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The evaluation of uncertainties in $[UTC - UTC(k)]$

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Abstract
This work presents a study of the determination of uncertainties in $[UTC - UTC(k)]$ needed for publication in the Bureau Inernational des Poids et Mesures's (BIPM's) *Circular T* and the Key Comparison Database, as required by the Mutual Recognition Arrangement. In the first part of the paper, an analytical solution based on the law of the propagation of uncertainty is derived. In the second part, the solution is verified numerically using the software used by the BIPM for the generation of UTC.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Coordinated Universal Time (UTC), the worldwide time standard, is disseminated monthly through publication of $[UTC - UTC(k)]$ in *Circular T* by the Bureau Inernational des Poids et Mesures (BIPM). UTC(k) is a realization of UTC by laboratory k. These values before 2005 have been published without their uncertainties, but the rules of the CIPM Mutual Recognition Arrangement (CIPM MRA) require the evaluation of this uncertainty. This paper reports the first steps towards computing these uncertainties.

UTC is derived from International Atomic Time (TAI) by the addition of leap seconds; while TAI is derived from the Free Atomic Timescale (EAL) through the addition of pre-announced frequency steers determined by comparison with a weighted set of primary frequency standards. EAL is a worldwide weighted average of a large number of free-running, effectively uncalibrated, frequency standards [1–3]. The uncertainty in the determination of EAL, TAI and UTC, as steps in the realization of Terrestrial Time (TT), is affected by three major elements: clock variations, the means of comparisons of remote clocks (time transfer) and the time-scale algorithm. The uncertainties of time transfer are particularly significant over averaging times of up to a few tens of days and also influence the uncertainty in the access of the participating laboratories k to UTC (in other words, the uncertainty of $[UTC - UTC(k)]$).

In this work, a study of the determination of the uncertainties of $[UTC - UTC(k)]$ is presented. In the first part of the paper, an analytical solution based on the law of the propagation of uncertainty is derived. The second part presents a numerical verification of the analytical results using the software used for the generation of UTC.

2. The algorithms of EAL, TAI and UTC and the working hypothesis

The uncertainty of $[UTC - UTC(k)]$ can be derived using the general equation of the Free Atomic Timescale (EAL), which is defined using the ALGOS algorithm [1–3] as

$$EAL(t) = \sum_{i=1}^{N} w_i[h_i(t) + h'_i(t)], \quad (1)$$

where N is the number of the atomic clocks, w_i the relative weight of the clock H_i, $h_i(t)$ is the reading of clock H_i at time t and $h'_i(t)$ is the prediction of the reading of clock H_i to guarantee the continuity of the time scale. The weight attributed to a given clock reflects its long-term stability, since the objective is to obtain a weighted average that is more stable in the long term than any of the contributing elements [4, 5].

The weights of the clocks obey the relation

$$\sum_{i=1}^{N} w_i = 1. \quad (2)$$

Subtracting the same quantity from both sides of equation (1),

$$EAL(t) - \sum_{i=1}^{N} w_i h_i(t) = \sum_{i=1}^{N} w_i[h_i(t) + h_i'(t)] - \sum_{i=1}^{N} w_i h_i(t).$$

Using equation (2) and rearranging,

$$\sum_{i=1}^{N} w_i(EAL(t) - h_i(t)) = \sum_{i=1}^{N} w_i h_i'(t). \quad (3)$$

Setting

$$x_i(t) = EAL(t) - h_i(t), \quad (4)$$

it is clear that equation (3) is of the form

$$\sum_{i=1}^{N} w_i x_i(t) = \sum_{i=1}^{N} w_i h_i'(t). \quad (5)$$

The software package termed ALGOS is used in the Time Section of the BIPM to generate UTC. Weights are determined from the variance of monthly average frequencies, subject to a maximum value [5]. The data used by ALGOS take the form of the time differences between readings of clocks, written as

$$x_{i,j}(t) = h_j(t) - h_i(t). \quad (6)$$

Equation (5) in conjunction with the $N-1$ equations (6) results in a system with N equations and N unknowns:

$$\begin{cases} \sum_{i=1}^{N} w_i x_i(t) = \sum_{i=1}^{N} w_i h_i'(t) \\ x_i(t) - x_j(t) = x_{i,j}(t) \end{cases}. \quad (7)$$

The solution is

$$x_j(t) = EAL(t) - h_j(t) = \sum_{i=1}^{N} w_i[h_i'(t) - x_{i,j}(t)]. \quad (8)$$

The difference between any clock H_j and EAL (8) depends on weights, clock prediction and measured clock differences. The clock H_j may also represent a UTC(j) time scale; therefore, $x_j(t)$ may also be interpreted as

$$x_j = EAL - UTC(j)$$

having dropped the time instant t in the notation for simplicity.

The predictions and the weights are fixed by appropriate algorithms based on past clock behaviour, and, in equation (8), they can be considered as time-varying deterministic parameters. Suboptimal estimation of these parameters would affect the uncertainty of TAI as a realization of the TT, but they do not affect the knowledge of the difference between EAL and clock H_j; the measures $x_{i,j}$ are thus the only contributors to the uncertainties in x_j. The contribution of the uncertainty given by measures of clocks located inside the same laboratory is considered negligible, and in terms of the ALGOS algorithm it can be ignored because it is inseparable from the noise of the clock.

Because $[UTC - EAL]$ depends only on pre-determined leap seconds and frequency steers that do not add uncertainty, and because UTC(k) is deterministically derived from the laboratory clock readings, the uncertainties of $[UTC - UTC(k)]$ are identical to the uncertainties of $[TAI - UTC(k)]$, $[EAL - UTC(k)]$ and $[EAL - H_k]$. Thus, the uncertainties of the links between laboratories are the only source of the uncertainty of $[UTC - UTC(k)]$. In the following sections its propagation will be studied.

2.1. The law of propagation of uncertainty

According to [6], the uncertainty in the $x_j(t)$ can be found using the law of the propagation of uncertainty. Defining y as a generic quantity indirectly measured by means of direct measurements of the input quantity x_i,

$$y = f(x_1, x_2, ..., x_M).$$

The expression of the law of the propagation of uncertainty [6] is given by

$$u_y^2 = \sum_{i=1}^{M} \left(\frac{\partial f}{\partial x_i}\right)^2 u_{x_i}^2 + 2\sum_{i=1}^{M-1}\sum_{k=i+1}^{M} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_k} u_{(x_i,x_k)}, \quad (9)$$

where the first term corresponds to the effect of the uncertainties on the input quantities x_i and the second term accounts for the correlation between them.

The uncertainty of the quantity $x_k = [EAL - UTC(k)]$ defined in equation (8) plays here the role of the indirect quantity y and the uncertainty contributions are only due to the measurement noise of the links $x_{i,j}(t)$. Applying equation (9) to the model (8) yields

$$\begin{aligned} u_{x_j}^2 &= \sum_{i=1}^{N} \left(\frac{\partial x_j}{\partial x_{i,j}}\right)^2 u_{x_{i,j}}^2 + 2\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} \frac{\partial x_j}{\partial x_{i,j}}\frac{\partial x_j}{\partial x_{k,j}} u_{(x_{i,j},x_{k,j})} \\ &= \sum_{i=1}^{N} w_i^2 u_{x_{i,j}}^2 + 2\sum_{i=1}^{N-1}\sum_{k=i+1}^{N} w_i w_k u_{(x_{i,j},x_{k,j})}, \end{aligned} \quad (10)$$

where $u_{x_j}^2 = u_{EAL-h_j}^2$.

The weights of the clocks are available from the BIPM website, and the uncertainties of links between the clocks [7] are published in *Circular T* (see the BIPM Time Section's FTP server at http://www.bipm.org/jsp/en/TimeFtp.jsp).

The propagation of uncertainty (10) could as well be expressed by a matrix formulation using a multivariate weighted approach [8, 9]; the scalar approach presented in this paper allows one to clearly understand how the different components contribute to the combined uncertainty.

It can be demonstrated that the ALGOS algorithm would generate the same results if each laboratory's clocks were replaced by a single 'equivalent' clock whose reading was the weighted average of the individual clock readings and whose weight in EAL was the sum of the individual clock weights. Therefore, the computations can be simplified by summing the weights of the clocks at each laboratory as follows:

$$W_{Lab_k} = \sum_{i=1}^{N_{Lab_k}} w_i, \quad (11)$$

where N_{Lab_k} is the number of clocks of the considered laboratory k and all the clocks at a laboratory are treated as one equivalent clock whose reading is the weighted average of individual clock readings. The formalism of equation (10) could still be applied without these simplifying assumptions, as the double summation would account for the 100% correlation of the time transfer noise between clocks in the same lab.

2.2. Extension of the computation

It is possible but not necessary to apply equation (10) to every laboratory. If the uncertainty for any one clock is known, for example, $x_j = EAL - h_j$, the evaluation of uncertainty on $x_i = EAL - h_i$ may be obtained by using the second equation in (7) and applying the property of the variance to obtain

$$u_{x_i}^2 = u_{x_j}^2 + u_{x_{i,j}}^2 + 2u_{x_j,x_{i,j}}. \tag{12}$$

The last term is the covariance of the measures $x_{i,j}$ and the quantity $x_j = EAL - h_j$. Since all the clocks are included in EAL, the measure $x_{i,j}$ will produce a nonzero covariance term by coupling to the same measures $x_{i,j}$, which enter into the EAL definition as many times as there are clocks inside or behind the laboratory i, obtaining

$$\begin{aligned} u_{x_j,x_{i,j}} &= u_{(\sum_{\ell=1}^{N} w_\ell (h'_\ell - x_{\ell,j}), x_{i,j})} = -u_{(\sum_{\ell=1}^{N} w_\ell x_{\ell,j}, x_{i,j})} \\ &= -\sum_{\ell=1}^{N_{eqlab_i}} w_\ell u_{x_{i,j}}^2 = -W_{eq_i} u_{x_{i,j}}^2 \end{aligned} \tag{13}$$

where N_{eqlab_i} is the equivalent number of clocks in laboratory i, also including the clock external to that laboratory, but which are connected to UTC through laboratory i. The equivalent weight W_{eq_i} can be defined as the sum of the N_{eqlab_i} clock weights. This will be explained further in the next section, where examples are given. In this case, equation (12) becomes

$$u_{x_i}^2 = u_{x_j}^2 + u_{x_{i,j}}^2 - 2u_{x_{i,j}}^2 W_{eq_i}. \tag{14}$$

Equation (14) gives the uncertainty of each clock H_i, given the uncertainty of any clock H_j and the uncertainties of the chains of measures linking clock H_i to clock H_j.

2.3. Correlations and anticorrelations

The BIPM provides the link uncertainties in *Circular T* (section 6), but every time a link $x_{i,j}$ appears in a multiple link such as $x_{i,k} = x_{i,j} + x_{j,k}$, it is necessary to estimate the correlations with respect to $x_{i,j}$, and this is the meaning of the last covariance term in equation (10).

A more refined evaluation should take into account the different correlation properties of the several links, including TWSTFT, GPS/Glonass common view (CV) and Melting Pot (also termed All-in-View) [10, 11]. For example, in some cases the uncertainty is dominated by calibration uncertainties between laboratory i and laboratory j, while in other cases the link uncertainty reported in BIPM *Circular T* is mostly due to the noise of the receivers at both sites.

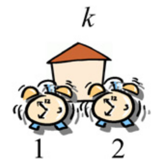
Correlations will always occur in situations where the same receiver or system is used to link two different external laboratories. The analysis of these effects requires more details than are readily available for correlation of the links. Further study is in progress and a preliminary evaluation indicates that the final uncertainty would change to a small extent. Some details are reported in the last section devoted to future work.

In this work, the noise affecting different nonoverlapping links is assumed to be uncorrelated. Correlations can and will appear when the same intermediate link appears in multiple links, as will be shown in the example below.

2.4. Examples of application of the method

Some examples are provided in order to illustrate the applied procedure. In each example, the uncertainty of each clock H_k with respect to UTC increases as the noise of the laboratory k link increases, but also decreases as the weight of the clock increases.

Case A1. Let us consider the simple case of only two clocks (labelled 1 and 2) forming UTC and maintained in the same laboratory k. External links are not needed. The case is depicted in the sketch below.



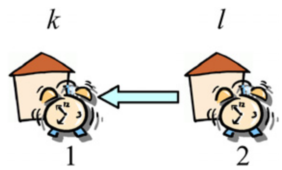
The ensemble time UTC would be computed with these two clocks inside the laboratory k. Suppose clock H_1 realizes UTC(k). The uncertainty of $x_k = [UTC - UTC(k)]$, by means of equation (10), would be

$$u_{[UTC-UTC(k)]}^2 = u_{[UTC-h_1]}^2 = w_2^2 u_{x_{1,2}}^2. \tag{15}$$

The uncertainty of the measure $x_{1,2}$ is negligible because the clocks are within the same laboratory. Thus

$$u_{[UTC-UTC(k)]}^2 \approx 0. \tag{16}$$

Case A2. Consider two clocks (labelled 1 and 2), but maintained in two different laboratories (k and l, respectively) connected by one measurement link $x_{k,l}$.

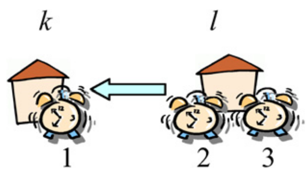


In this case, the uncertainty of the measurement link would be non-negligible and the final uncertainty for UTC(k), realized in laboratory k with clock H_1, would be

$$u_{[UTC-UTC(k)]}^2 = w_2^2 u_{x_{l,k}}^2 = (1 - w_1)^2 u_{x_{l,k}}^2. \tag{17}$$

Equation (17) shows that increasing the weight of clock H_1 from laboratory k decreases the uncertainty of $[UTC - UTC(k)]$.

Case B1. Consider three clocks (labelled 1, 2 and 3), maintained in two different laboratories (k and l) connected by a measurement link $x_{k,l}$ according to the sketch below.

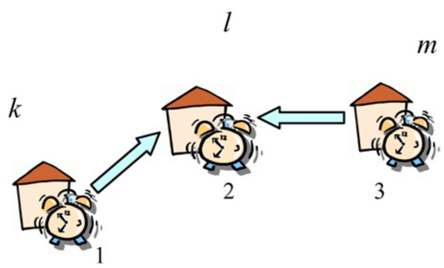


For site k, with UTC(k) realized by clock H_1, the uncertainty on $[UTC - UTC(k)]$ would be

$$u^2_{[UTC-UTC(k)]} = w_2^2 u^2_{x_{2,k}} + w_3^2 u^2_{x_{3,k}} + 2w_2 w_3 u_{(x_{2,k},x_{3,k})}$$
$$= (w_2 + w_3)^2 u^2_{x_{l,k}} = (1 - w_1)^2 u^2_{x_{l,k}}, \quad (18)$$

since $u^2_{2,k} = u^2_{3,k} = u_{(x_{2,k},x_{3,k})} = u^2_{l,k}$, because clocks H_2 and H_3 are in the same laboratory.

Case B2. We compute the uncertainty for UTC(l) realized with clock H_2 inside laboratory l for a system of three clocks (labelled 1, 2 and 3) in three different laboratories (k, l and m) connected according to the sketch below.



If the links between laboratories (l, k) and laboratories (l, m) are uncorrelated, then the double summation in equation (10) is zero and

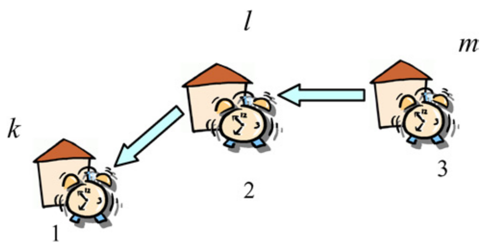
$$u^2_{[UTC-UTC(l)]} = w_1^2 u^2_{x_{k,l}} + w_3^2 u^2_{x_{m,l}}. \quad (19)$$

If the links between laboratories (l, k) and laboratories (l, m) are correlated through site-based noise in the receiver in laboratory l, but the total noise in both links is the same and equally contributed by the two laboratories: $u_{(x_{l,k},x_{l,m})} = 0.5\,u^2_{l,k} = 0.5\,u^2_{l,m}$ and

$$u^2_{[UTC-UTC(l)]} = w_1^2 u^2_{x_{l,k}} + w_3^2 u^2_{x_{l,m}} + 2w_1 w_3 u_{(x_{l,k},x_{l,m})}$$
$$= (w_1^2 + w_1 w_3 + w_3^2) u^2_{x_{l,k}} \quad (20)$$

Comparing equation (19) with equation (20), the effect of considering partial correlations between the link noises becomes evident.

Case B3. The same configuration as Case B2, except that the uncertainty for laboratory k is to be evaluated. The clocks in laboratory m are connected to lab k through the pivot lab l with $x_{k,m} = x_{k,l} + x_{l,m}$:



Assuming the two links $x_{k,l}$ and $x_{l,m}$ are uncorrelated, one obtains from equation (10)

$$u^2_{[UTC-UTC(k)]} = w_2^2 u^2_{x_{l,k}} + w_3^2 [u^2_{x_{l,k}} + u^2_{x_{m,l}}] + 2w_2 w_3 u_{(x_{l,k},x_{m,k})}$$
$$= (w_2 + w_3)^2 u^2_{x_{l,k}} + w_3^2 u^2_{x_{m,l}}, \quad (21)$$

where the correlation between $x_{k,l}$ and $x_{k,m}$ is due to the common path (l, k):

$$u_{(x_{l,k},x_{m,k})} = u^2_{x_{l,k}}. \quad (22)$$

This example shows that, for uncorrelated links, the weight of laboratories behind a pivot laboratory is added to the weight of the pivot laboratory itself and this is the motivation behind the definition of 'equivalent weight' W_{eq_i} as the weight of the pivot laboratory i plus the weights of the clocks behind that pivot laboratory, which was introduced in equation (13). The last term in equation (21) takes into account the noise from the pivot (laboratory l) to the remote laboratory m, weighted by the weight of clocks in laboratory m.

In the case of links correlated through site-based noise, with the same assumptions leading to equation (20) and assuming that the noise of the receiver in laboratory l is cancelled while connecting laboratory m to laboratory k with contemporaneous measurements, the same equations would apply, and

$$u^2_{[UTC-UTC(k)]} = (w_2^2 + w_2 w_3 + w_3^2) u^2_{x_{l,k}}. \quad (23)$$

This would be expected for links dominated by site-based noise, because the site-based noise of each site combines in such a way that all possible combinations of links yield the same results.

2.5. *Formulae for the UTC network*

Starting from the examples above, the uncertainty corresponding to the network of links currently used for the computation of UTC as reported in figure 1 can be evaluated.

Figure 1 shows that USNO, NIST and NICT[1], and NTSC[2] act as 'intermediate' pivots and the central role of the PTB.

Since the PTB plays a central role, equation (10) can be used to obtain the uncertainty for $[UTC - UTC(\text{PTB})]$ considering the equivalent weights W_{eq_i} for the intermediate pivot laboratories USNO, NIST, NICT and NTSC. The final expression turns out to be

$$\begin{aligned} u^2_{[UTC-UTC(\text{PTB})]} = & \sum_{i=2}^{N_1+1} W_i^2 u^2_{x_{i,\text{UTC(PTB)}}} + \sum_{k=N_1+2}^{(N_1+N_2)+1} W_k^2 u^2_{x_{k,\text{UTC(USNO)}}} \\ & + \sum_{k=(N_1+N_2)+2}^{(N_1+N_2+N_3)+1} W_k^2 u^2_{x_{k,\text{UTC(NIST)}}} + \sum_{k=(N_1+N_2+N_3)+2}^{(N_1+N_2+N_3+N_4)+1} W_k^2 u^2_{x_{k,\text{UTC(NICT)}}} \\ & + W^2_{eq_{\text{USNO}}} u^2_{x_{\text{UTC(PTB),UTC(USNO)}}} + W^2_{eq_{\text{NIST}}} u^2_{x_{\text{UTC(PTB),UTC(NIST)}}} \\ & + W^2_{eq_{\text{NICT}}} u^2_{x_{\text{UTC(PTB),UTC(NICT)}}} + W^2_{eq_{\text{NTSC}}} u^2_{x_{\text{UTC(PTB),UTC(NTSC)}}} \\ & + W^2_{\text{JACT}} u^2_{x_{\text{UTC(NTSC),UTC(JACT)}}} \end{aligned} \quad (24)$$

Here $N_1 = 25$ is the number of laboratories directly linked to PTB with one single link (IEN, AOS, SP, LT, etc) excluding

[1] NICT is the new name for CLR.

[2] All the acronyms appearing in this paper are in agreement with the list published by the BIPM Time Section at http://www.bipm.org/jsp/en/TimeFtp.jsp?TypePub=publication.

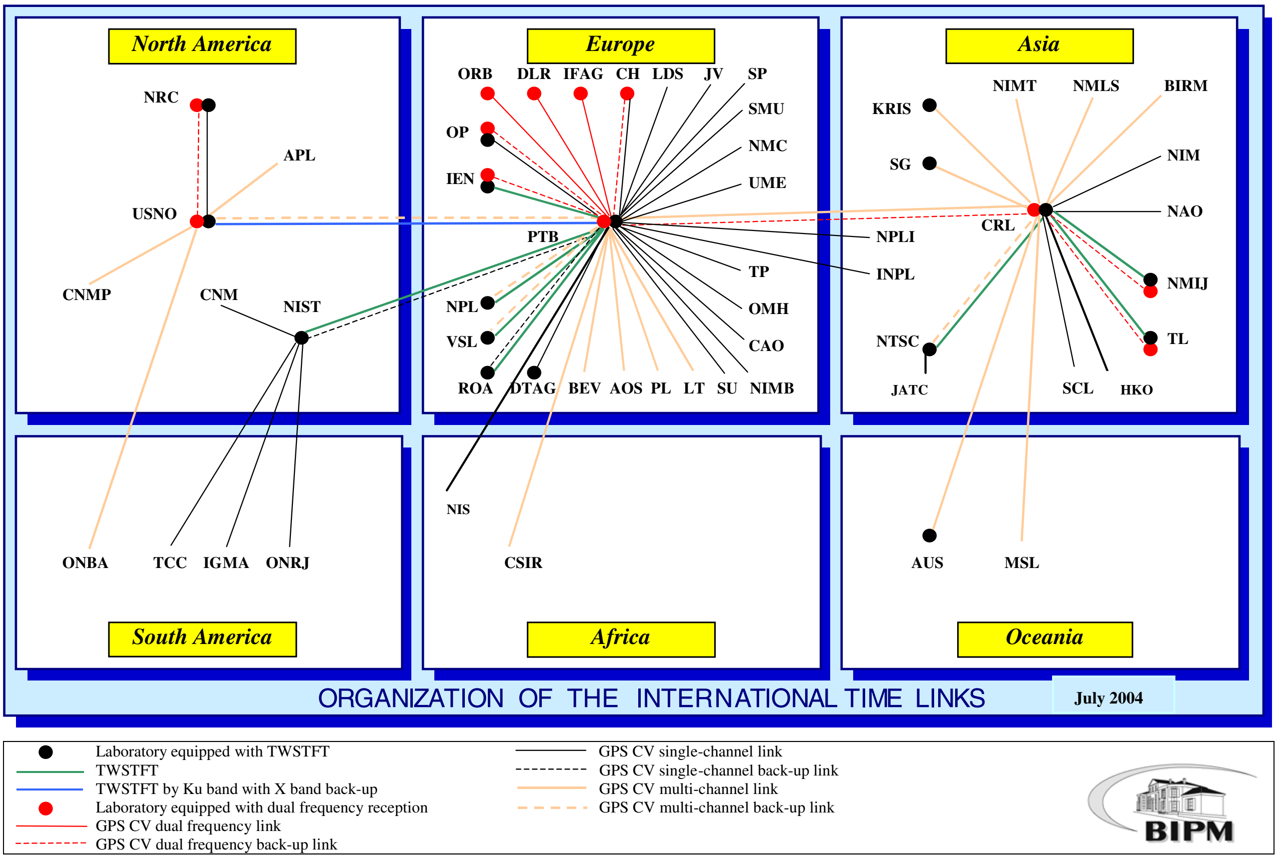


Figure 1. International time links used for computation of UTC (July 2004).

the four laboratories acting as intermediate pivots; $N_2 = 4$ is the number of laboratories linked to USNO (namely NRC, CNMP, APL, ONBA); $N_3 = 4$ is the number of laboratories linked to NIST (namely TCC, ONRJ, CNM, IGMA); $N_4 = 13$ is the number of laboratories linked to NICT (SG, TL, NAO, MSL, etc) and W_{eq_i} are equivalent weights of laboratories as introduced in equation (13).

The equivalent weight of the USNO is, for example,

$$W_{eq_{\mathrm{USNO}}} = W_{\mathrm{NRC}} + W_{\mathrm{CNMP}} + W_{\mathrm{APL}} + W_{\mathrm{ONBA}} + W_{\mathrm{USNO}}.$$

Similar expressions hold for the other intermediate pivots, NIST, NICT and NTSC and W_{Lab} was defined in equation (11).

Once the uncertainty of $[UTC - UTC(\mathrm{PTB})]$ is evaluated, the expression (14) can be used to compute the uncertainty of $[UTC - UTC(k)]$ for every other laboratory k.

3. A Monte Carlo simulation

The analytical approach presented in the previous section was tested by a Monte Carlo simulation using the BIPM's software ALGOS, on the full set of clock and time transfer data used to generate *Circular T* published in July 2004. For that month, the pivot laboratories were PTB, USNO, NIST, NICT and NTSC.

Since the source of uncertainty in $[UTC - UTC(k)]$ is the links whose uncertainty $u_{x_{i,j}}$ is listed in *Circular T*, a simulation was performed assuming that every link measure is described by a random variable with a Gaussian distribution, mean value equal to the obtained measurement value a and standard deviation equal to $u_{x_{i,j}}$.

The Monte Carlo simulation consists of picking up different values for the measure $x_{i,j}$ from its statistical distribution and computing $[UTC - UTC(k)]$ with the different simulated measure values. This gives an indication of the variability of the results $[UTC - UTC(k)]$ due to the variability of the measures $x_{i,j}$, which have been shown to be the only source of variability and, hence, of uncertainty.

The simulation proceeded by first evaluating the effect on $[UTC - UTC(k)]$ of only one noisy link (all the other links are considered with negligible noise); then the noise was inserted link by link. This 'experimental' variability was compared with the expected theoretical results coming from the analytical estimation (10).

3.1. Example of a single link

The effect of the uncertainty of the link between the PTB and the LT laboratories using the data of July 2004 was evaluated. Laboratory LT has one clock of percentage weight $w_{\mathrm{LT}} = 0.119$; the squared uncertainty of the link $[UTC(\mathrm{LT}) - UTC(\mathrm{PTB})]$ is $u^2_{[UTC(\mathrm{LT})-UTC(\mathrm{PTB})]} = 27.25\ \mathrm{ns}^2$ and on MJD 53144 $[UTC(\mathrm{LT}) - UTC(\mathrm{PTB})]$ was found to be -242.8 ns. A large number, N, of values of $[UTC(\mathrm{LT}) - UTC(\mathrm{PTB})]$ was generated normally distributed around the central value $[UTC(\mathrm{LT}) - UTC(\mathrm{PTB})] = -242.8$ ns and with a variance of $u^2_{[UTC(\mathrm{LT})-UTC(\mathrm{PTB})]} = 27.25\ \mathrm{ns}^2$.

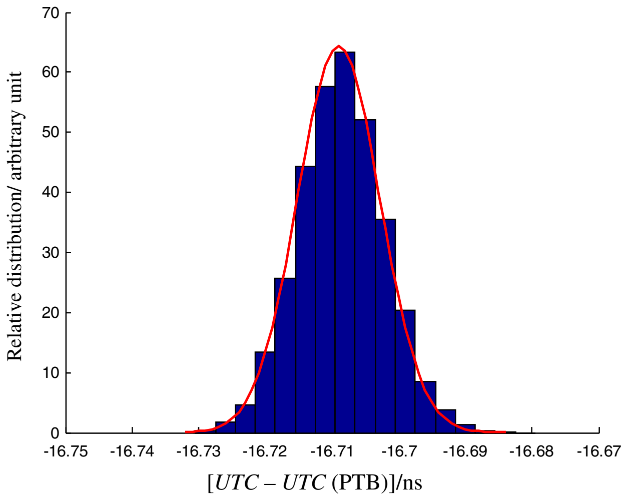


Figure 2. Theoretical and experimental distribution of the values [$UTC - UTC$(PTB)] considering only one link with noise.

Using equation (10), with only the link PTB/LT corrupted with noise

$$u^2_{[UTC-UTC(\mathrm{PTB})]} = w^2_{\mathrm{LT}} u^2_{[UTC(\mathrm{LT})-UTC(\mathrm{PTB})]}$$
$$= \left(\frac{0.119}{100}\right)^2 \times (27.25\,\mathrm{ns}^2) = 3.85 \times 10^{-5}\,\mathrm{ns}^2.$$

Thus

$$u_{[UTC-UTC(\mathrm{PTB})]} = 0.0062\,\mathrm{ns},$$

and from equation (14)

$$u_{[UTC-UTC(\mathrm{LT})]} = 5.21\,\mathrm{ns}.$$

Alternatively, using equation (10) directly referenced to laboratory LT, which corresponds to having the total clock weight in PTB with the exception of the LT clock, the same theoretical result was obtained:

$$u_{[UTC-UTC(\mathrm{LT})]} = \left(1 - \frac{0.119}{100}\right) \times \sqrt{27.25\,\mathrm{ns}^2} = 5.21\,\mathrm{ns}.$$

Using $N = 10\,000$ simulated measurement values, a normal distribution of values [$UTC-UTC$(PTB)] was obtained with a mean value equal to −16.7087 ns and standard deviation equal to 0.0063 ns. The value [$UTC - UTC$(PTB)] for that date published in *Circular T* was equal to −16.7 ns and the standard uncertainty from the above computation was $u_{[UTC-UTC(\mathrm{PTB})]} = 0.0062$ ns. Therefore, the mean value and the standard deviation of [$UTC - UTC$(PTB)] obtained by simulating the noisy link PTB/LT corresponded to the published value and to the expected theoretical uncertainty.

For the laboratory LT a normal distribution of mean value equal to 226.161 ns was obtained, with a standard deviation of 5.2 ns. The published value in *Circular T* was [$UTC - UTC$(LT)] = 226.1 ns and the standard uncertainty from the above theoretical computation was $u_{[UTC-UTC(\mathrm{LT})]} = 5.21$ ns.

The results are depicted in figures 2–4, where the histogram reports the results obtained from the Monte Carlo simulation, while the solid line (in red) represents the expected statistical distribution of the values according to the theoretical analysis.

As can be seen, the analytical and simulation results agree.

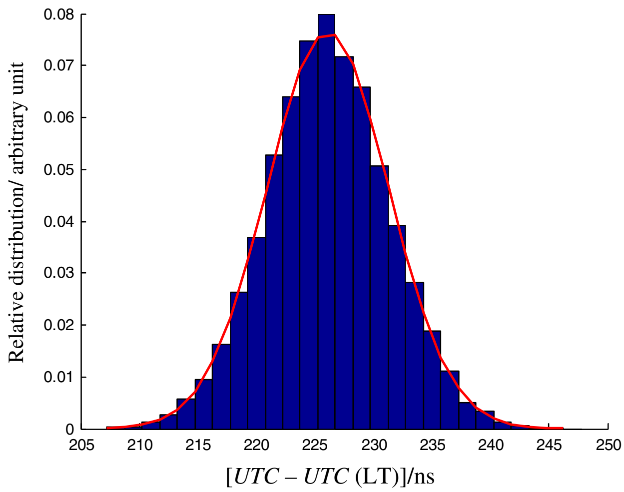


Figure 3. Theoretical and experimental distribution of the values [$UTC - UTC$(LT)] considering only one link with noise.

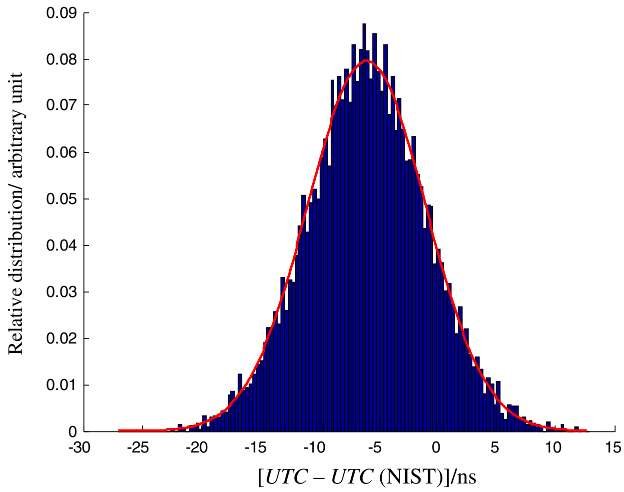


Figure 4. Theoretical and experimental distribution of the values of [$UTC - UTC$(NIST)].

3.2. Complete system

The Monte Carlo results are reported here for the complete system of links, using $N = 20\,000$ simulated data for any link. Figure 4 shows the close agreement, for example, between the theoretical and experimental results for the values of [$UTC - UTC$(NIST)]. In table 1 the analytical results are compared to the simulations assuming that every link had the noise corresponding to the uncertainty listed in *Circular T*. The agreement between the two estimation methods is evident and gives confidence to the analysis.

4. Application to the computation of *Circular T*

The formulae developed in section 2.5 for the evaluation of [$UTC - UTC(k)$] uncertainties have been applied using the link uncertainties as listed in section 6 of *Circular T*. They have been used operationally since January 2005, reporting the uncertainties in *Circular T*. Results for *Circular T* no 205 of January 2005 are reported in figure 5.

Table 1. Analytical and numerically estimated uncertainties of all the UTC participating laboratories considering every link with noise.

k	$u_{[UTC-UTC(k)]}$/ns Analytical method	$u_{[UTC-UTC(k)]}$/ns Numerical method	k	$u_{[UTC-UTC(k)]}$/ns Analytical method	$u_{[UTC-UTC(k)]}$/ns Numerical method
AOS	5.6	5.6	NIS	20.3	20.6
APL	5.7	5.7	NIST	5.0	5.0
AUS	7.3	7.3	NMC	20.7	20.8
BEV	5.5	5.5	NMIJ	7.0	7.0
BIRM	20.6	20.6	NPL	5.3	5.2
CAO	21.2	21.0	NPLI	20.3	20.4
CH	5.4	5.3	NRC	15.2	15.1
CNM	21.0	21.1	NTSC	7.1	7.0
CNMP	8.4	8.4	OMH	20.2	20.5
CSIR	20.3	20.6	ONBA	8.9	8.9
DLR	5.4	5.4	ONRJ	21.2	21.3
DTAG	10.6	10.5	OP	5.4	5.4
HKO	7.3	7.3	ORB	5.3	5.4
IEN	2.4	2.4	PL	5.4	5.3
IFAG	5.3	5.3	PTB	1.9	1.9
IGMA	20.8	20.9	ROA	5.4	5.4
INPL	10.9	11.0	SCL	11.6	11.6
JATC	21.2	21.3	SG	20.5	20.6
JV	20.7	20.6	SMU	20.7	20.7
KRIS	7.1	7.2	SP	10.4	10.5
LDS	20.3	20.2	SU	6.1	6.1
LT	5.6	5.6	TCC	21.2	20.9
MSL	20.7	20.8	TL	6.7	6.7
NAO	20.6	20.7	TP	5.8	5.8
NICT	4.4	4.4	UME	25.1	25.1
NIM	20.4	20.3	USNO	2.3	2.3
NIMT	20.7	20.9	VSL	5.3	5.3

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1 - Coordinated Universal Time UTC and its local realizations UTC(k). Computed values of [*UTC-UTC(k)*].
From 1999 January 1, 0h UTC, *TAI-UTC* = 32 s.

Date 2004/05 0h UTC	DEC 30	JAN 4	JAN 9	JAN 14	JAN 19	JAN 24	JAN 29	Uncertainty/ns			Notes
MJD	53369	53374	53379	53384	53389	53394	53399				
Laboratory *k*				[*UTC-UTC(k)*]/ns				u_A	u_B	u	
AOS (Borowiec)	-21.2	-14.8	-18.7	-13.6	-11.2	-12.1	-17.9	1.6	5.4	5.6	
BEV (Wien)	84.9	87.5	90.2	90.2	94.8	96.9	99.9	1.5	5.3	5.5	
CH (Bern)	-27.4	-25.2	-27.4	-31.1	-28.6	-23.8	-21.1	0.8	5.3	5.4	
DLR (Oberpfaffenhofen)	1.4	-3.1	-7.8	-14.1	-19.1	-22.3	-31.3	0.8	5.4	5.5	
IEN (Torino)	-58.2	-71.8	-77.9	-79.8	-78.5	-84.6	-87.5	0.7	2.2	2.3	
LT (Vilnius)	486.0	519.1	547.4	549.6	528.4	533.6	542.7	1.6	5.3	5.5	
NICT (Tokyo)	4.5	-0.7	6.8	6.1	-1.7	1.1	3.9	1.2	4.3	4.5	
NIST (Boulder)	3.5	2.6	1.3	1.5	0.8	-0.6	-1.7	0.6	5.1	5.1	
NPL (Teddington)	8.0	11.2	14.6	18.5	22.7	25.7	33.7	0.7	2.2	2.3	
NTSC (Lintong)	17.7	20.4	26.7	31.1	27.4	19.6	19.1	2.7	6.5	7.0	
ONBA (Buenos Aires)	-923.0	-967.0	-1200.8	-1407.0	-1666.2	-3418.7	-3392.1	5.0	7.3	8.8	
ONRJ (Rio de Janeiro)	155.1	153.1	162.1	159.6	161.7	158.4	176.0	5.0	20.6	21.2	
OP (Paris)	41.7	45.8	49.4	43.6	40.3	39.8	36.4	0.6	2.1	2.2	
ORB (Bruxelles)	-42.9	-41.2	-39.3	-37.5	-35.2	-34.5	-32.3	0.8	5.3	5.4	
PL (Warszawa)	-3.9	-1.8	-5.7	-6.5	-4.5	-4.1	-7.8	1.5	5.2	5.4	
PTB (Braunschweig)	-0.6	1.2	2.5	3.0	5.5	5.9	4.7	0.4	1.9	1.9	
ROA (San Fernando)	-63.1	-62.0	-68.3	-67.9	-67.1	-65.8	-74.1	0.8	5.3	5.4	
UME (Gebze-Kocaeli)	599.8	595.1	593.2	603.0	608.3	611.2	612.5	15.0	20.1	25.1	
USNO (Washington DC)	-4.9	-4.7	-5.1	-4.6	-3.5	-3.5	-2.5	0.5	2.2	2.3	
VSL (Delft)	-76.9	-77.0	-77.9	-77.4	-56.4	-41.8	-21.2	0.7	2.2	2.3	

Figure 5. A sample of the first section of *Circular T* no. 205 showing uncertainties in $[UTC - UTC(k)]$.

In *Circular T*, the last column (see figure 5), denoted by u, corresponds to the combined standard uncertainty of $[UTC - UTC(k)]$ for any laboratory k. The values u_A and u_B are the components of the combined standard uncertainty u originating from type A and type B evaluations listed in section 6, respectively. The following relationship holds:

$$u = \sqrt{u_A^2 + u_B^2}.$$

It is noted that the application of equations (24) and (14) separately to the type A and type B evaluations is permitted only when all the other quantities involved are constant, in particular the number of clocks and their weights are constant; this is currently true only within each separate month of UTC computation.

The application of the above theory to the actual case of $[UTC - UTC(k)]$ suggests an additional experimental test to check the validity of the obtained estimates. In fact, under the assumptions that the values of $[UTC - UTC(k)]$ are almost constant or very slowly varying in time, which is a reasonable assumption in case of the UTC(k) time scale realized with hydrogen masers, one can ascribe the observed instability of the values $[UTC - UTC(k)]$, at a five-day observation interval, mostly to the combined effect of time transfer noise rather than to clock noise. By estimating the Allan deviation or the Time Deviation (TDEV) of the quantity $[UTC - UTC(k)]$ one has thus a rough idea of the combined time transfer noise of all the links in TAI as observed from laboratory k and therefore gets an independent estimate of the u_A uncertainty for that laboratory. A simple test evaluating the TDEV for a few time laboratories gave reasonably good agreement with the calculated values of u_A reported here.

5. Future extension

To refine the evaluation of uncertainties, it is necessary to have more details than are readily available about the correlation of the links. This is largely because, as noted in the examples, the correlated noise between two sites should include the contribution from site-based uncertainties at sites along the path of links and because the correlation between two links with a common site that uses the same equipment for both links should include the site-based noise contribution from that site. This is difficult to estimate when calibrations are done by links rather than by sites. In computations assuming site-based noise, which supplemented the link uncertainties reported in *Circular T*, it was found that on the whole the effects of these simplifications are small.

Further extension of this evaluation would be based on the following consideration. If all time transfers were achieved using a single system per site and if all sources of noise were site-based, such as mostly happens for Melting Pot GPS, also termed All-in-View (AV) [10,11], then all possible links would obey the following closure relation:

$$x_{i,j}(t) + x_{j,k}(t) + x_{k,i}(t) = 0. \qquad (25)$$

In this situation, the noise of each site's time transfer system would be indistinguishable from the noise of its clocks and the dominant uncertainty would be given by the site noise, and all the external clocks would be seen through that dominant noise independently of their location. If different satellite schedules are used in the relevant links, uncertainties in time transfer using GPS CV are largely, though not entirely, site-based. Even for CV observations made using every available satellite, closure violations will only arise if simultaneous satellite observations are recorded at only two of the three sites. In such cases, orbit misestimation and receiver noise will contribute uncertainties, and any azimuth or elevation-dependent asymmetries in the multipath environment would cause both uncertainties and biases. Since calibration is achieved by an all-sky sampling that is systematically different from the sky-sampling of CV, a different multipath will also lead to uncertainties. Despite these noise sources, the closure relation largely holds for CV, and the largest source of uncertainty is typically due to variations of the receiver system that are common to all data.

For Two-Way Satellite Time and Frequency Transfer (TWSTFT), the noise is again largely site-dependent. Some closure violations can occur because the observations between pairs of sites are typically made with different codes and slightly different frequencies. The largest source of closure errors is probably due to the fact that the received signals are shaped by the product of the transmitting and receiving bandpasses, while the delay and certain noise components such as the cable-dependent multipath can systematically vary over the bandpass [12]. While TWSTFT closure violations are seen at the 1 ns level in the data sent to the BIPM, they could be reduced through baseline-dependent calibrations. In the cases where a TWSTFT system is calibrated with GPS, the uncertainties in the calibration are determined by the uncertainties in the GPS calibration.

Special situations arise when one site is a pivot site, connected to some sites by one technique and other sites by a different technique. By means of an illustration, we can consider a very simplified situation in which every site is directly linked to one central pivot site, either by AV GPS or TWSTFT; this can easily be generalized to describe more complex topologies. We will assume that variations between any two links are completely uncorrelated with respect to variations at the link extremities, but that the links are 100% correlated with respect to variations of the equipment at the central pivot site, provided both links are by either TWSTFT or GPS. Let us also assume a bias B exists in the GPS equipment at the pivot site. In this case, it is easy to show that the bias would affect those laboratories k linked by GPS to the pivot as follows:

$$\Delta_{[UTC-UTC(k)]} = (1 - W_G)B$$

where W_G is the sum of all the weights of the laboratories linked to the pivot site by GPS.

The bias would affect the pivot laboratory and those linked to it by TWSTFT as follows:

$$\Delta_{[UTC-UTC(k)]} = -W_G B.$$

Under the normal circumstances described in this paper, the existence of any biases would not carry any significant statistical implications, as they would be directly related to the tabulated uncertainties in the links themselves. However, the above equations illustrate the dependence of TAI and UTC upon the equipment at any central pivot, which may not always follow the Gaussian behaviour assumed here, particularly in the case of equipment failure.

6. Conclusions

This paper presents a study of the determination of the uncertainties in $[UTC - UTC(k)]$. An analytical solution is derived from the law of the propagation of uncertainty, taking into account that leap seconds and deterministic frequency steering of EAL do not affect these uncertainties. The analytical results were verified through Monte Carlo simulations using the software used to generate UTC, and agreement was found, giving confidence in the analytical estimation.

A more detailed analysis is in progress, including full inclusion of site-based noise contributions, all available calibration information, more details for the correlation between the links, methods for optimizing the link structure, given uncertainty information, non-Gaussian behaviour and different correlation properties of uncertainties due to calibration or due to random noise.

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