediction with fresh measures (or mean of measures) and we raise an alarm if the observed behaviour is too different from the predicted one

Nw = full data set

Nw_1 = estimation of deterministic behaviour

Nw_2 = check on jump

Nw_3 = moving average to smooth noise

$\hat{\mu}$ = estimated value of deterministic trend

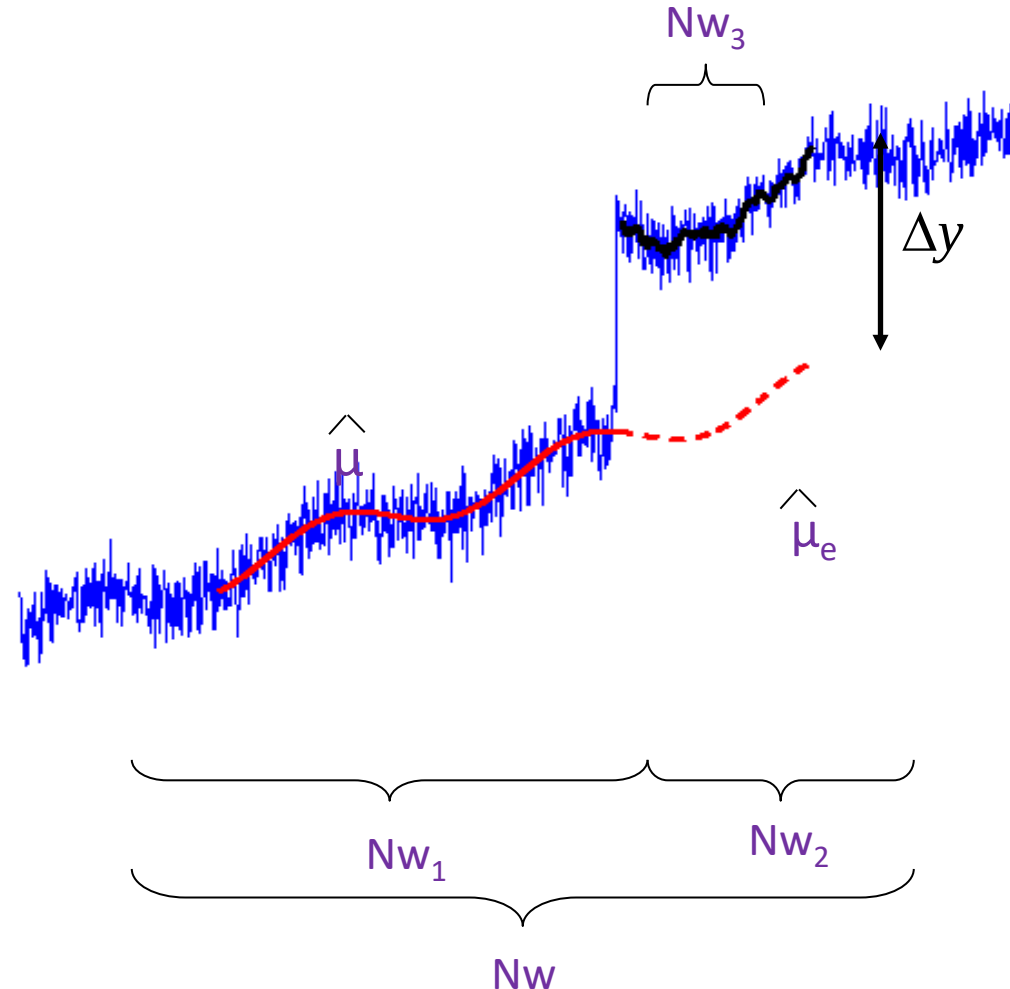
$\hat{\mu}_e$ = extrapolated deterministic trend

y_s = frequency values smoothed by a moving average on Nw_3 samples (sliding in window Nw_2)

$$\Delta y = |y_s - \hat{\mu}_e|$$

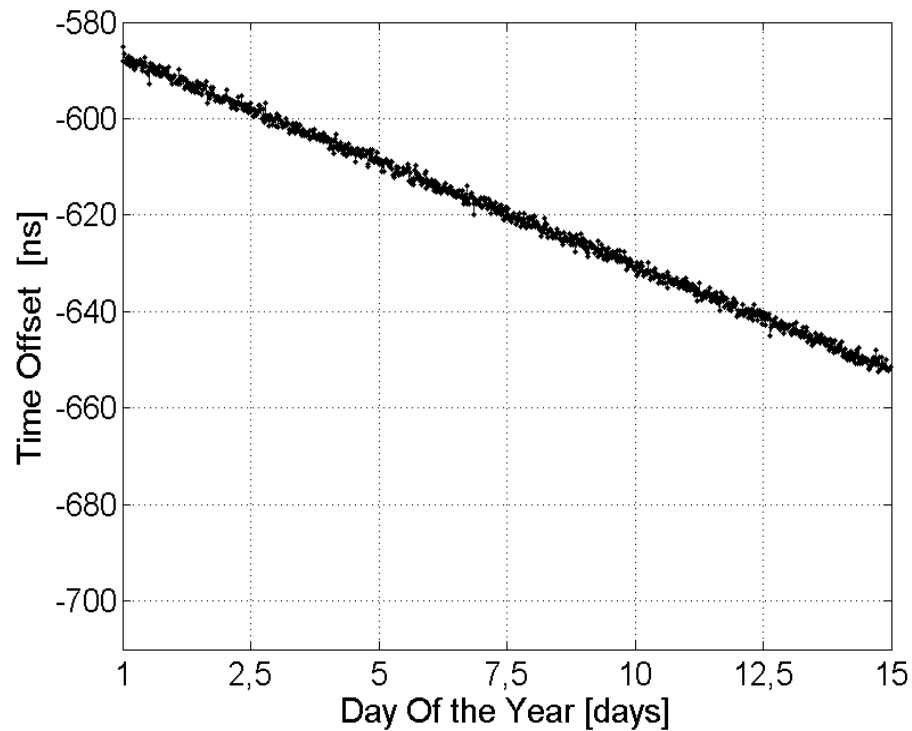
If $\Delta y > \text{threshold}$

➡ Alarm

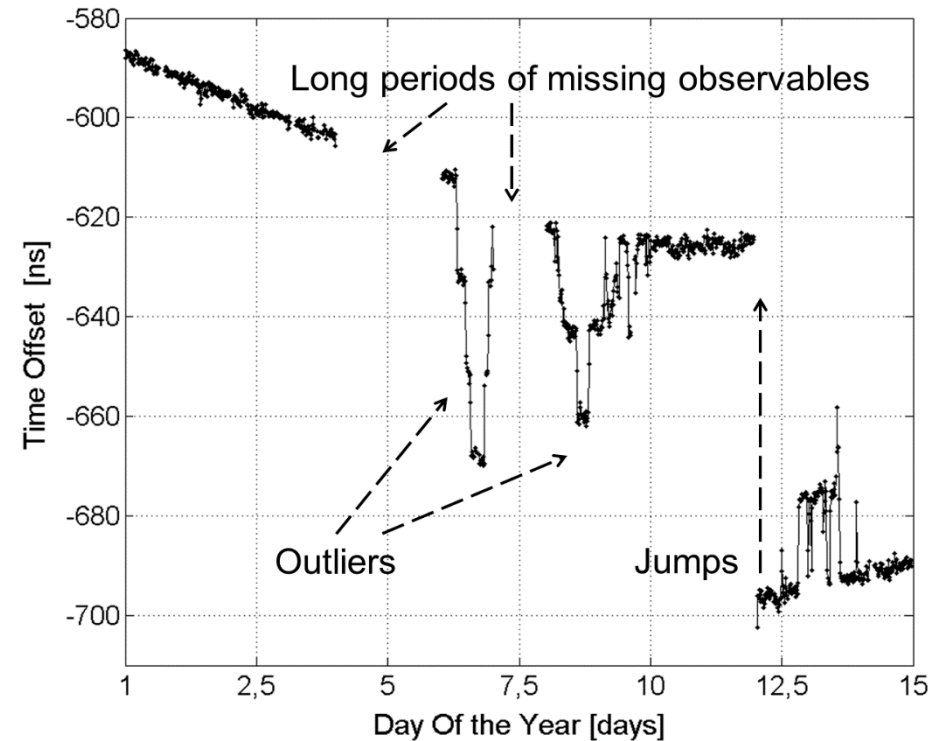


Sometimes real experimental data are not as we would like

ideal case



real case

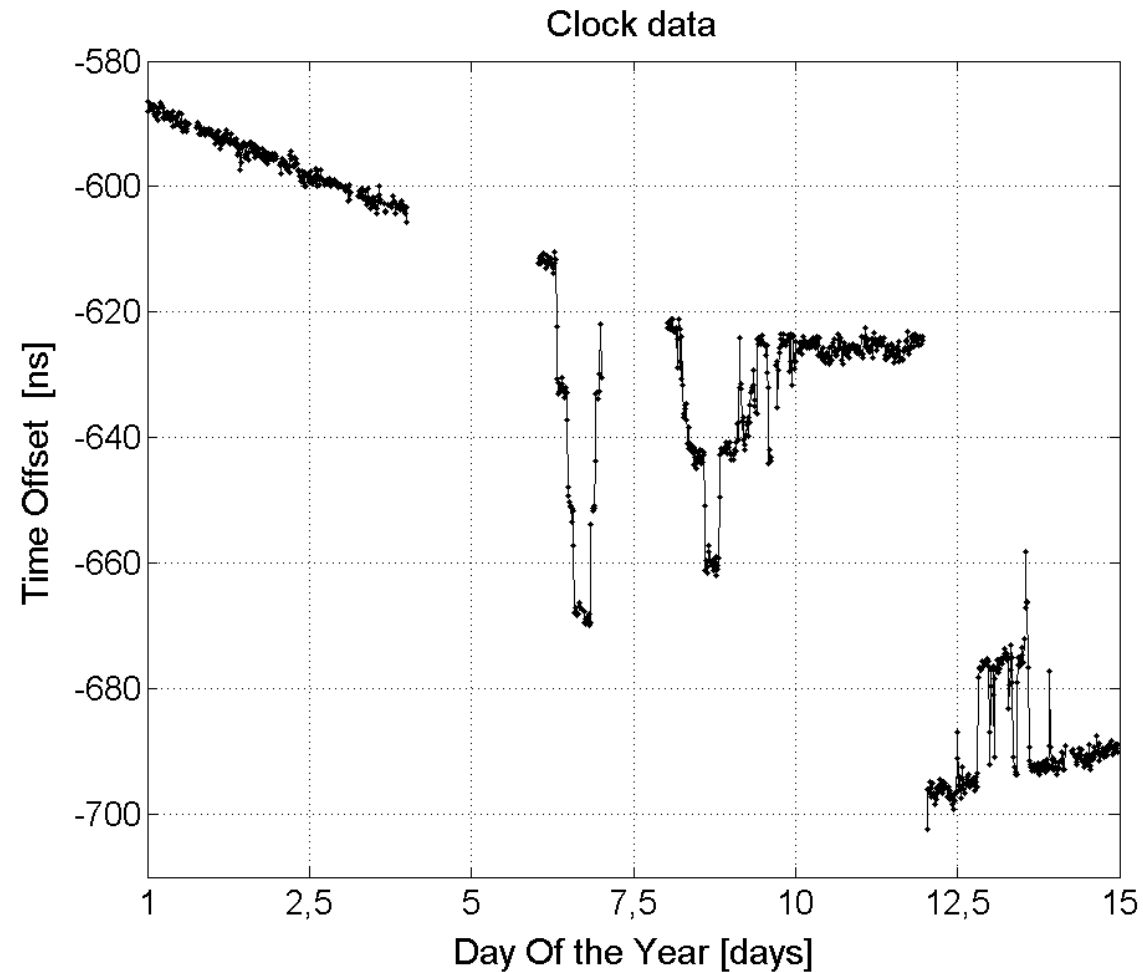


The importance of data preprocessing

- ✓ Data series may present gaps, long periods of missing observation, outliers and jumps
- ✓ Anomalies have to be detected and properly removed before the data processing and analysis steps
- ✓ An efficient preprocessing is fundamental to ensure a proper data analysis

Outliers filtering

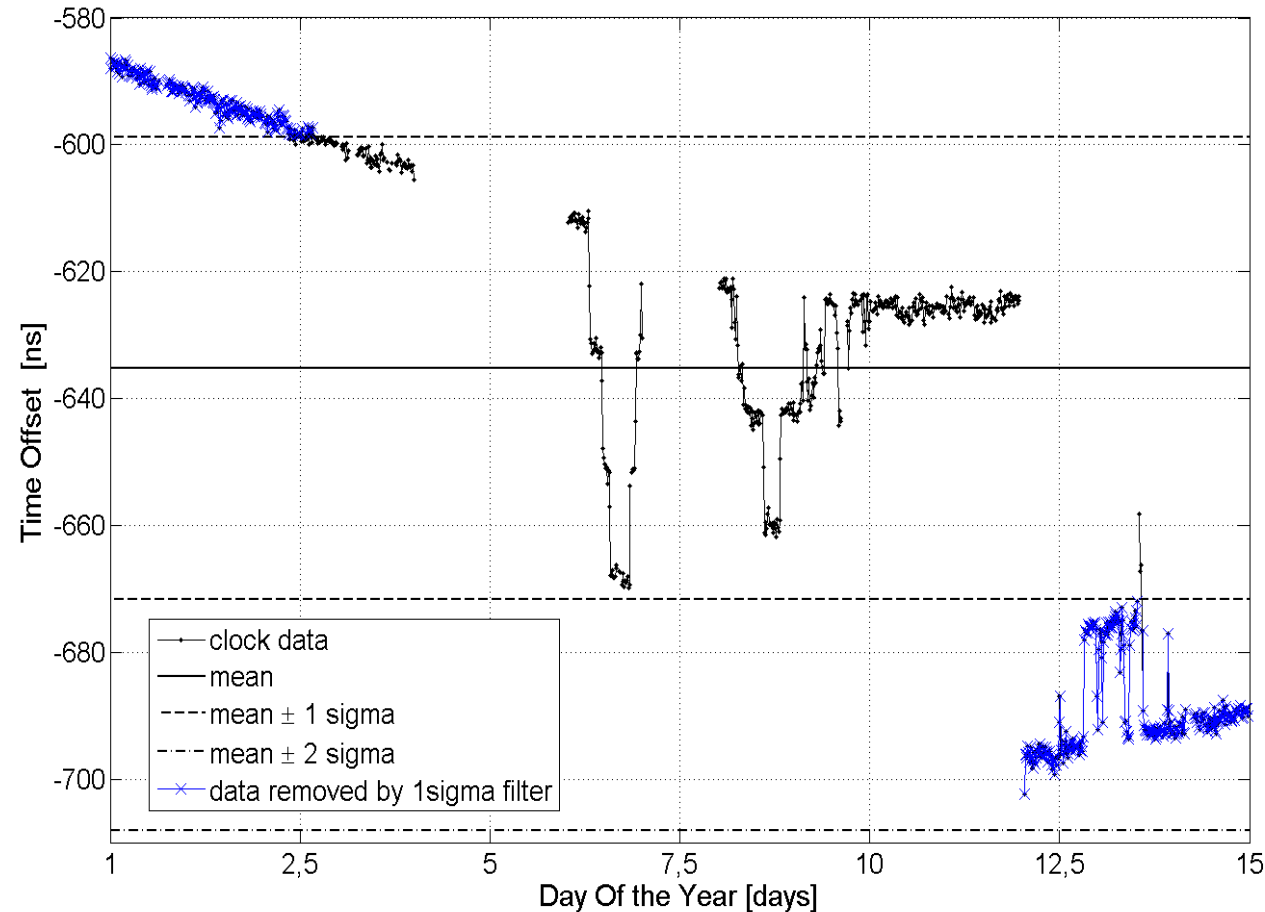
Examples of applications to experimental clock data



Outliers filtering

Examples of applications: batch sigma filter

Data outside $(\text{mean} \pm k \sigma)$ are removed



Outliers filtering

Examples of applications: batch MAD-based filter
Data outside (median $\pm$ k MAD) are removed

Median

Middle value separating the greater and lesser halves of a data set

Ex: 2, 4, 7, 2, 9, 3, 1

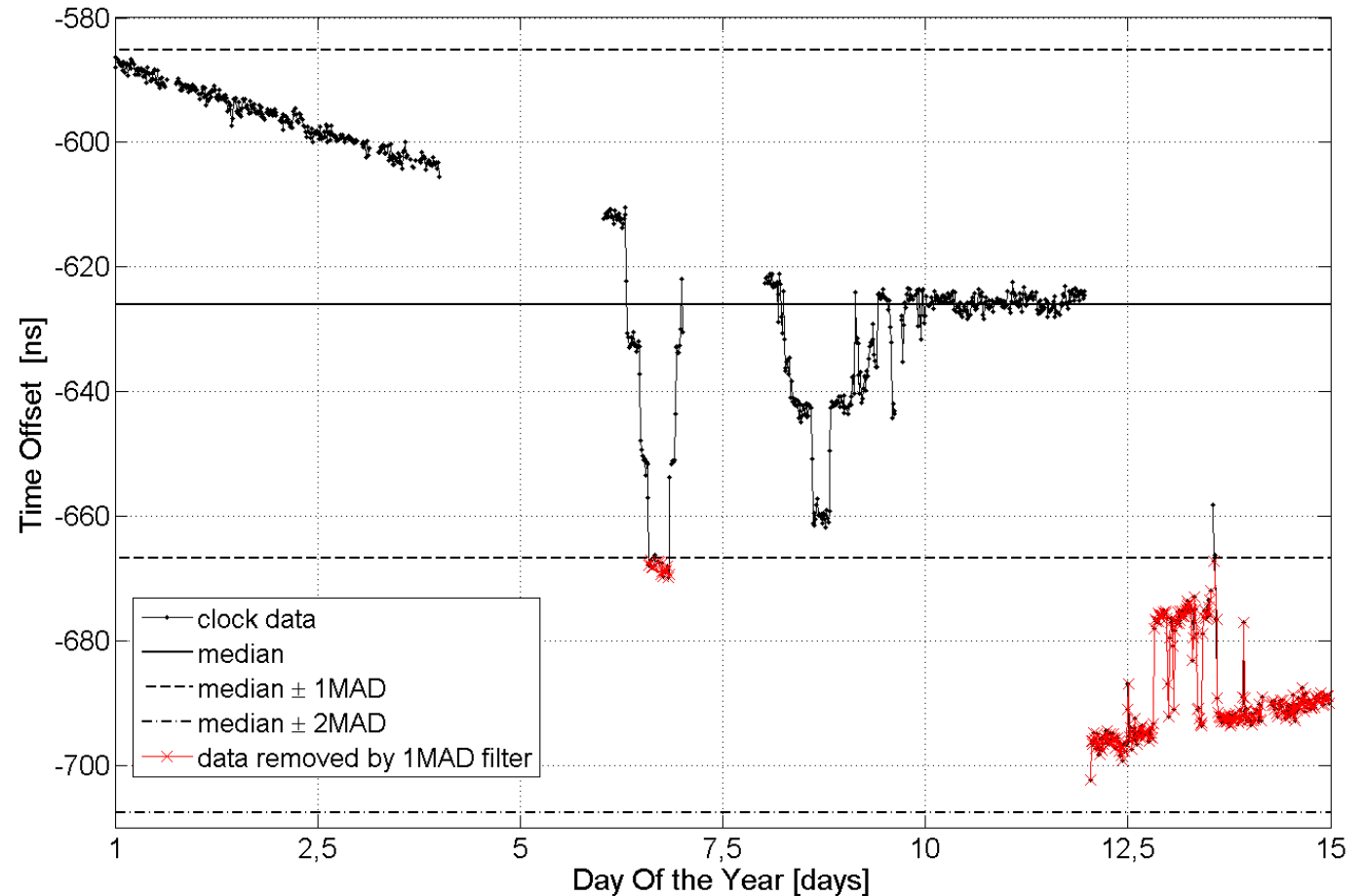
1, 2, 2, **3**, 4, 7, 9

MAD

median of the absolute deviations

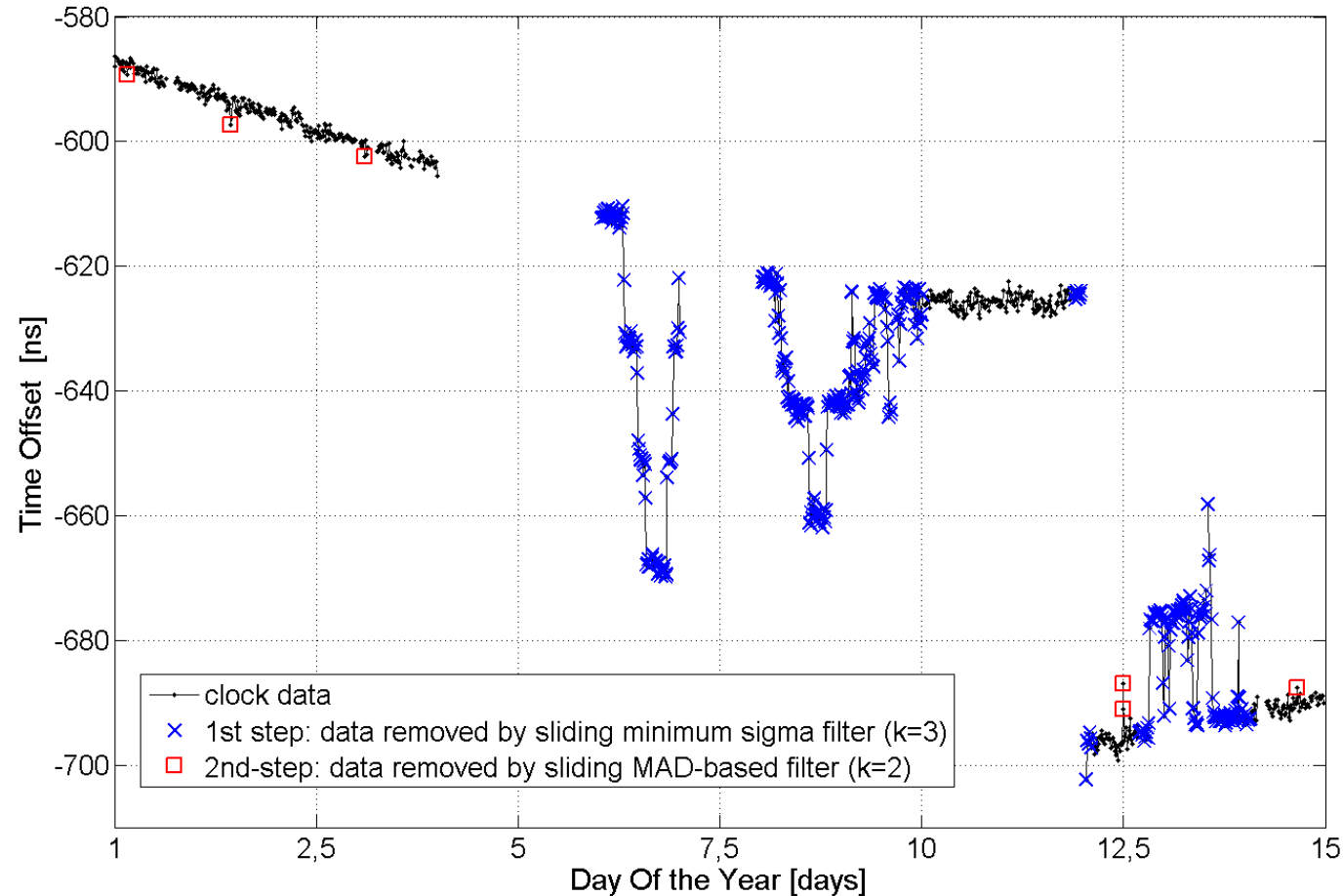
from the data's median

$$\text{MAD} = \text{Median} (|X_i - \text{Median}(X_i)|)$$



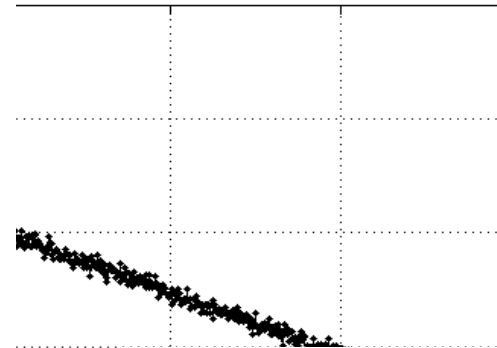
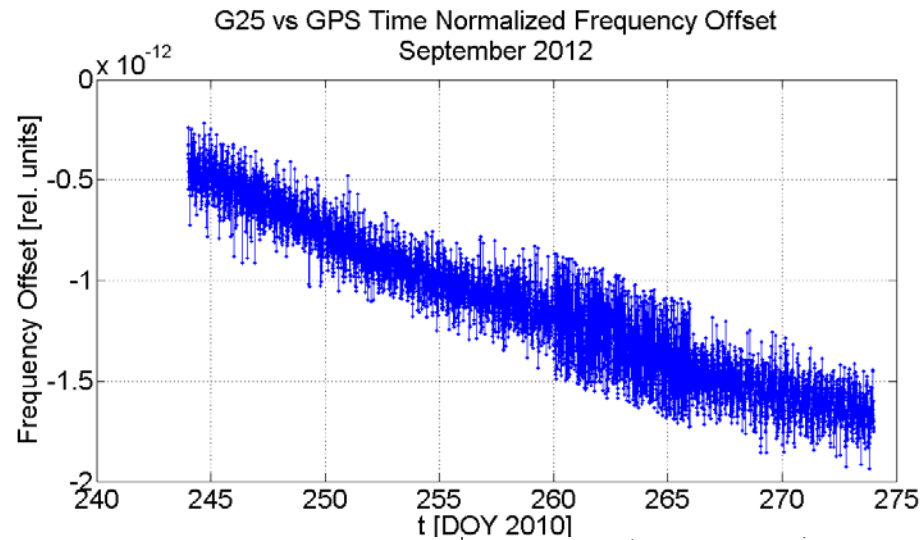
Outliers filtering

A moving window re evaluating sigma and MAD and validating outliers on different windows may be useful

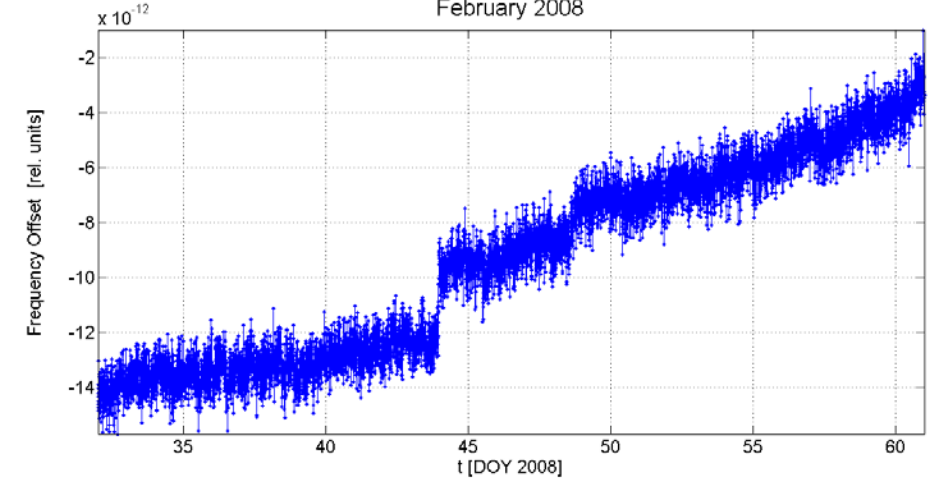
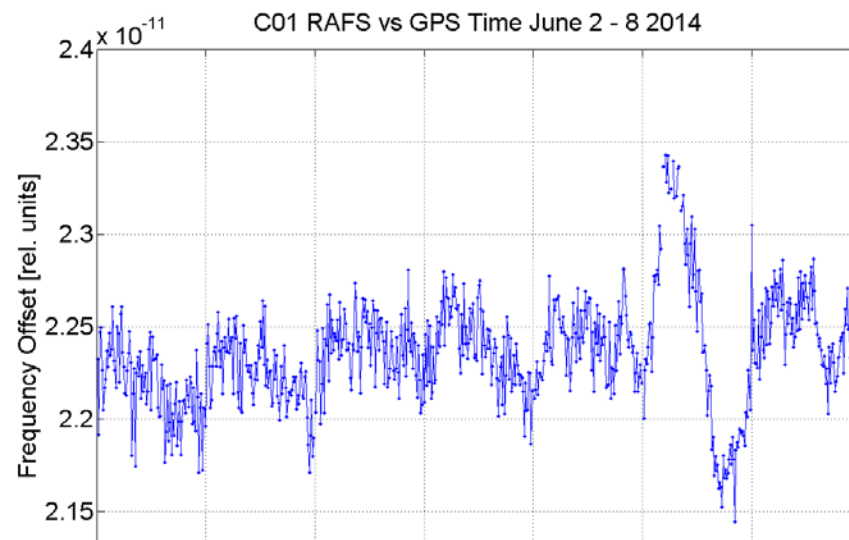


Filters are applied on a sliding window of 5 hours. Outliers are validated before removal.

Will the well understood and modeled behaviour last forever?



G10 vs GPS Time Normalized Frequency Offset
February 2008



We need a
dynamical characterization of the noise

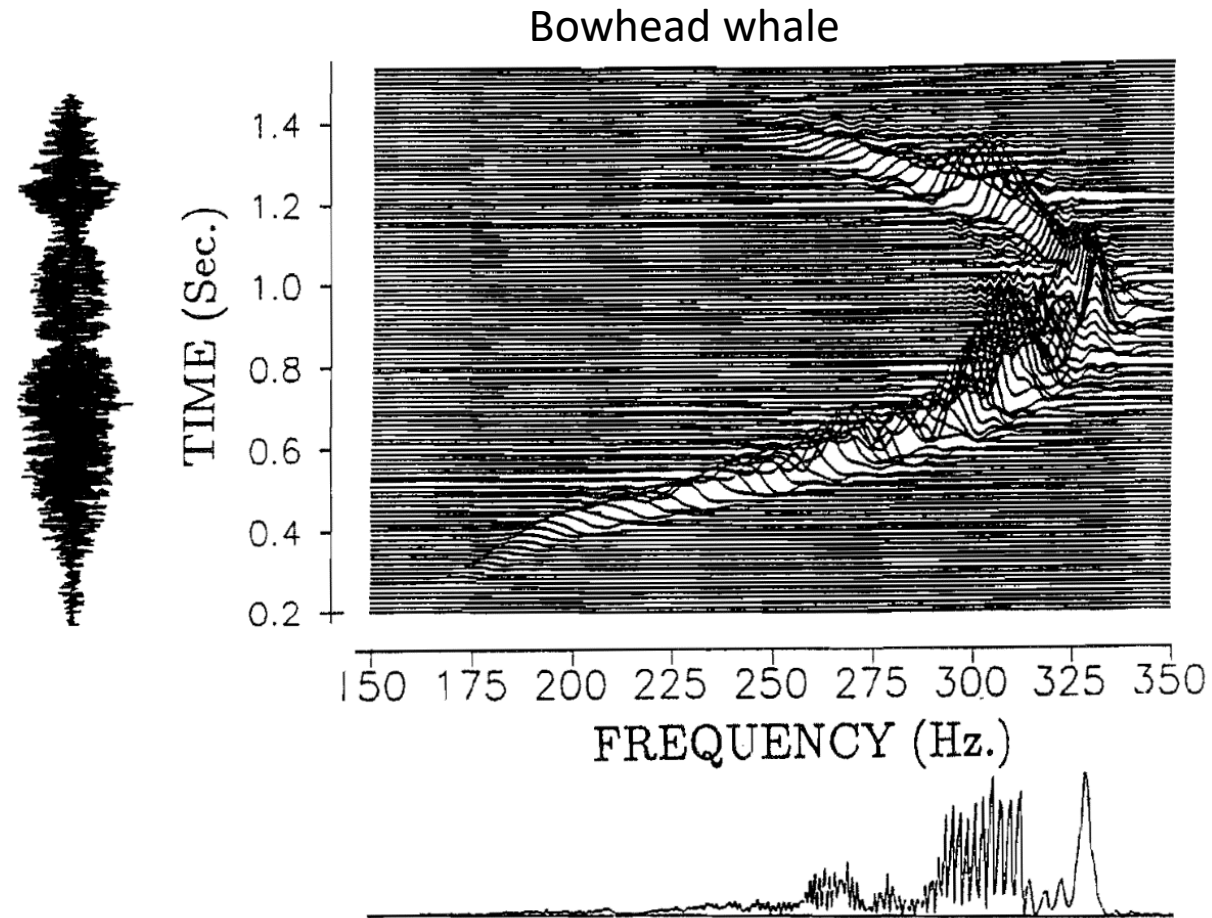
Time and Frequency spectral analysis
for example

Not only estimating **which** frequencies
existed

But also estimating **when** they existed

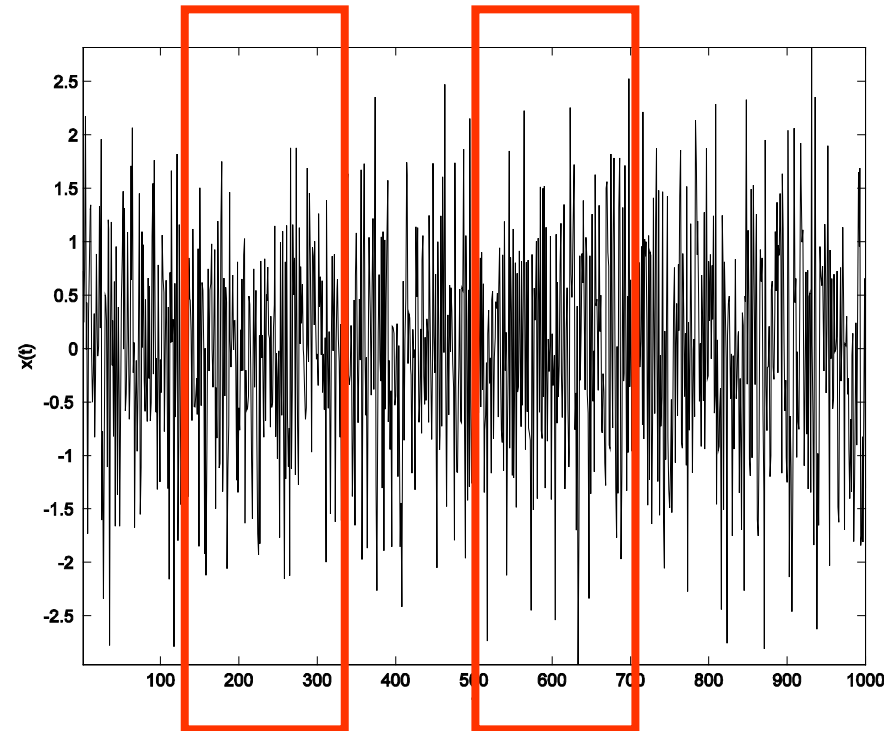
Time-frequency analysis

It describes how the frequencies of a signal change with time



A Dynamic Allan variance

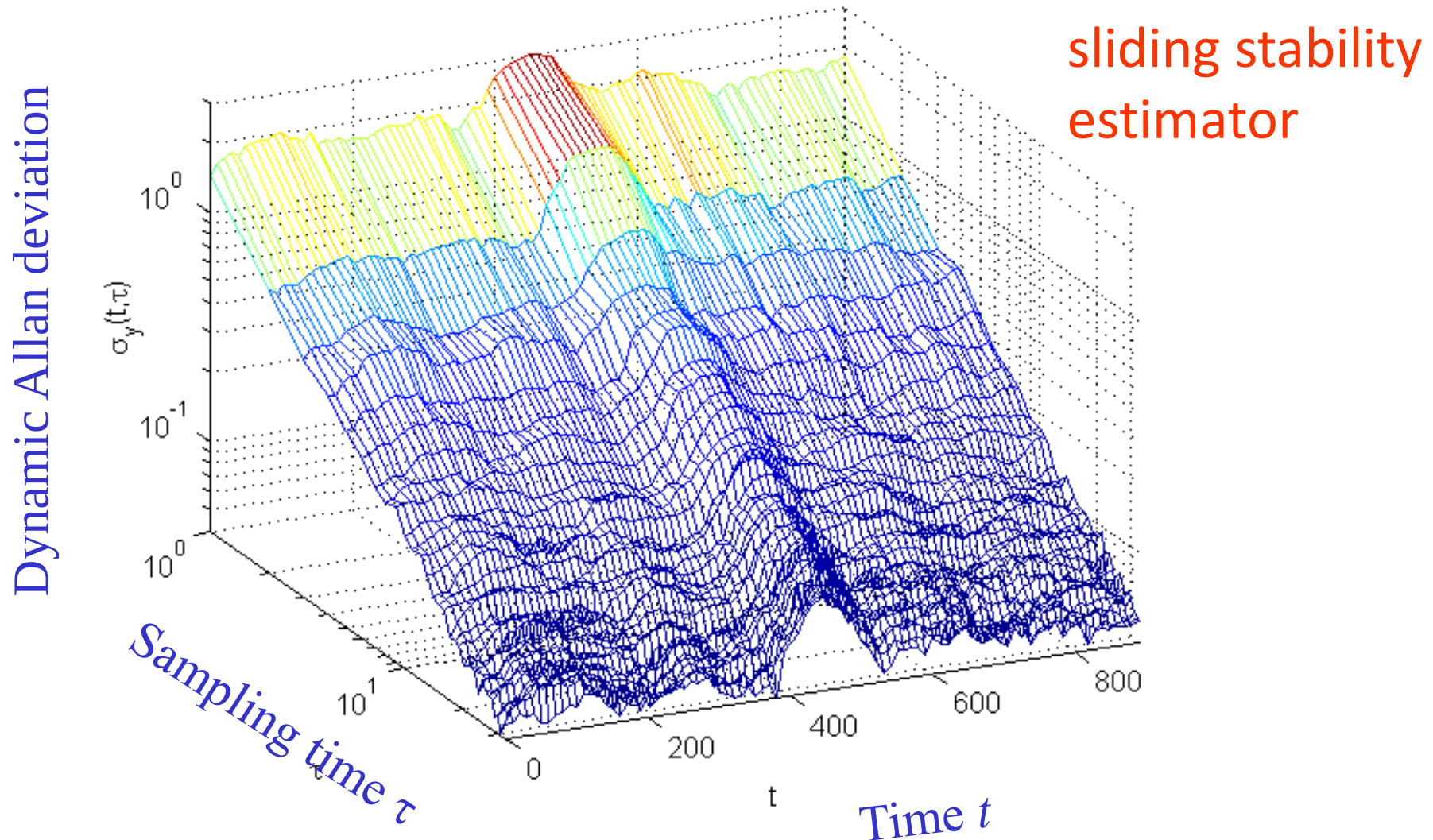
sliding the Allan variance estimator on the data



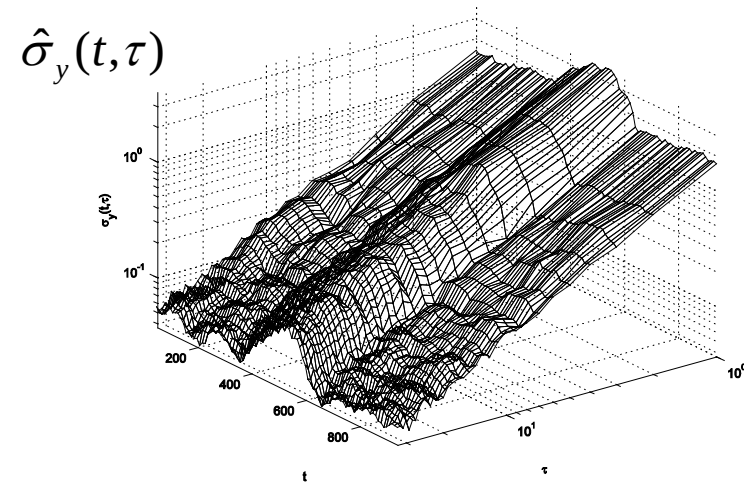
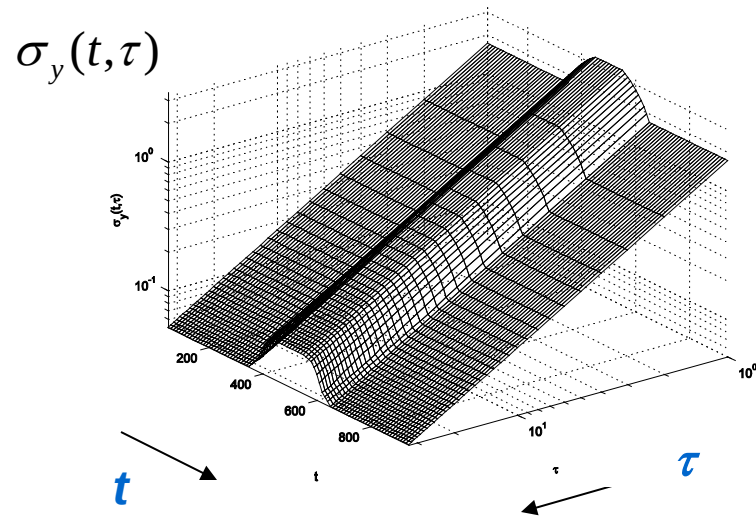
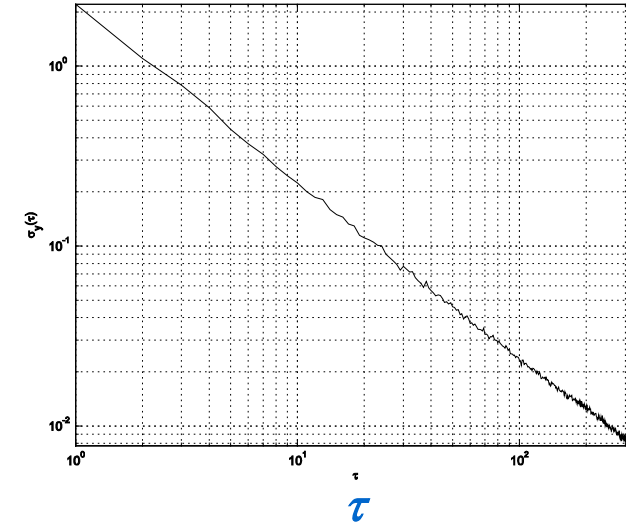
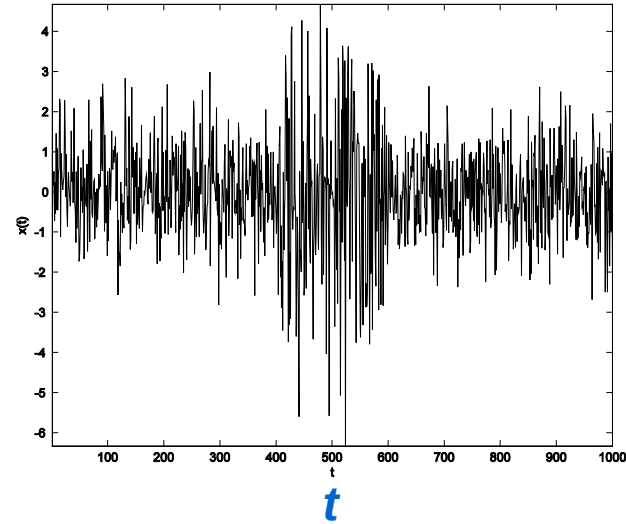
t_1 t_2 ... $\Rightarrow \sigma_y(t, \tau)$

$\sigma_y(t_1, \tau)$ $\sigma_y(t_2, \tau)$

stability may vary with time



Simulation results : Bump



The Dynamic Allan variance

Discrete time formulation from the phase samples $x[n]$

$$\sigma_y^2[n, k] = \frac{1}{2k^2 \tau_0^2} \frac{1}{Nw - 2k} \sum_{m=n-Nw/2+k}^{n+Nw/2-k-1} E \left[(x_N[m+k] - 2x_N[m] + x_N[m-k])^2 \right]$$

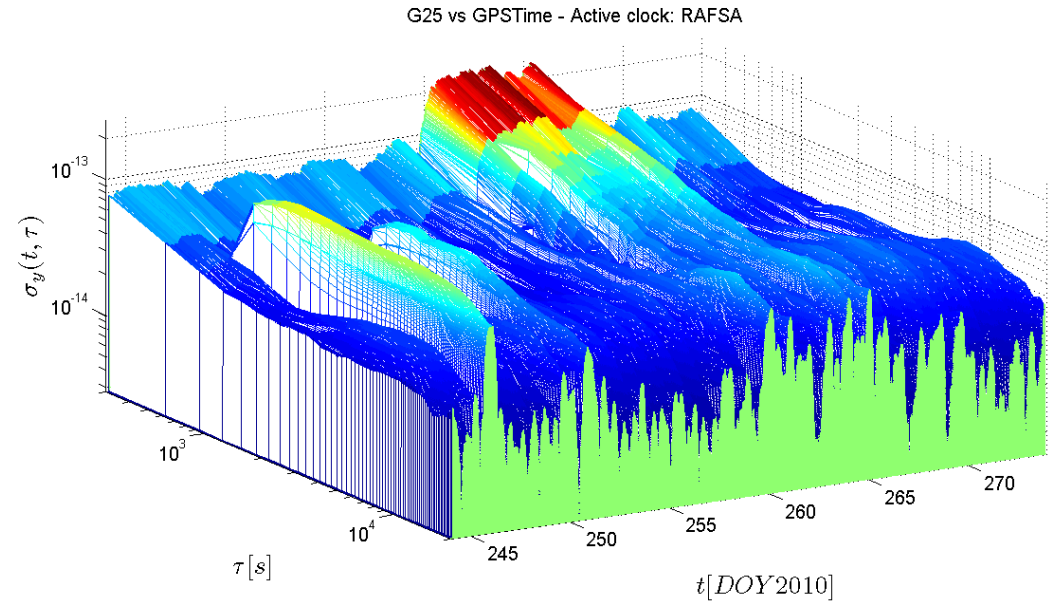
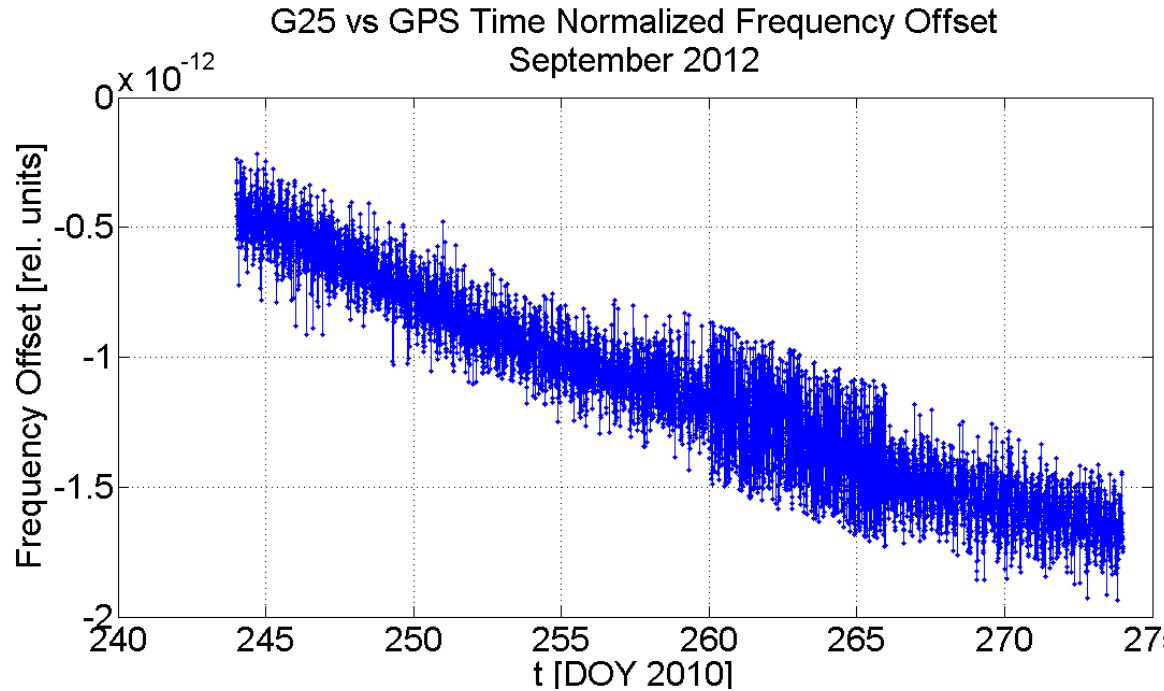
where:

- ▶ Nw is the window length
- ▶ x_N is the phase signal in the window Nw
- ▶ τ_0 is the sampling time

the DAVAR estimator

has no expectation value E because we have one realization only

The change in instability is easily detected



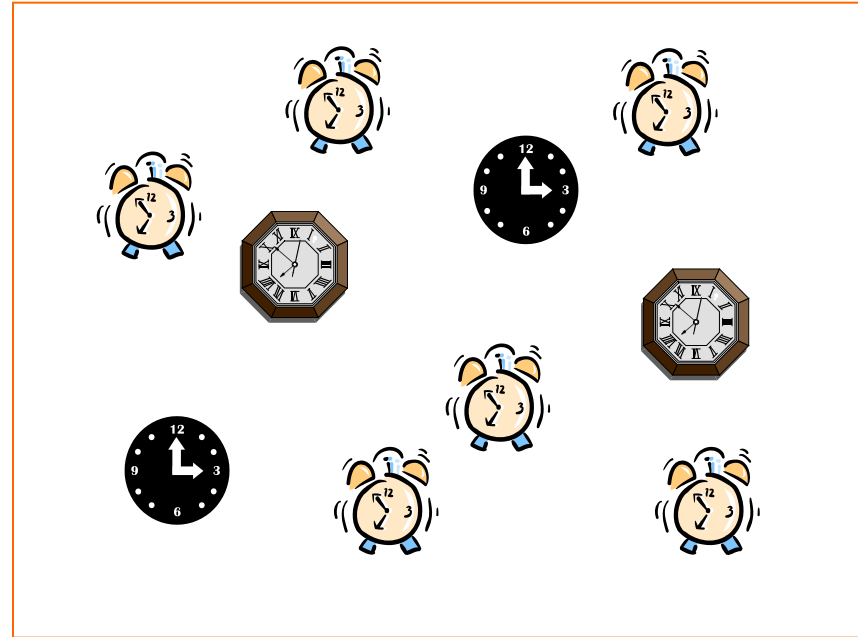
The computation of the DADEV was available in STABLE 32

What we learnt

1. The (in)stability of time varying quantities may be highly impacting
2. Instability may be estimated by appropriate tools mathematical models including noises can be written
3. Models allow estimation, prediction, simulation, control
4. The behaviour may change due to ageing, failures, wearing... These changes are to be detected (rapidly) and the model is be dynamically updated
5. Some «preprocessing» on the data is fundamental to identify non typical behaviours

How can an ensemble time be defined?

Clock ensemble algorithm



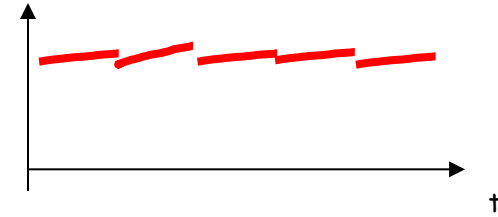
Optimising the contribution of each clock

Ensemble algorithms

Two main steps:

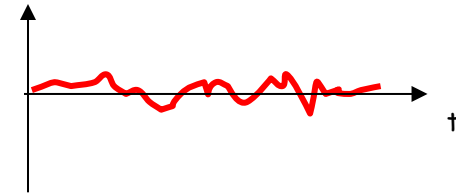
1. Average of clock reading

Stability



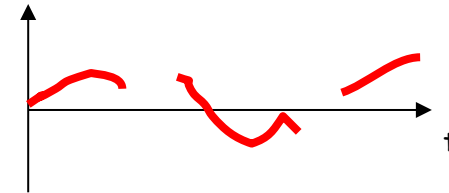
2. Steering

Accuracy

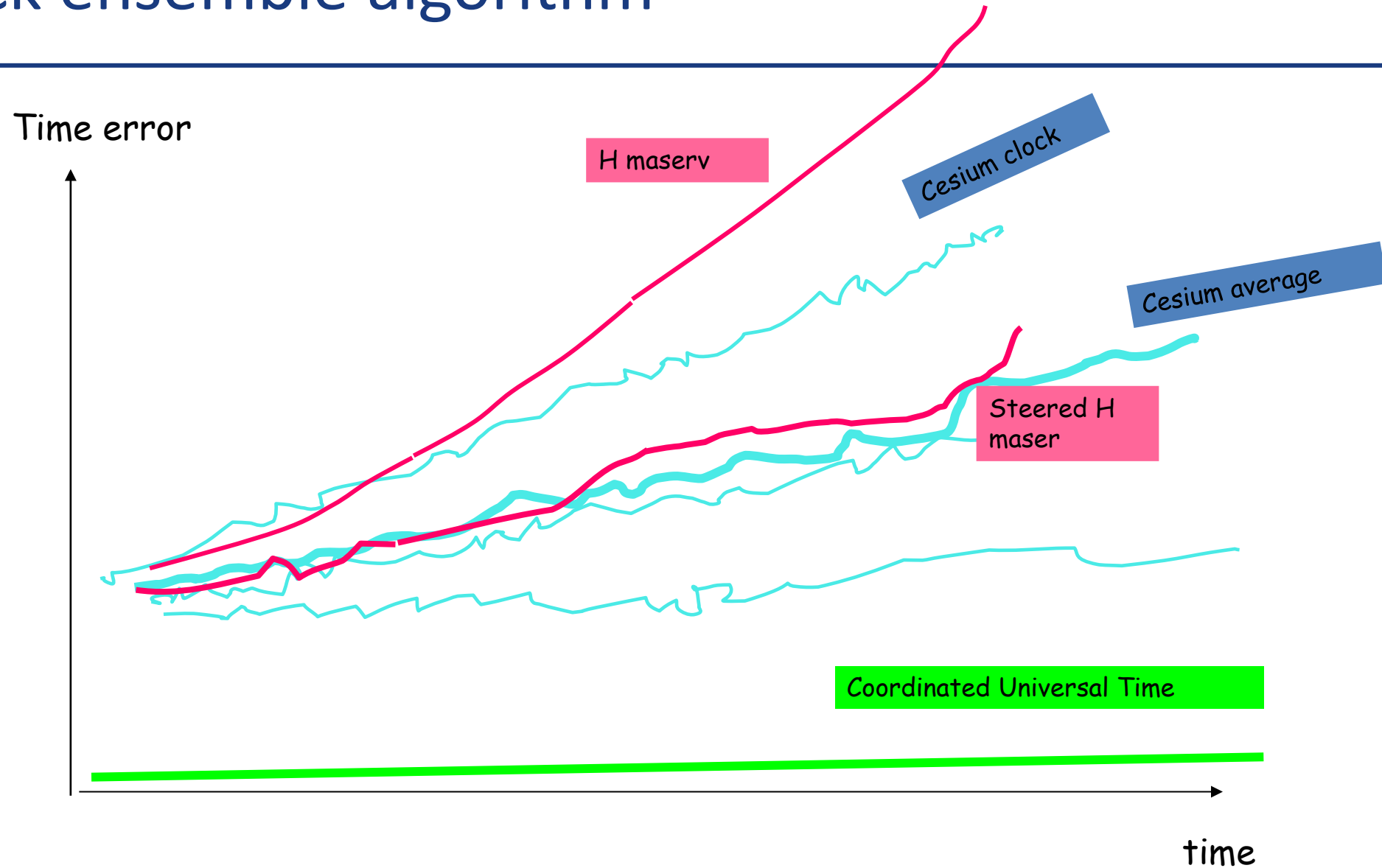


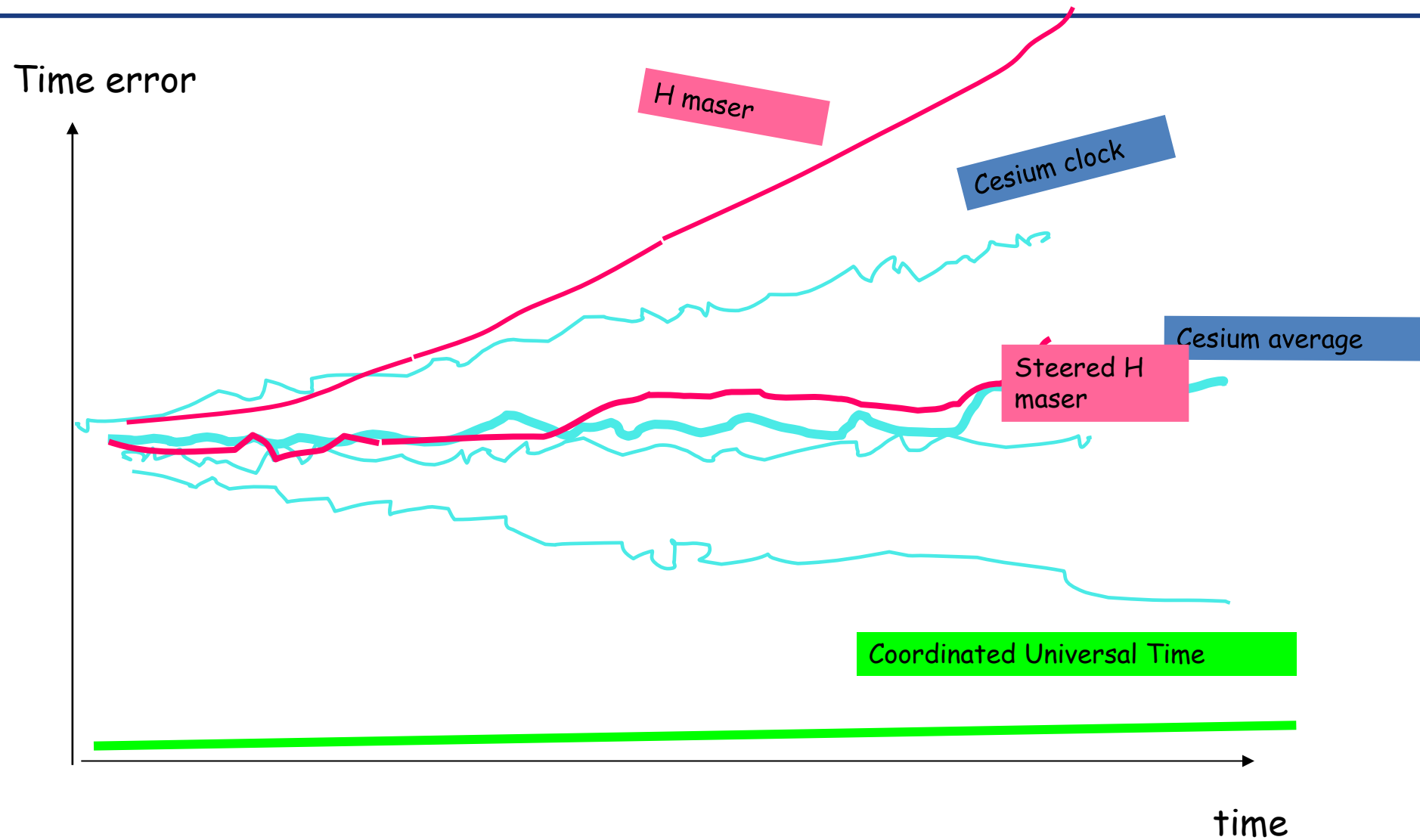
with care to

Reliability



Clock ensemble algorithm





Ensemble algorithms

Weighted average time scale TA

$$TA = \frac{\sum_{i=1}^N p_i [h_i(t)]}{\sum_{i=1}^N p_i}$$

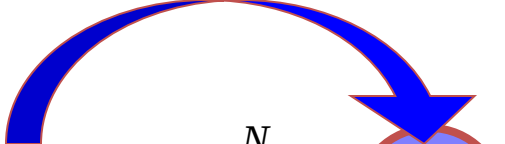
TA = ensemble time

h_i = reading of i -th clock

p_i = weight of i -th clock

$x_{ij} = h_j - h_i$ = measured clock difference

The reading h_i of i -th clock is not observable


$$TA = \frac{\sum_{i=1}^N p_i [h_i(t)]}{\sum_{i=1}^N p_i}$$

Only difference between clock reading are measurable

$$x_{ij} = h_j - h_i$$

The time scale TA is defined

through the difference versus each clock reading

$$x_k = TA - h_k$$

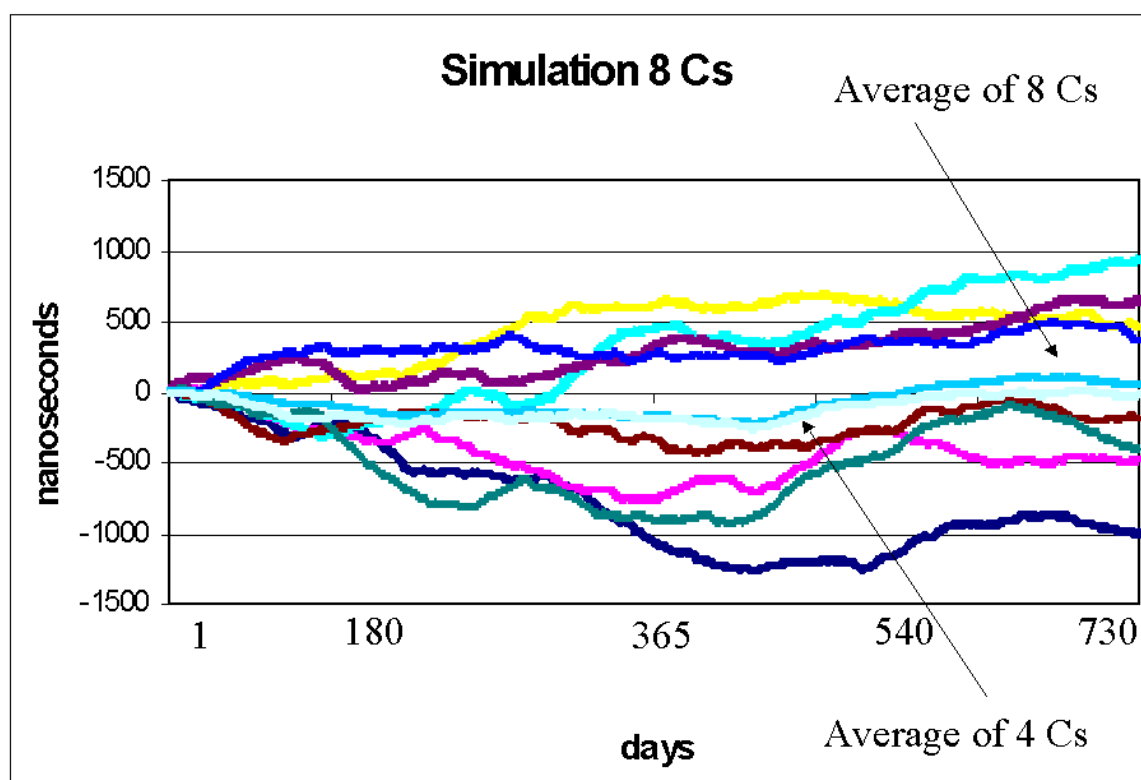
TA = ensemble time

h_i = reading of i -th clock

p_i = weight of i -th clock

$x_{ij} = h_j - h_i$ = measured clock difference

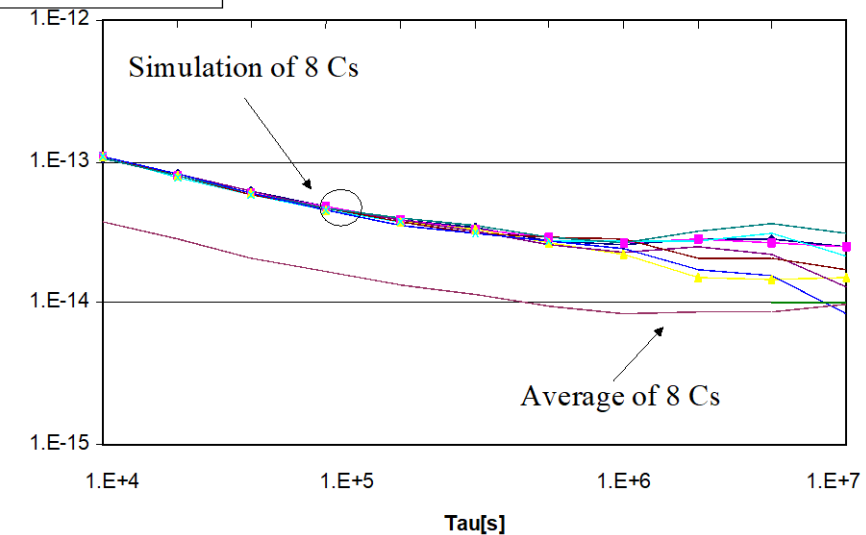
$$TA - h_k = \frac{\sum_{i=1}^N p_i [h_i(t) - h_k(t)]}{\sum_{i=1}^N p_i}$$



Stability improves

$\sigma_y(\tau)$ Frequency stability

The average time scale TA is
in the middle of clock
readings
the weighted sum of distance
between TA and clocks is
zero



Weight

Ensemble Time Scale

$$TA - h_k = \frac{\sum_{i=1}^N p_i [h_i(t) - h_k(t)]}{\sum_{i=1}^N p_i}$$

weight p_i
assignment

←

From the theory of least squares, the instability of TA is minimal if each clock is weighted as inversely proportional to its instability

$$p_i = \frac{1}{\sigma_y^2(\tau)}$$

in case of independent clock

τ

← using complete

TA = ensemble time

h_i = reading of i -th clock

p_i = weight of i -th clock

$x_{ij} = h_j - h_i$ = measured clock difference

$$p_i = \frac{\mathbf{R}^{-1} \cdot \mathbf{u}}{\mathbf{u}^T \cdot \mathbf{R}^{-1} \cdot \mathbf{u}}$$

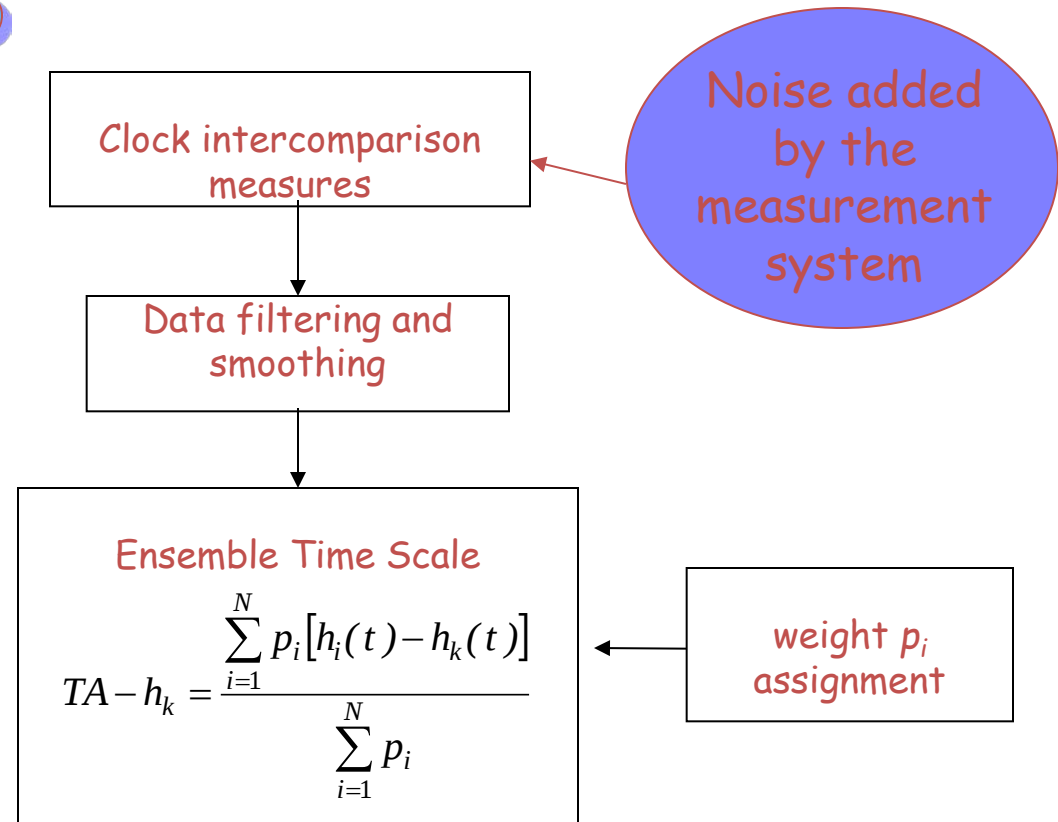
clock covariance matrix $\mathbf{R}$

in case of not independent clock

$\mathbf{u}$ is unit vector

Ensemble algorithms

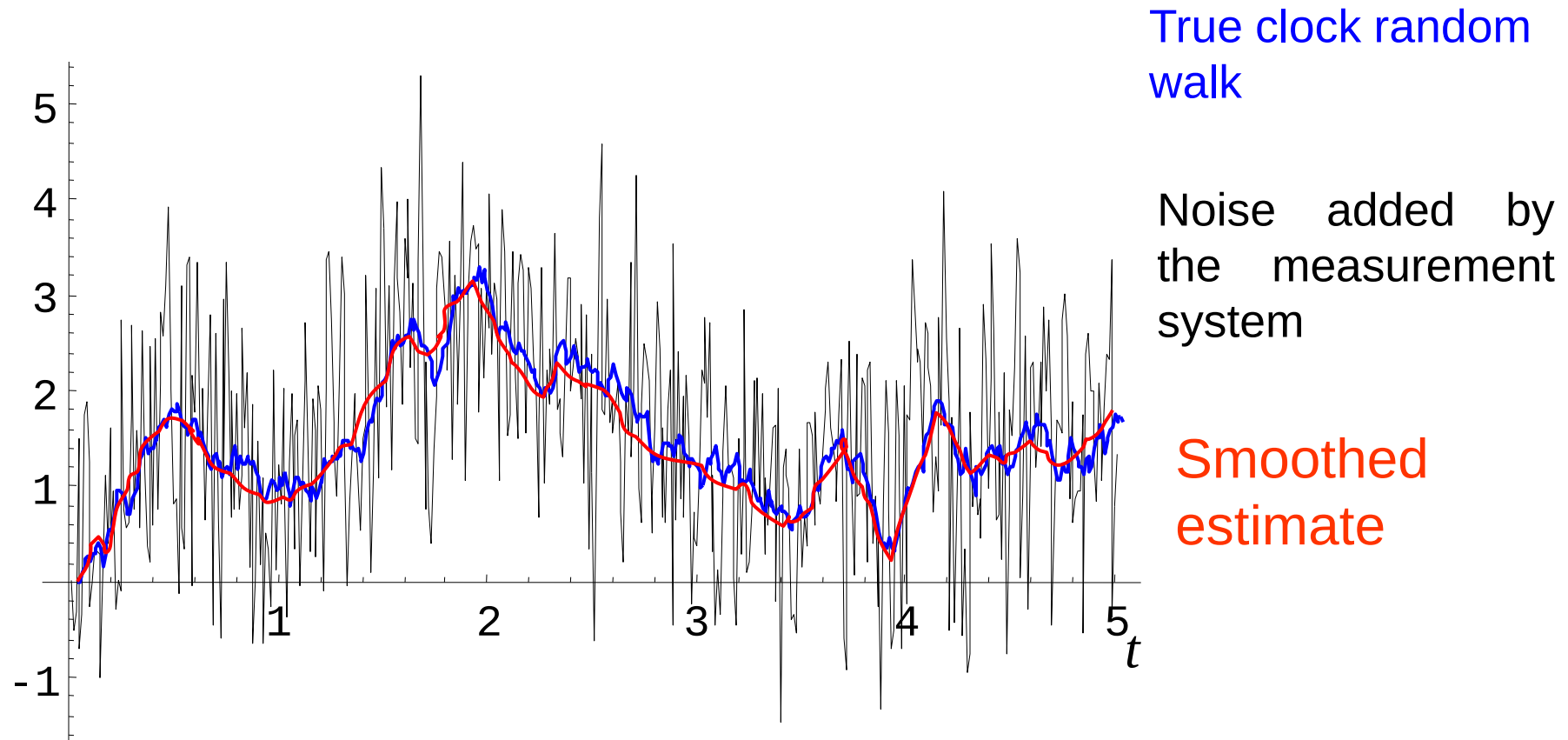
Data filtering and smoothing



Using a clock model and appropriate filtering techniques as

Kalman filter, AutoRegressive, Moving Average,

The noise added by the measurement system needs to be filtered by a smoothing technique



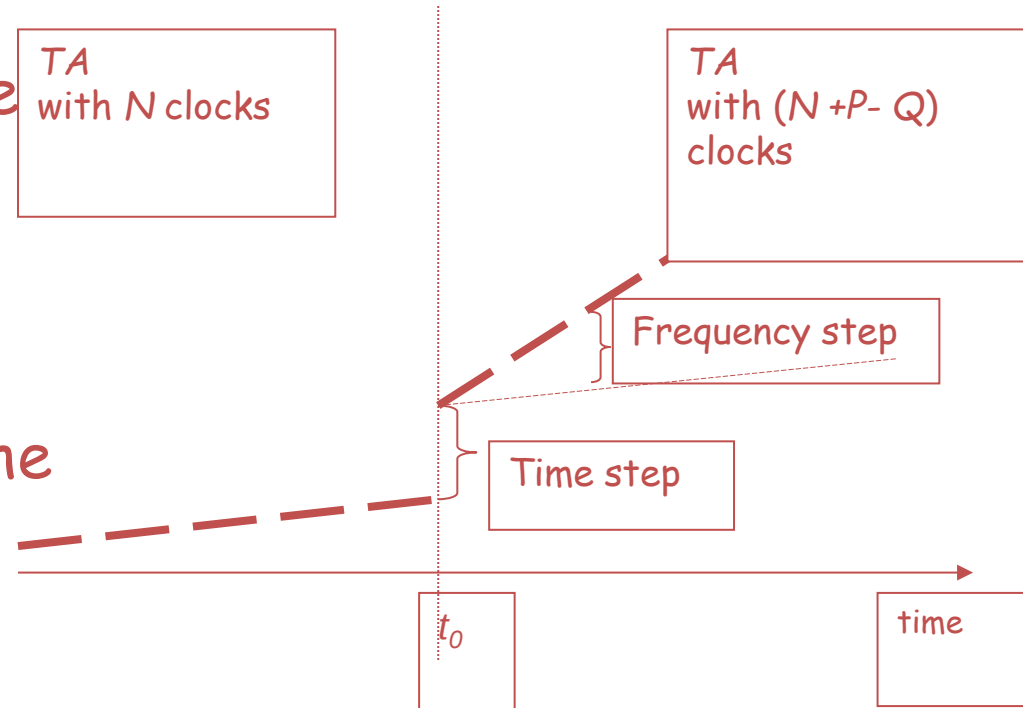
Stability asks for

to changing number of clocks

or weights in the ensemble

insensitivity
insensitivity

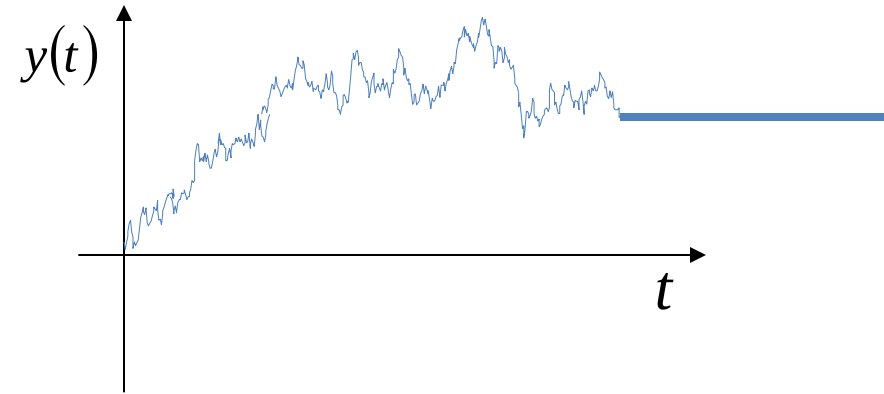
This means subtracting the
predictable behaviour of
the new (changed) clock



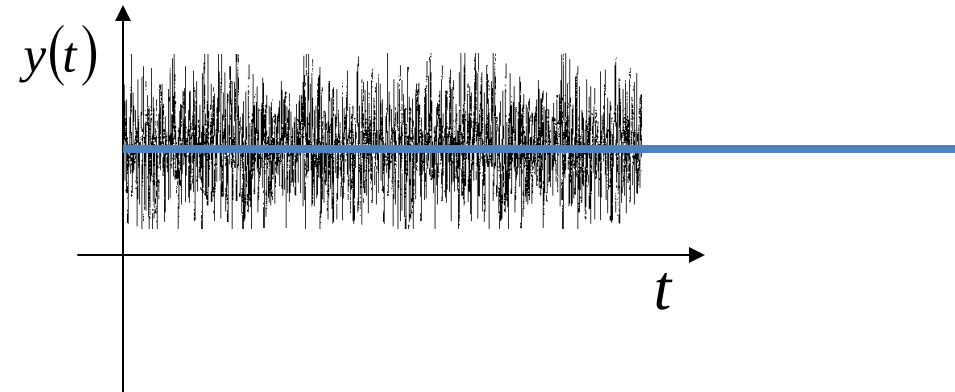
Prediction by means of clock model

Prediction

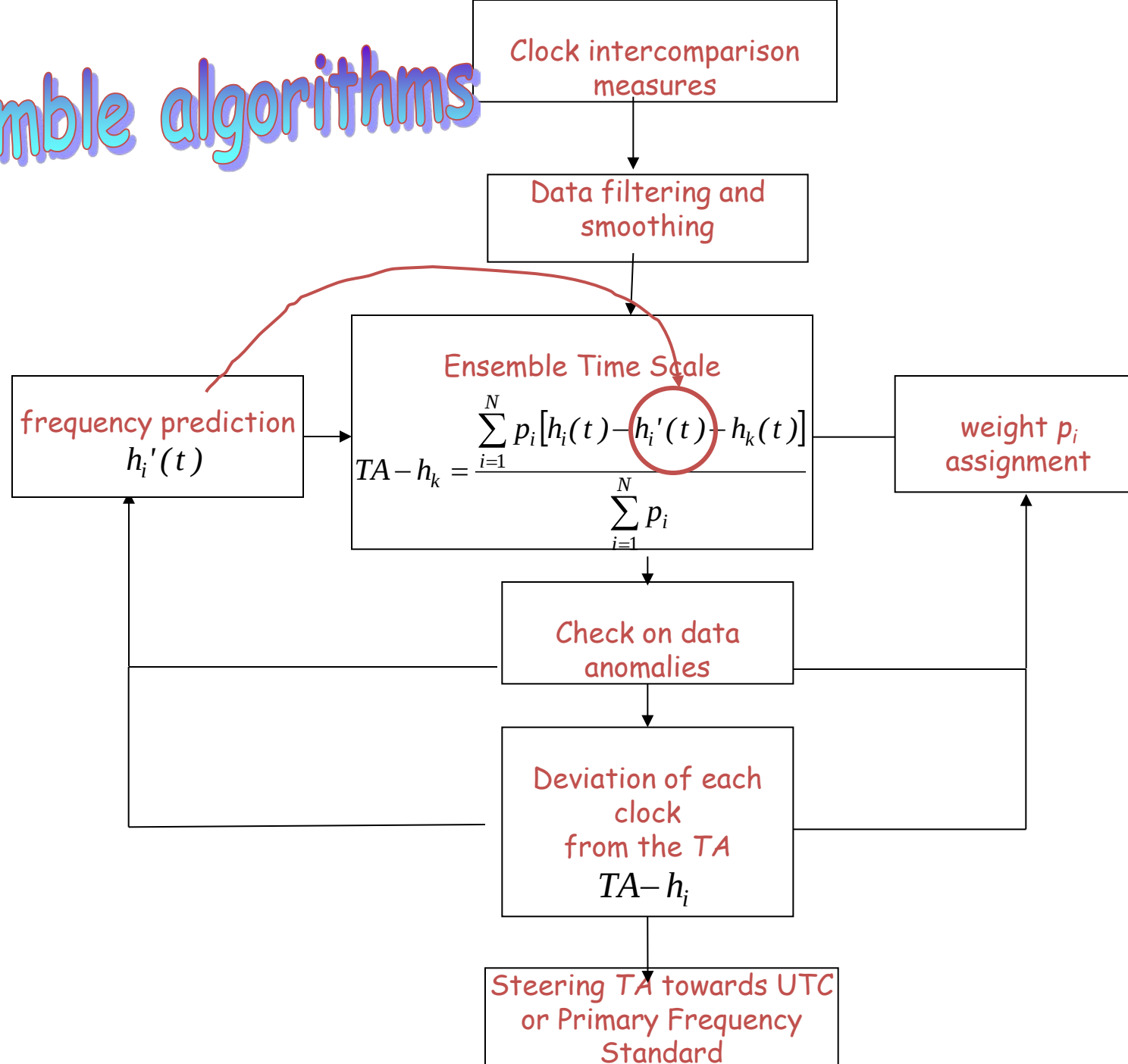
In case of Random Walk,
the best prediction is the
last value



In case of White Noise, the
best prediction is the
average value



Ensemble algorithms



Steering

versus UTC or primary frequency standard

From past measurement
the frequency of the ensemble time is **predicted**
(or the frequency of an individual clock is predicted)

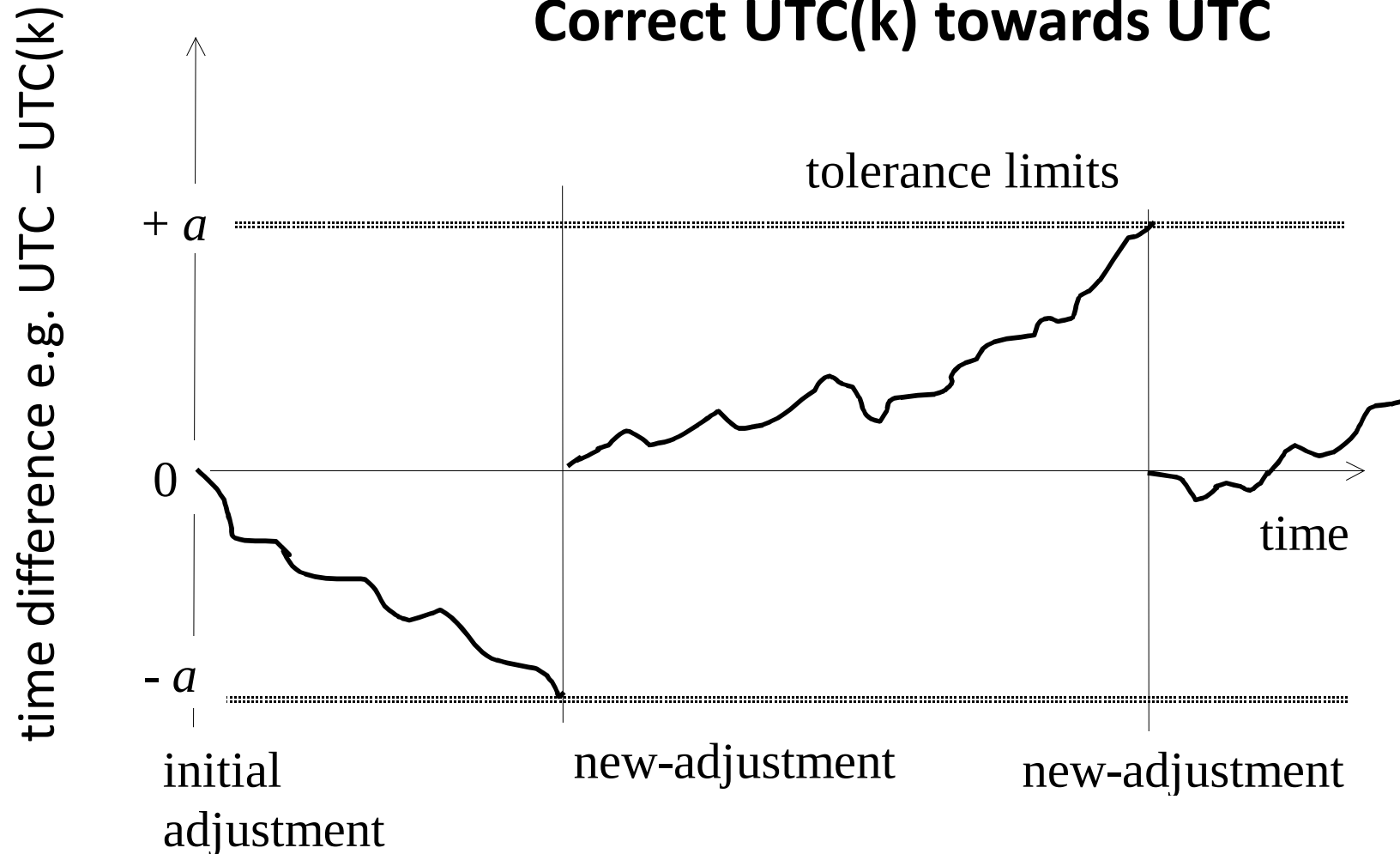
In **real time** the predicted frequency offset is compensated
to generate a **Steered Ensemble Time**
(or to generate a steered individual clock)

Steering

Example of time adjustment

When $|\text{UTC} - \text{UTC}(k)| > a$

Correct $\text{UTC}(k)$ towards UTC

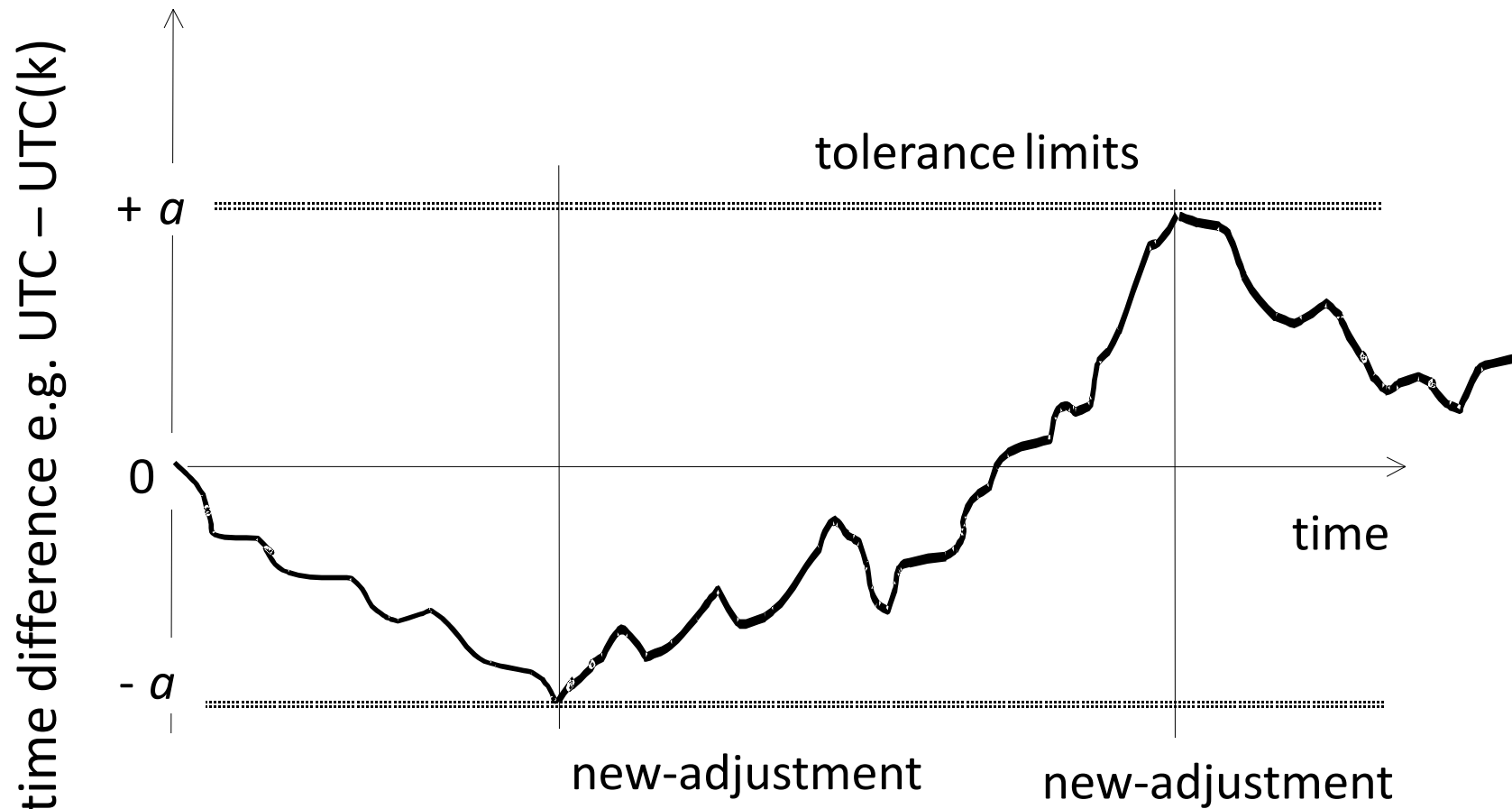


Steering

Example of frequency adjustment

When $|\text{UTC} - \text{UTC}(k)| > a$

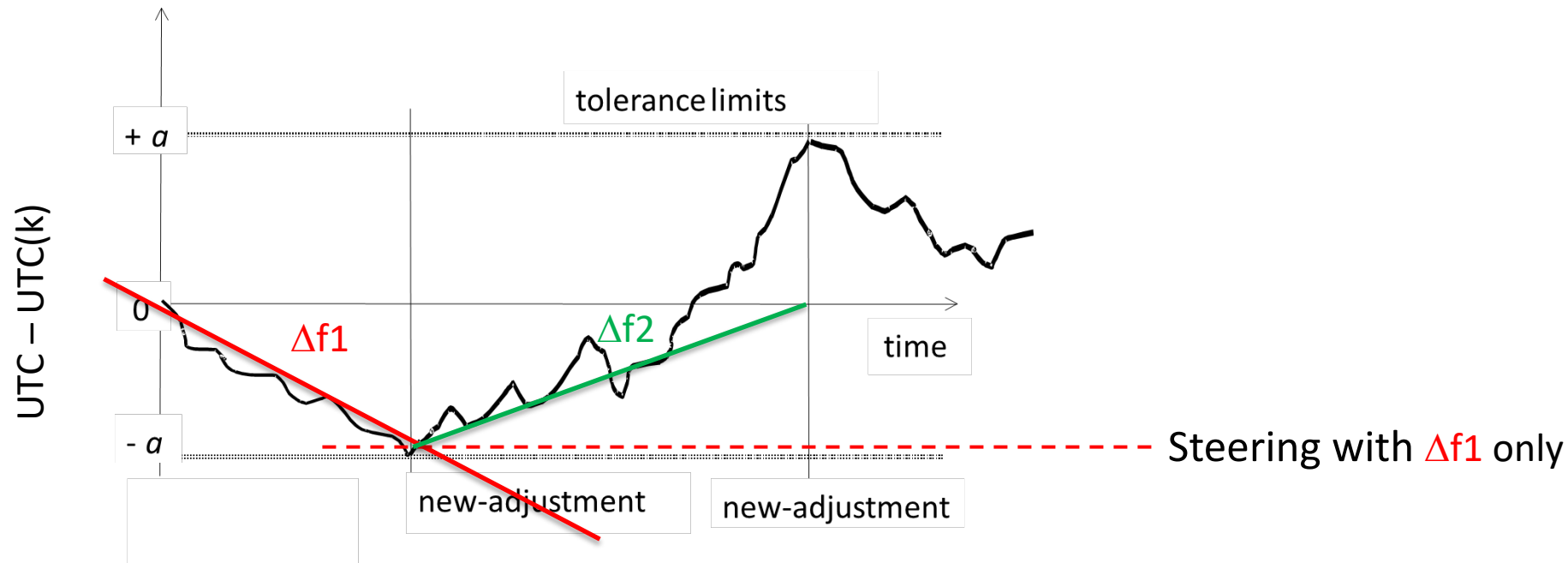
Correct $\text{UTC}(k)$ frequency towards UTC



1. Use last months “UTC – UTC(k)” data
2. estimate the linear slope as $\Delta f1$
3. estimate the frequency correction to reduce UTC-UTC(k) over 30 days

$$\Delta f2 = (\text{UTC} - \text{UTC}(k))_{\text{now}} / 30 \text{ days}$$

4. Apply to the microstepper (or to the master clock) $\Delta f1 + \Delta f2$
5. Repeat again at the next Circular T
6. The reference can be UTC or UTCr or the ensemble time TA or ...

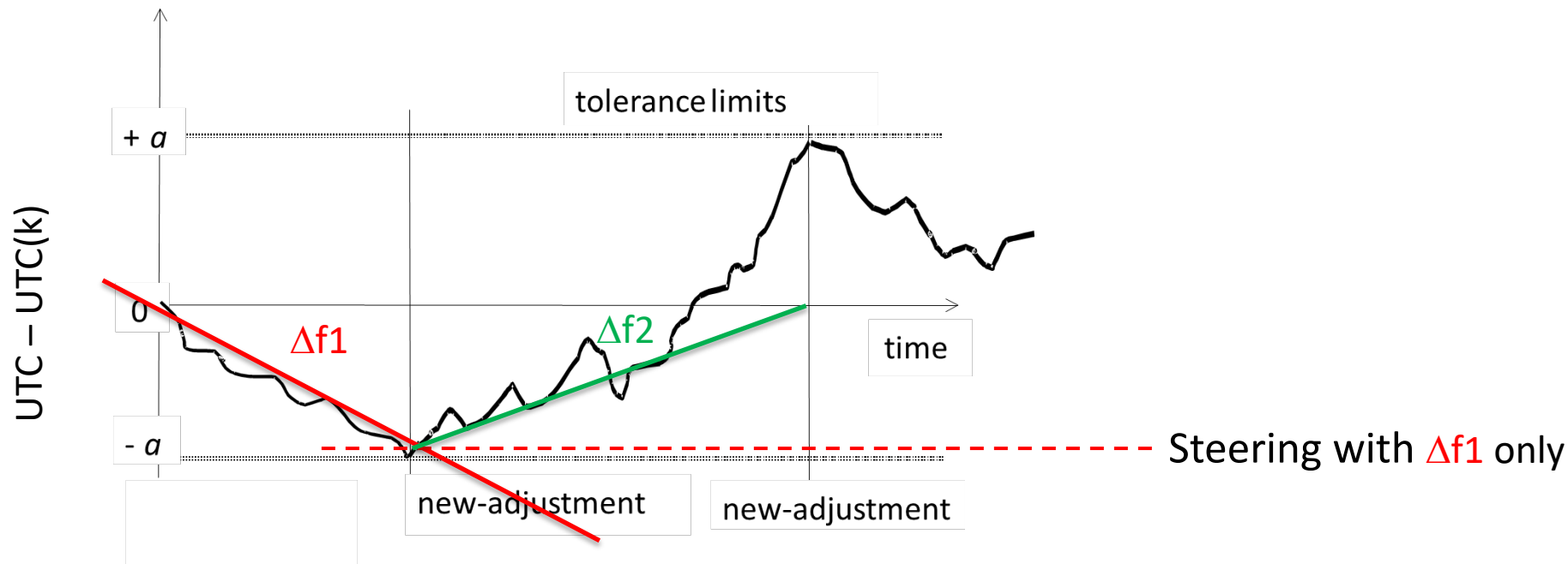


Actually the estimation of $\Delta f1$ should be on the master clock

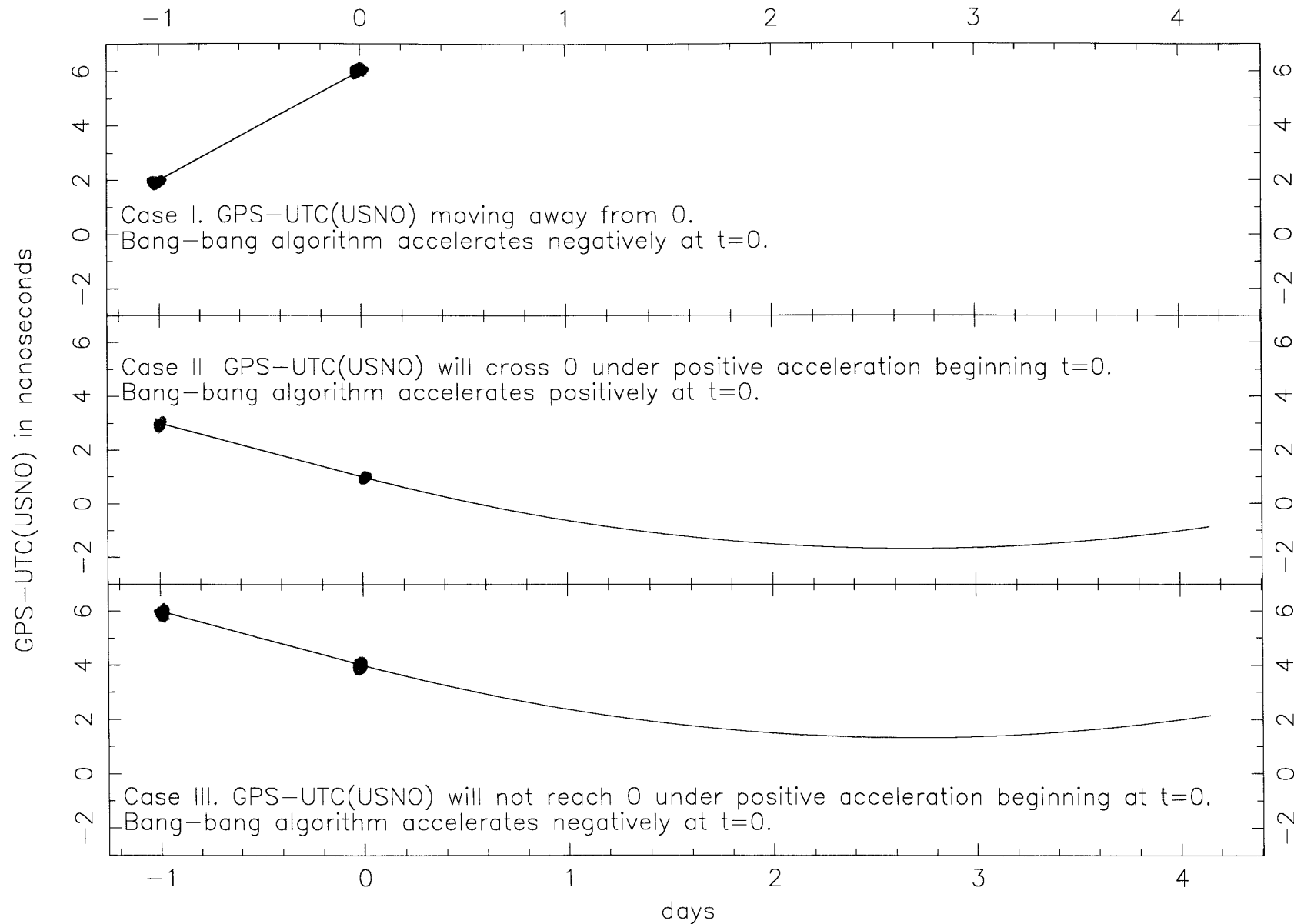
1. Use last months “UTC – UTC(k)” data and “UTC(k) – master clock” measures to get “UTC – Master clock”.
2. estimate the linear slope of master clock as $\Delta f1$
3. estimate the frequency correction to reduce UTC-UTC(k) over 30 days

$$\Delta f2 = (\text{UTC} - \text{UTC}(k))_{\text{now}} / 30 \text{ days}$$

4. Apply to the microstepper (or to the master clock) $\Delta f1 + \Delta f2$



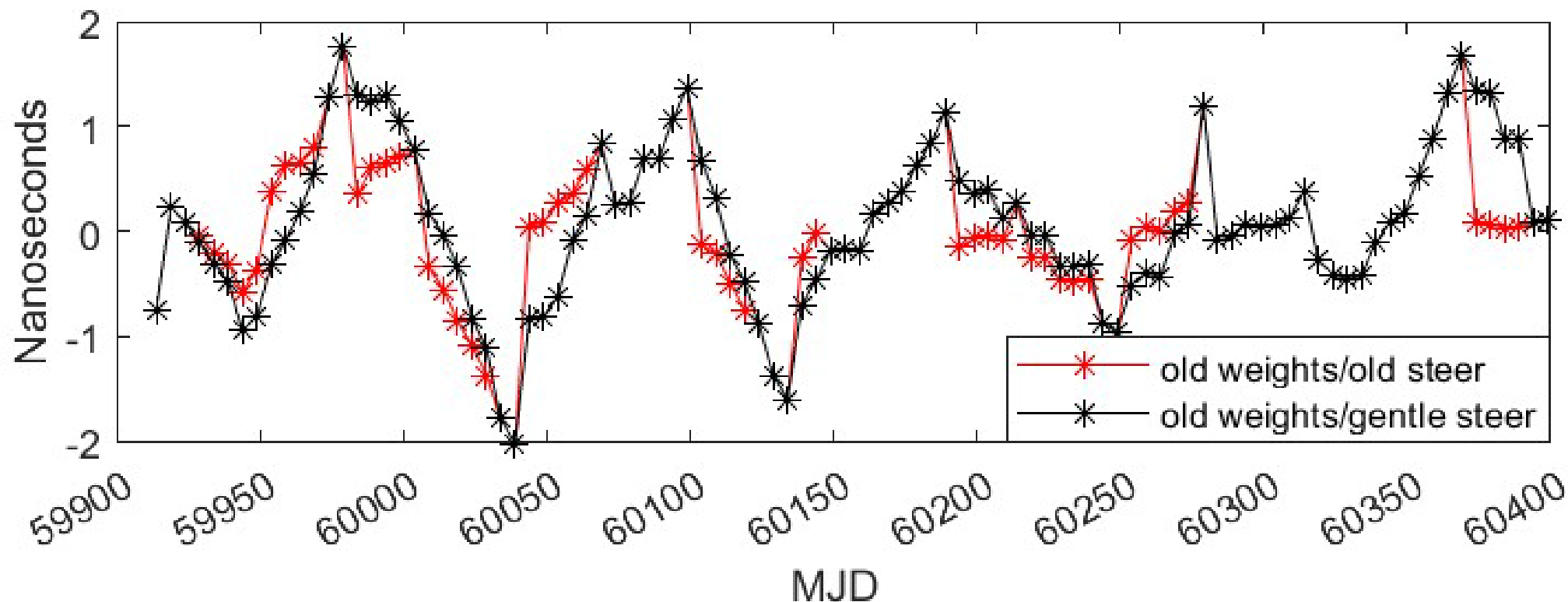
GPS “Bang-bang” steering the freq drift



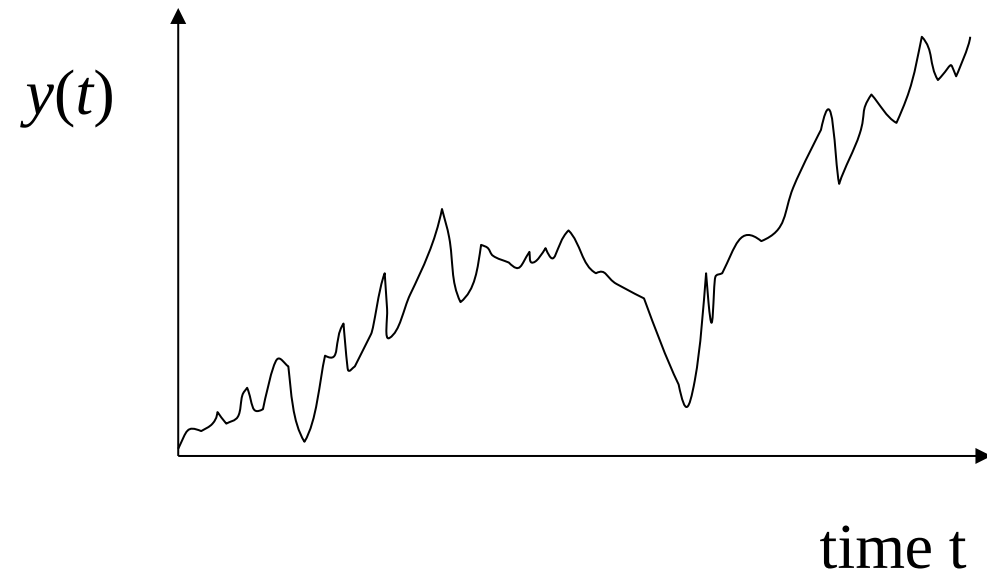
Steering

Example of the new steering algorithm of Rapid UTC

from time step to frequency steering



Is there a dynamic technique to estimate a time varying quantity?



KALMAN filter:

Recursive version of least squares

Optimal linear estimation of a dynamical system with Gaussian noise

By Kalman - Bucy 1961

Kalman estimation needs:

1) a mathematical model of the evolution in time of the physical system + Gaussian noise

2) recursive measurement + Gaussian noise

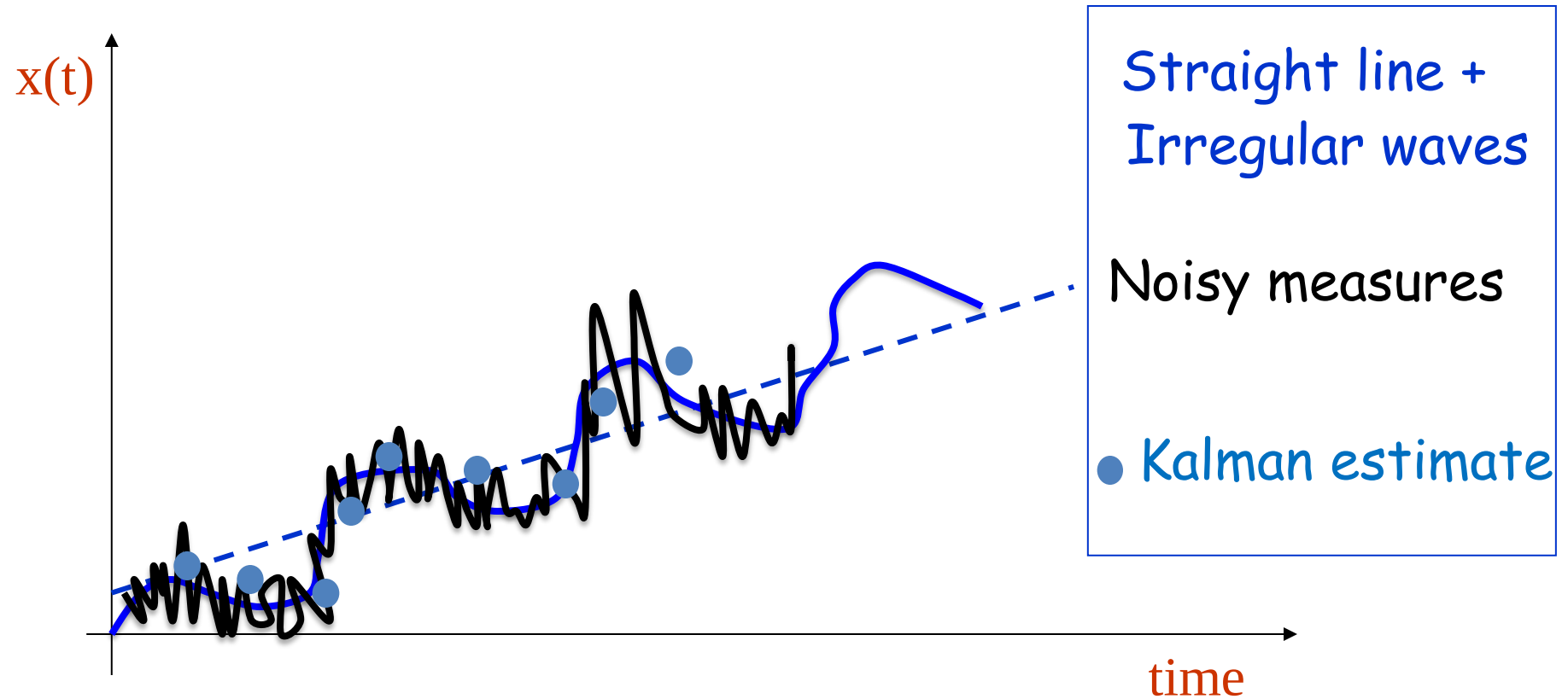
Gaussian
noise:

$N(0, \Sigma)$

Example: ship position

Constant speed + waves $x(t) = v t + \xi(t)$

Measures: $z(t) = x(t) + \varepsilon(t)$



The Kalman filter mixes two information

1. from past measures and estimates at t_{k-1} ,

knowing the dynamics of the system

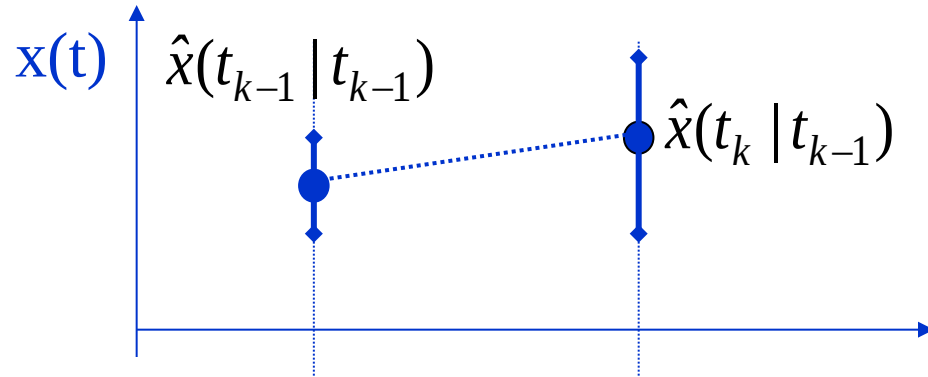


A prediction of the system at time t_k
is possible $\hat{x}(t_k | t_{k-1})$

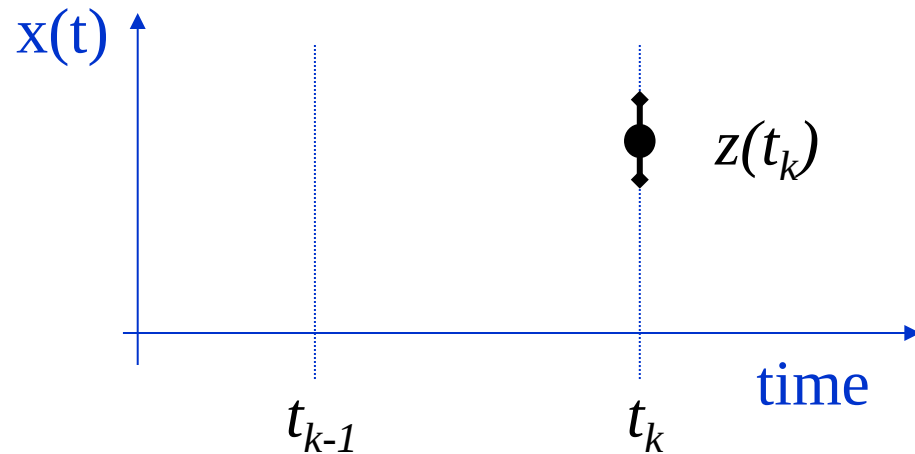
2. from the measurement system



A measure of the system at time t_k
is available $z(t_k)$

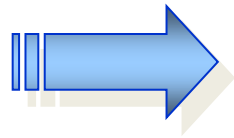


The prediction has an uncertainty due to the random disturbances and to the uncertainty of the previous estimate



The measure has noise and thus uncertainty

$$z(t) = x(t) + \varepsilon(t)$$



The two information are combined in a weighted average to obtain the best estimate at t_k

Estimate:

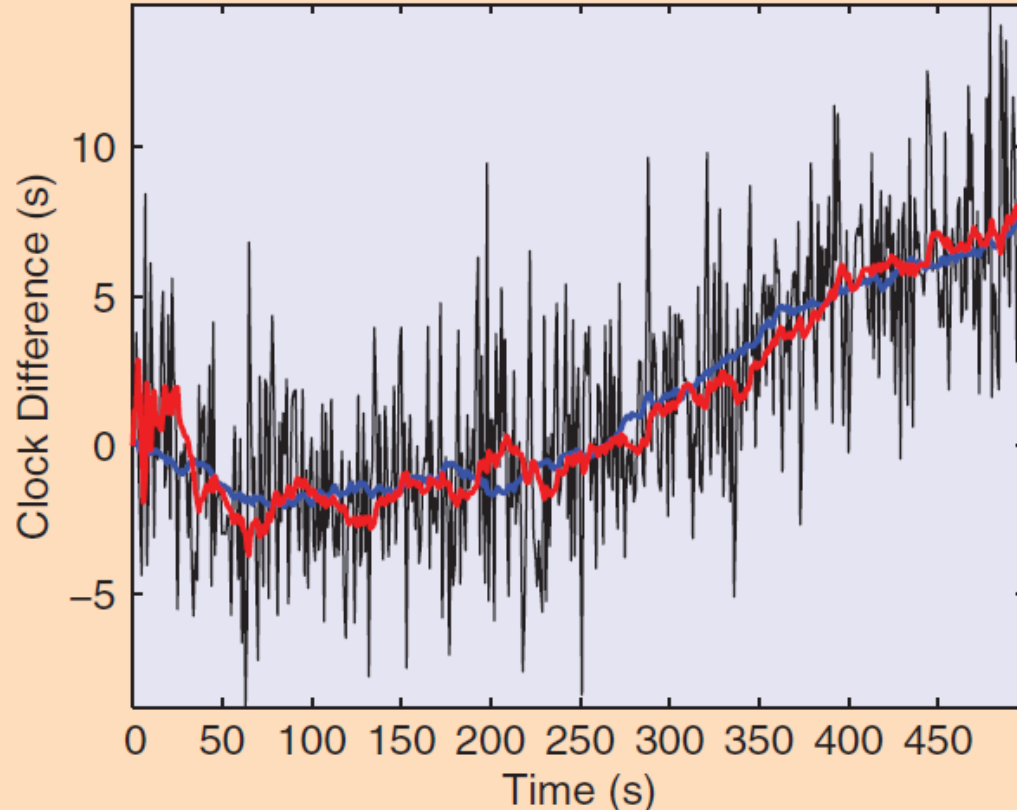
$$\hat{x}(t_k | t_k) = \text{weight}_{\text{prediction}} \hat{x}(t_k | t_{k-1}) + \text{weight}_{\text{measure}} z(t_k)$$

Uncertainty:

$\Sigma_{\hat{x}}(t_k | t_k)$ = combination of the prediction and
measure uncertainty

Application to clock difference estimation:

e.g. $\text{UTC}(i) - \text{UTC}(j)$



- the clock difference is observable,
- the measures have an added noise
- estimate of the Kalman filter

Additionally:

Multidimensional problem to be treated with matrices,

optimal parameter tuning,

the model has to be correct and stationary

No outliers, no jumps,...

What we learnt

1. Don't be scared by the mathematics. After some dedication, things become easier...
2. The quality of the clock, or of clock comparison method, can be estimated, monitored, and analysed. This can help in getting better data, better UTC(k), and better UTC
3. Next talks will give more details, and will explain how some available software tools can be used
4. The BIPM with the secondes is developing tools to help the application of these analysis techniques, please try!
5. Let us know what you would like to see in the next CBKT courses

Thank you for listening till the end!

patrizia.tavella@bipm.org

