On the status of the two-loop self-energy calculations

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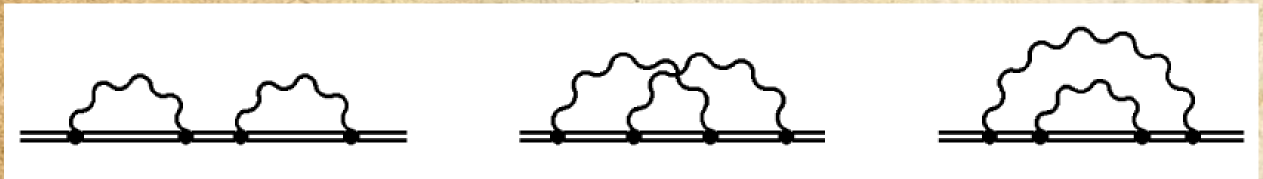
Two-loop self-energy for the Lamb shift of hydrogen-like ions

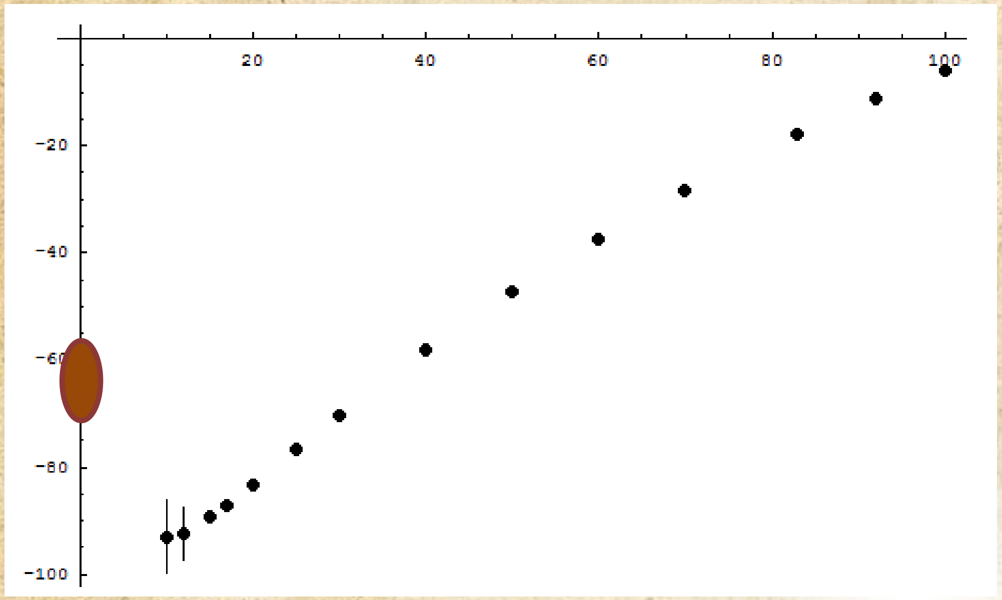


$$E_{\mathrm{SESE}} = mc^2 \left(\frac{\alpha}{\pi}\right)^2 \frac{(Z\alpha)^4}{n^3} \left[B_{40} + (Z\alpha)\, B_{50} + (Z\alpha)^2 \left[B_{63}\, \ln^3 (Z\alpha)^{-2} + B_{62}\, \ln^2 (Z\alpha)^{-2} + B_{61}\, \ln (Z\alpha)^{-2} + G_{\mathrm{SESE}}(Z\alpha)\right]\right],$$

Higher-order remainder:

$$G_{\mathrm{SESE}}(Z\alpha) = B_{60} + \ldots (\text{higher terms in } Z\alpha) \ldots$$

SESE: Zα expansion versus all-order, 1s state



Zα-expansion calculation [K. Pachucki and U. D. Jentschura, Phys. Rev. Lett. 91, 113005 (2003)]:

$G_{SESE}(Z = 0) == B_{60} = -61.6(9.2)$

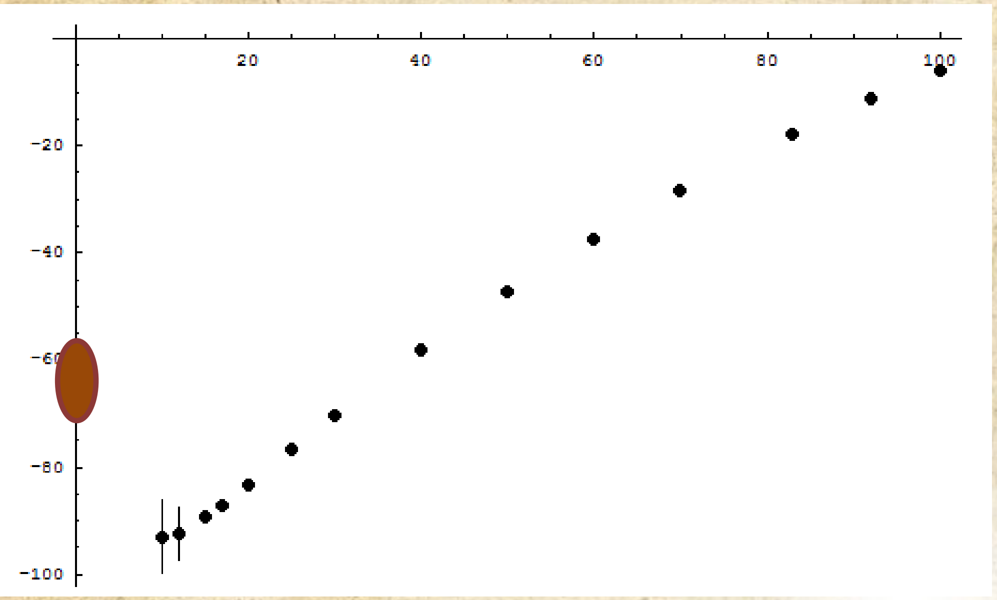
All-order (in Zα) numerical calculation

[V.A. Yerokhin, P. Indelicato, V.M. Shabaev, Phys. Rev. Lett. 91, 073001 (2003); Phys. Rev. A 71, 040101(R) (2005)]

[V.A. Yerokhin, Phys. Rev. A 80, 040501(R) (2009)]

$G_{SESE}(Z = 1, \text{extrapolation}) = -86(15)$

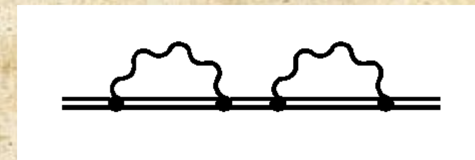
SESE: Zα expansion versus all-order, 1s state

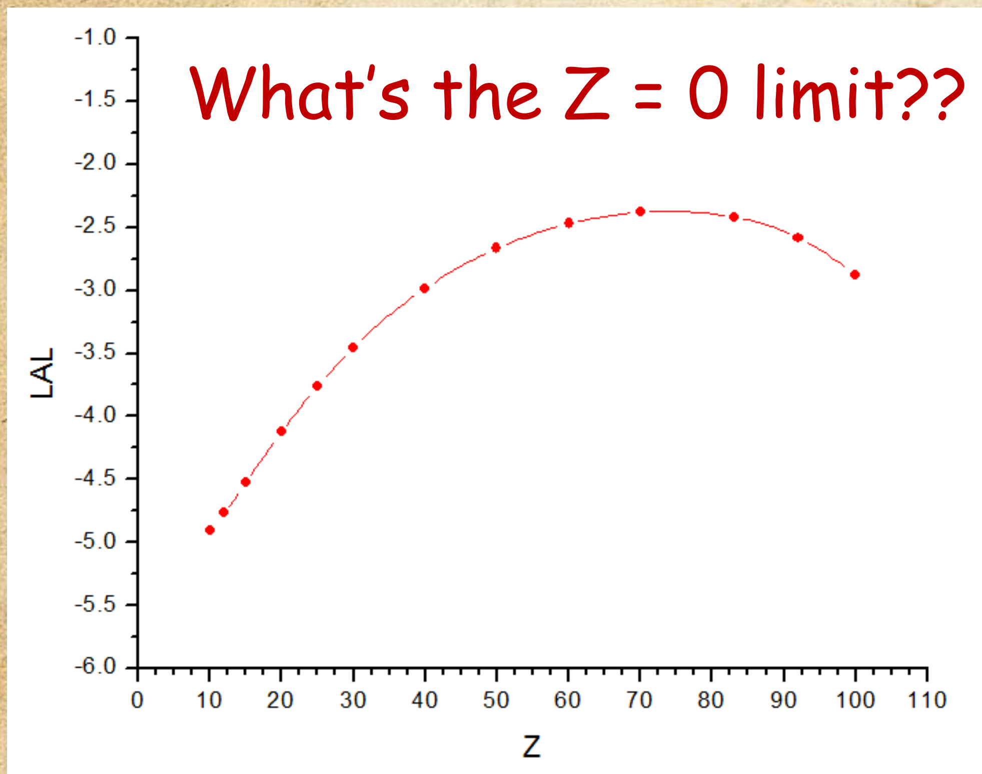


Should we call this agreement or disagreement??

Lesson #1: Z dependence can be tricky

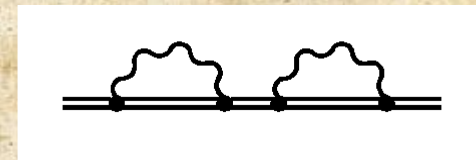
Loop-after-loop correction:

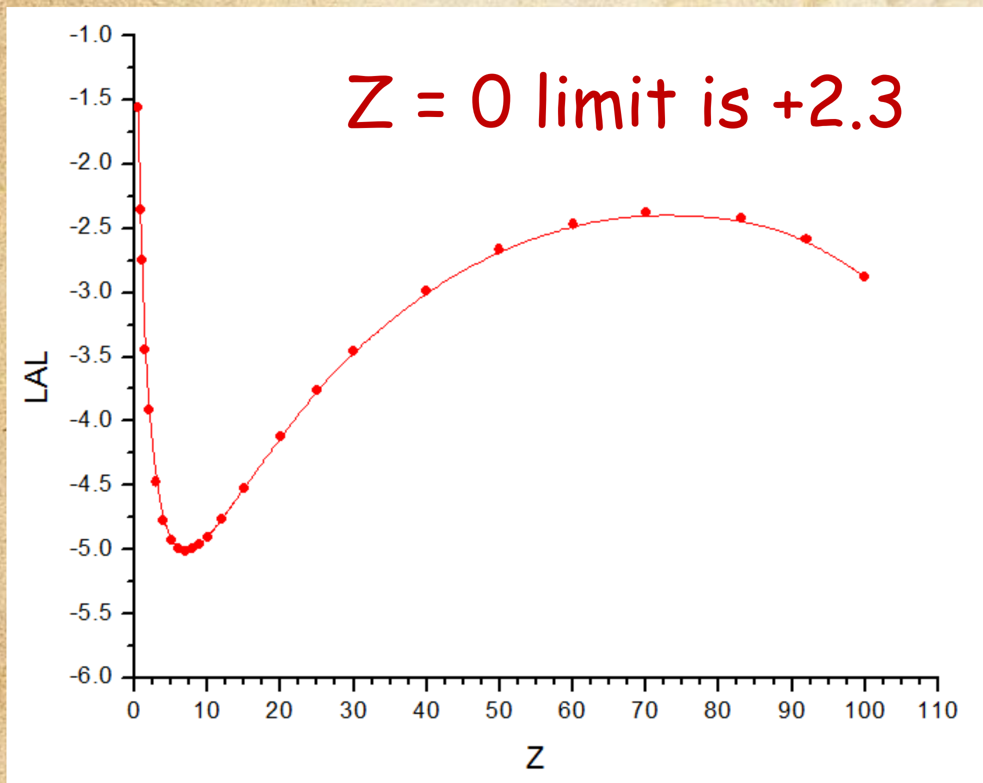




Lesson #1: Z dependence can be tricky

Loop-after-loop correction:





$$
\begin{aligned}
G_{\mathrm{LAL}}(Z\alpha) = a_{50} &\\
&+ (Z\alpha)\,\ln^3(Z\alpha)^{-2}\\
&+ (Z\alpha)\,\ln^2(Z\alpha)^{-2}\\
&+ (Z\alpha)\,\ln(Z\alpha)^{-2}\\
&+ (Z\alpha) + \ldots
\end{aligned}
$$

Logarithms !!!

All-order results:

V. A. Yerokhin, Phys. Rev. A 62, 012508 (2000); Phys. Rev. Lett. 86, 1990 (2001)

S. Mallampalli and J. Sapirstein. Phys. Rev. Lett. 80, 5297 (1998)

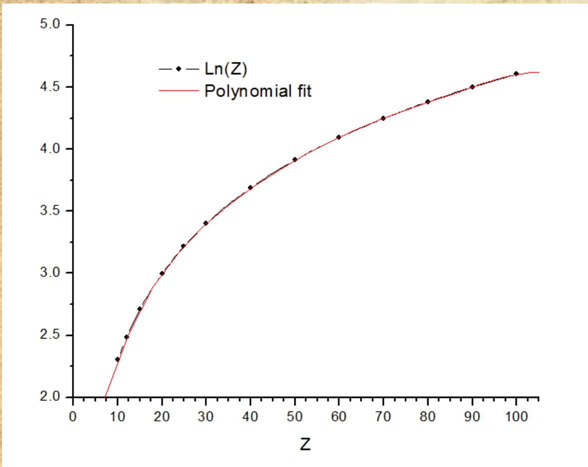
Lesson #2: Extrapolation of small-Z limit in QED

QED corrections are semi-analytic functions of Za:

$$F(Z\alpha) = a_{00} + \sum_{k=1}^{\infty} \sum_{s=0}^{?} a_{ks}\,(Z\alpha)^k \ln^s(Z\alpha)$$

Typical task: we have numerical results for F(Z_i) for varios Z_i; we are looking for the coefficients a_{kl}.

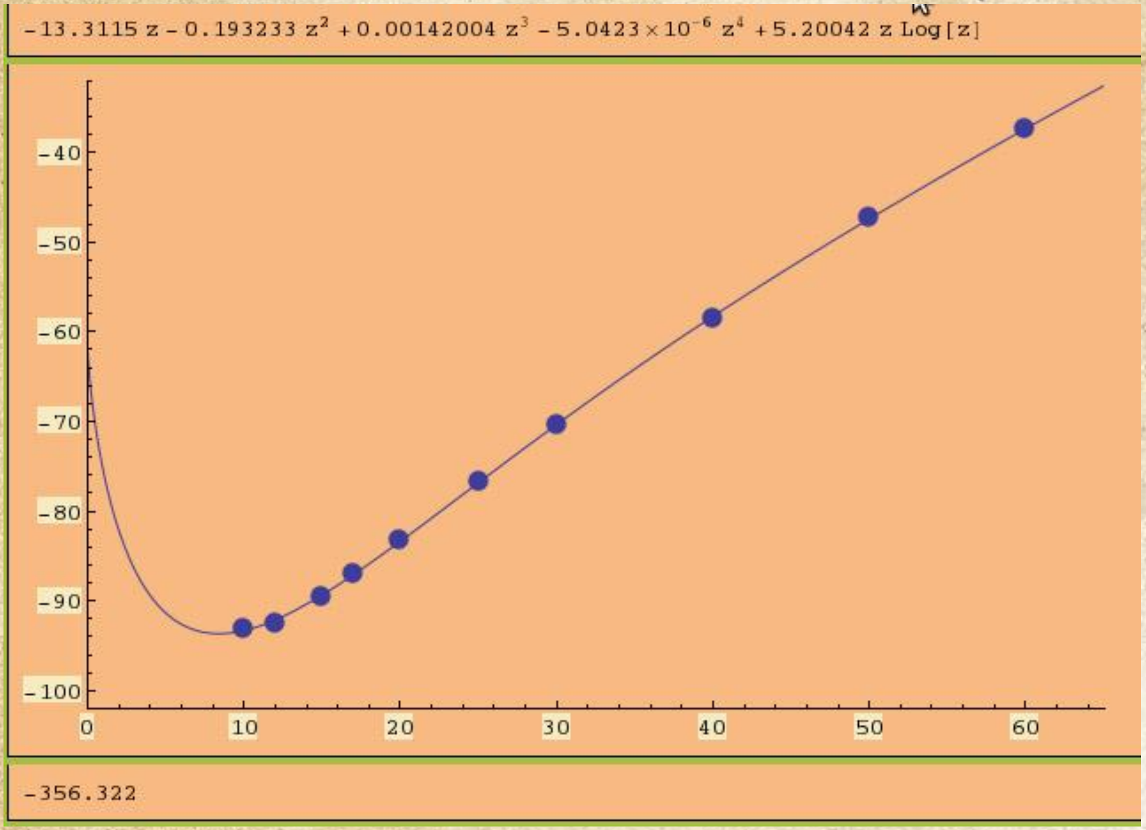
The main problem: in the region Z >= 10, ln(Z) can be well approximated by polynomials!



Conclusion: if we have only results in the medium-Z region, we are in trouble

SESE: Can we fit an agreement? (1)

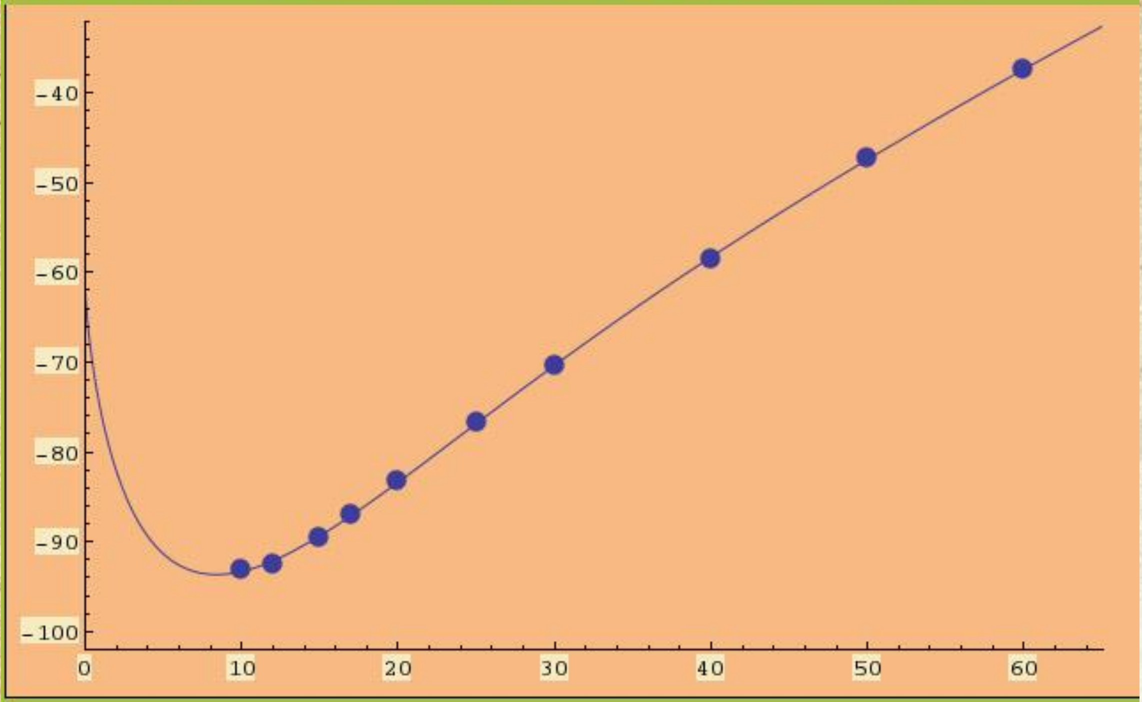
Basis: $\{z, z \, Log[z], z^2, z^3, z^4\}$



SESE: Can we fit an agreement? (1)

Basis: {z, z Log[z], z^2, z^3, z^4}

$-13.3115\,z - 0.193233\,z^2 + 0.00142004\,z^3 - 5.0423\times10^{-6}\,z^4 + 5.20042\,z\,\mathrm{Log}[z]$



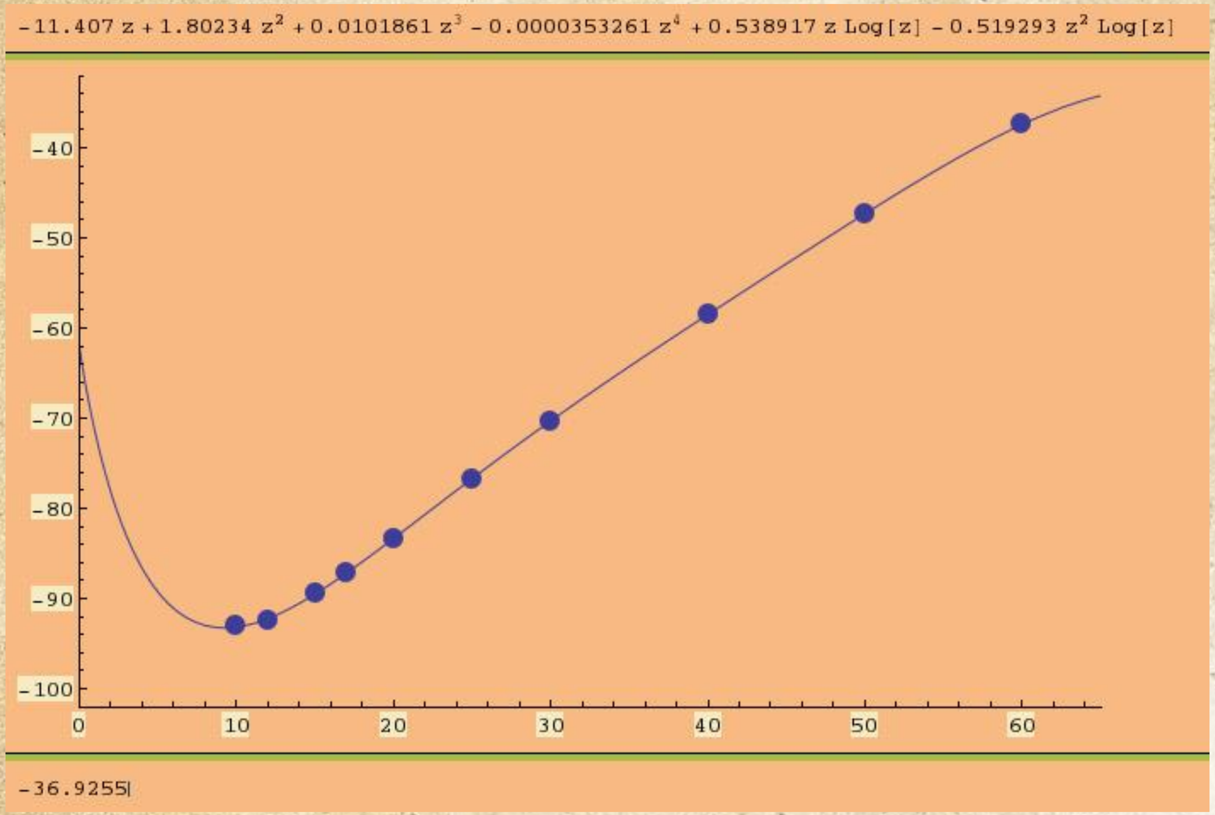
-356.322

The logarithmic term: -356 $(Z\alpha)\ \ln(Z\alpha)^{-2}$

Too large! Disagreement?

SESE: Can we fit an agreement? (2)

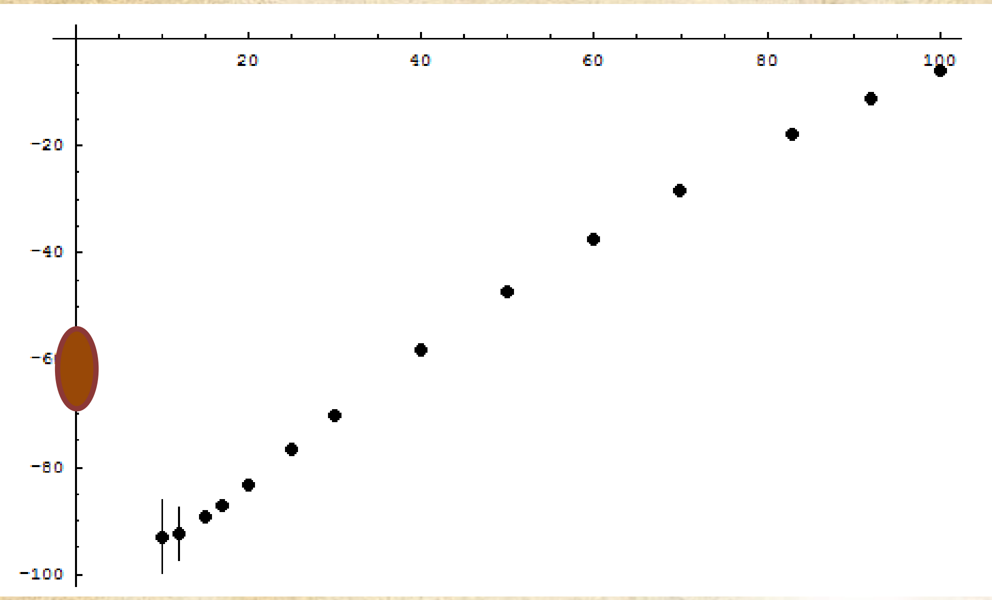
Basis: {z, z Log[z], z^2, z^2 Log[z], z^3, z^4}



The logarithmic term: $-37\,(Z\alpha)\ln(Z\alpha)^{-2}$

Any conclusion?

SESE: Zα expansion versus all-order, 1s state



Should we call

this agreement

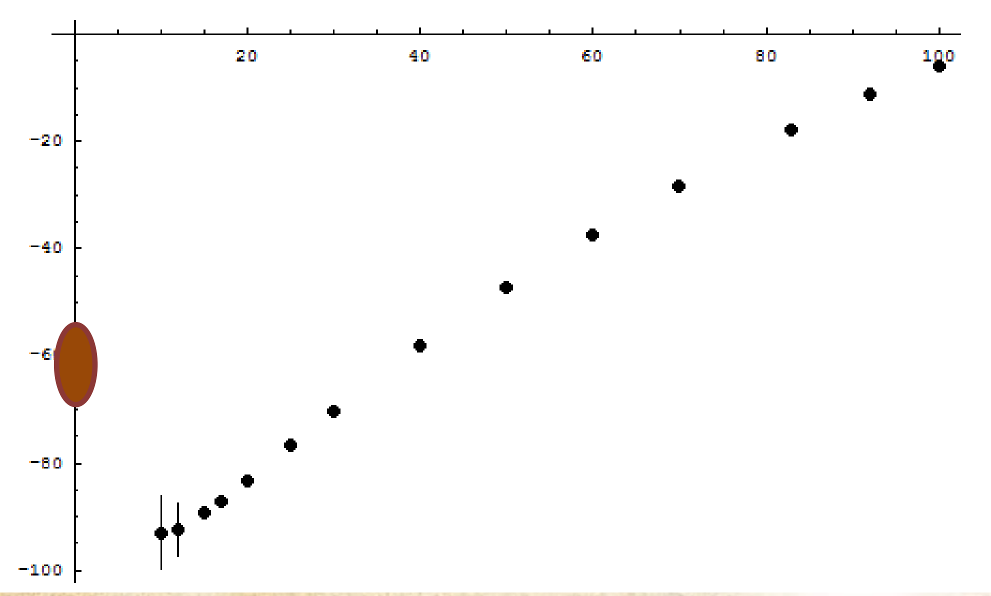
or

disagreement??

Conservative answer:

No comments

SESE: Zα expansion versus all-order, 1s state

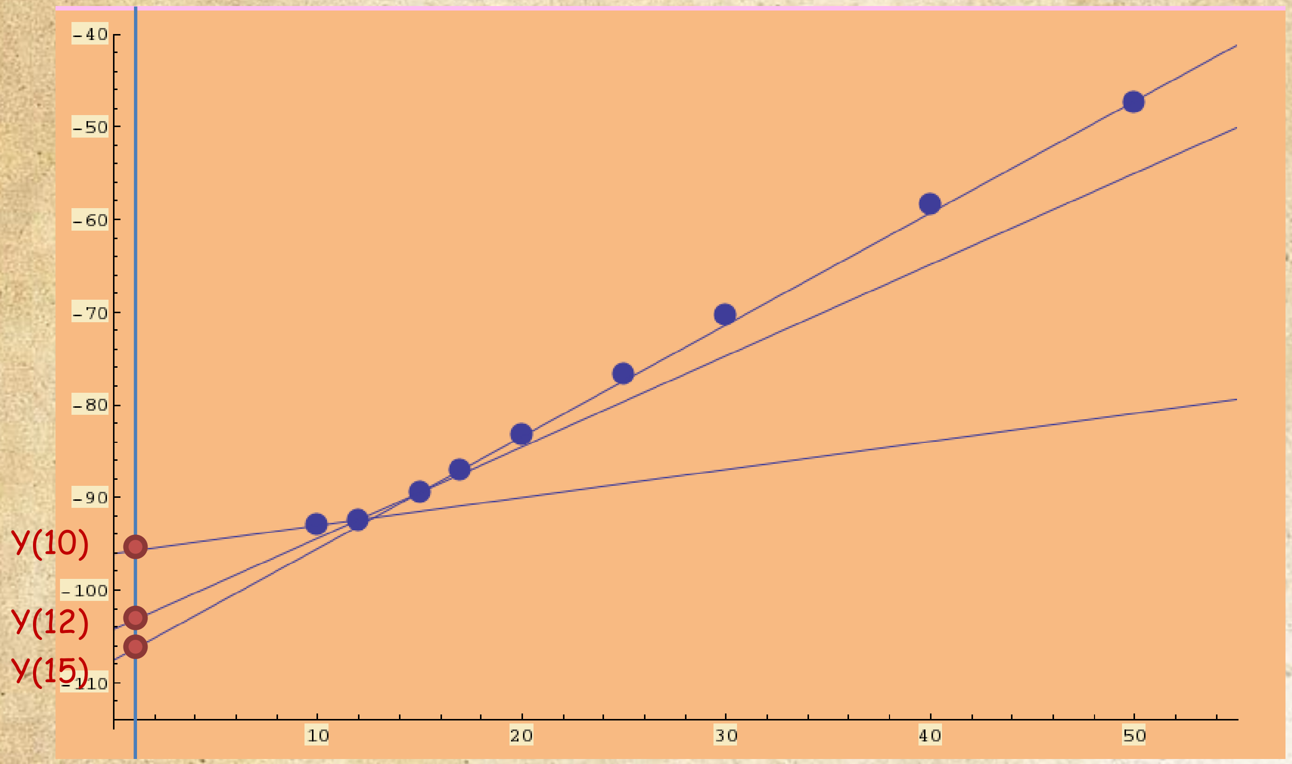


Should we call this agreement or disagreement??

Let us try our best guess

Fit of numerical results to Z = 1

Step #1: linear fit of each two consecutive data points to Z = 1 => new set of data y(x)

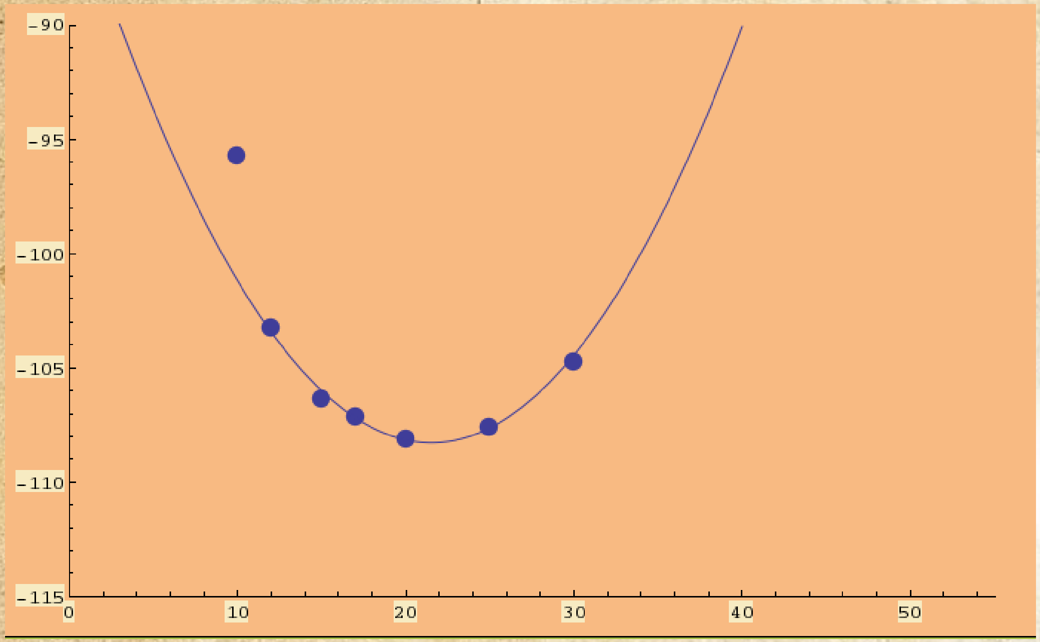


Step #2: set of data y(x) is plotted and fitted again. The procedure may be repeated iteratively.

Inspired by extrapolation procedure by P.J. Mohr

Fit of numerical results to Z = 1

Step #2: set of data y(x) is plotted and fitted again.



Result (Z = 1): -85.8

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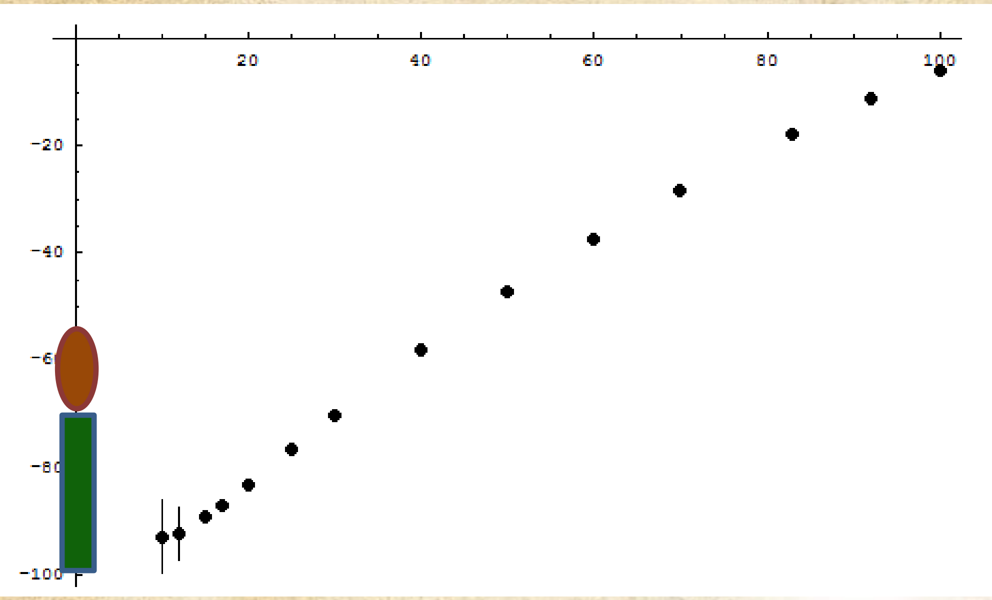

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"best guess" answer: marginal agreement