# Present Status of SI Values of $K_J$ and $R_K$

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The 2002 Committee on Data for Science and Technology (CODATA) least-squares adjustment of the values of the fundamental physical constants was completed in December 2003. Carried out by the authors under the auspices of the CODATA Task Group on Fundamental Constants, it took into account all relevant data available through 2002 December 31, plus a few especially important data that became available by the Fall of 2003. A lengthy paper that gives the 2002 CODATA recommended values of the constants and describes in detail the data and their treatment is now available [1].

The 2002 CODATA recommended values of the Josephson constant $K_J$ (assumed equal to $2e/h$) and von Klitzing constant $R_K$ (assumed equal to $h/e^2 = \mu_0 c/2\alpha$, where $\mu_0$ is the magnetic constant and $c$ is the speed of light in vacuum, both of which are exactly known in the International System of Units or SI, and $\alpha$ is the fine-structure constant), are

$$K_J = K_{J-90}[1 - 4.3(8.5) \times 10^{-8}] \tag{1}$$

$$R_K = R_{K-90}[1 + 1.74(33) \times 10^{-8}]. \tag{2}$$

Here $K_{J-90} = 483\,597.9$ GHz/V exactly and $R_{K-90} = 25\,812.807\ \Omega$ exactly are the conventional values of $K_J$ and $R_K$ adopted by the CIPM in 1988 for the purpose of basing a representation of the volt and of the ohm on the Josephson effect (JE) and on the quantum Hall effect (QHE), respectively, starting 1990 January 1.

The standard uncertainty at the level of one (SI) volt assigned by the CIPM in 1988 to a “perfectly” realized (i.e., no experimental uncertainty) representation of the volt based on the JE and $K_{J-90}$ is $u = 0.4\ \mu$V, corresponding to a relative standard uncertainty $u_r = 40 \times 10^{-8}$, and the standard uncertainty at the level of one (SI) ohm assigned by the CIPM in 1988 to a “perfectly” realized (i.e., no experimental uncertainty) representation of the ohm based on the QHE and $R_{K-90}$ is $u = 0.2\ \mu\Omega$, corresponding to a relative standard uncertainty $u_r = 20 \times 10^{-8}$. The conventional values of $K_J$ and $R_K$ and these uncertainties were deduced by the CCEM (then the CCE) in 1988 June from all of the data available by 1988 June 15.

Based on the version of Eq. (2) that resulted from the 1998 CODATA adjustment of the values of the constants [2] and a review of the relevant $R_K$ data as it existed at the time, the 22nd CCEM in 2000 September decided that the assigned standard uncertainty $u$ of a “perfectly” realized ohm representation based on the QHE and $R_{K-90}$ should be reduced by a factor of two to $0.1\ \mu\Omega$, corresponding to a relative standard uncertainty $u_r = 10 \times 10^{-8}$. This uncertainty reduction was subsequently approved by the CIPM in

2000 October. On the other hand, the 22nd CCEM concluded that it was premature to reduce the value $u = 0.4$ µV, corresponding to $u_r = 40 \times 10^{-8}$, assigned by the CIPM in 1988 to a "perfectly" realized volt representation based on the JE and $K_{J-90}$.

The data relevant to the determination of the values of $K_J$ and $R_K$ in SI units are graphically compared in the two attached figures. These data are based on the information that was available by the Fall of 2003 and were considered for inclusion in the 2002 CODATA adjustment. In the first, or $K_J$, figure, the $K_J$ NML-89 and $K_J$ PTB-91 values are direct determinations of $K_J$, hence they do not depend on either of the assumptions $K_J = 2e/h$ or $R_K = h/e^2$, or on the values of any other constants. Similarly, in the second, or $R_K$, figure, the $R_K$ NIST-97, $R_K$ NML-97, $R_K$ BNM-01, and $R_K$ NPL-88 values are direct determinations of $R_K$ and also do not require either of these assumptions or the values of any other constants. It should be noted, however, that as part of the data analysis for the 2002 CODATA adjustment, least-squares studies were carried out in which one or the other, or both, of these assumptions were relaxed, but no statistically significant evidence was found to indicate that either of these fundamental relations is invalid.

The most troublesome problem in the 2002 adjustment is evident from the $K_J$ figure. As can be seen, the value of $K_J$ obtained from the Avogadro constant $N_A$, denoted $N_A$ P/N/I-03, through a measured value of the molar volume of silicon $V_m(\mathrm{Si})$ and the 2002 CODATA recommended value of the 220 silicon lattice spacing $d_{220}$, is not in agreement with the $K_J$ values denoted $K_J^2 R_K$ NPL-90 and $K_J^2 R_K$ NIST-98, where in each case $K_J^2 R_K = 4/h$ was directly obtained from a moving-coil watt balance experiment, nor is it in particularly good agreement with the directly measured values $K_J$ NML-89 and $K_J$ PTB-91. [The P/N/I (abbreviated from PTB/NMIJ/IRMM) value of $N_A$ derived from $V_m(\mathrm{Si})$ and $d_{220}$ is a combined value based on density measurements carried out at PTB and NMIJ, molar mass measurements carried out at the Institute for Reference Materials and Measurements or IRMM, and x-ray related measurements carried out at PTB and NMIJ.]

It should be noted that $K_J = (8\alpha/\mu_0 ch)^{1/2}$, and since the relative standard uncertainty $u_r$ of the fine-structure constant $\alpha$ is $3.3 \times 10^{-9}$, a value of $K_J$ can be readily obtained from a value of $h$ ($\mu_0$ and $c$ are, of course, exactly known). The relations between $h$, $N_A$, and $V_m(\mathrm{Si})$ are

$$h = \frac{cA_r(\mathrm{e})M_u\alpha^2}{2R_\infty N_A} \quad \text{with} \quad N_A = \frac{V_m(\mathrm{Si})}{\sqrt{8}d_{220}^3}, \tag{3}$$

where $M_u$ is the molar mass constant and is equal to $10^{-3}$ kg/mol exactly, $A_r(\mathrm{e})$ is the relative atomic mass of the electron ($u_r = 4.4 \times 10^{-10}$), $d_{220}$ is the 220 lattice spacing of an ideal silicon crystal in vacuum and at a temperature of 25 °C ($u_r = 3.6 \times 10^{-8}$), and $R_\infty$ is the Rydberg constant ($u_r = 6.6 \times 10^{-12}$). Values of $d_{220}$ are discussed further below in connection with the $R_K$ figure. In regard to that discussion, it should be noted that the NMIJ-97 value of $d_{220}$ yields a value of $\alpha$ from the PTB determination of $h/m_n d_{220}$ in better agreement with other values of $\alpha$ than do the PTB-81 and IMGC-94 values of $d_{220}$, while the PTB-81 and IMGC-94 values of $d_{220}$ yield values of $K_J$ from $V_m(\mathrm{Si})$ in slightly better agreement with other values of $K_J$ than does the NMIJ-97 value of $d_{220}$.

Because of the inconsistency of the value of $V_{\mathrm{m}}(\mathrm{Si})$ with the two watt-balance values of $K_{\mathrm{J}}^2 R_{\mathrm{K}}$, and to a somewhat lesser extent with the two direct measurements of $K_{\mathrm{J}}$, the *a priori* assigned uncertainty of each of these five data was weighted by the factor 2.325 in the final least-squares adjustment from which the 2002 CODATA recommended values were obtained in order to reduce the inconsistency to an acceptable level. Notwithstanding this difficulty, it is clear from the figures that the $40 \times 10^{-8}$ or 0.4 μV and the $10 \times 10^{-8}$ or 0.1 μΩ uncertainties discussed above remain quite reasonable, if not in fact somewhat conservative.

The next CODATA adjustment of the values of the constants will be the 2006 adjustment and the closing date for data will be 2006 December 31.

## References


[1] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **77**(1), 1-107 (2005).
[2] P. J. Mohr and B. N. Taylor, Rev. Mod. Phys. **72**(2), 351-495 (2000).


## Notes on the Figures

Those values of $R_{\mathrm{K}}$ that depend on QED and/or atomic physics theory, that is, the University of Washington 1987 (Uwash-87) value of $R_{\mathrm{K}}$ inferred from the value of $\alpha$ derived from the electron magnetic moment anomaly $a_{\mathrm{e}}$, and the Los Alamos Meson Physics Facility 1999 (LAMPF-99) value of $R_{\mathrm{K}}$ inferred from the value of the fine-structure constant $\alpha$ derived from the muonium ground-state hyperfine splitting $\Delta\nu_{\mathrm{Mu}}$, are based on the theory of $a_{\mathrm{e}}$ and $\Delta\nu_{\mathrm{Mu}}$ as it existed in the Fall of 2003. However, nothing of major significance has since occurred. In those cases where inexactly known constants are required to calculate $K_{\mathrm{J}}$ or $R_{\mathrm{K}}$ from the quantity actually measured, the 2002 CODATA recommended values of these constants are used. Nonetheless, in each case the uncertainty of the measured quantity exceeds the combined uncertainty of the required constants. It is, therefore, reasonable to expect that any future changes in the recommended values of these constants will have an insignificant impact on the derived values of $K_{\mathrm{J}}$ and $R_{\mathrm{K}}$, including their uncertainties, as presented in the figures.

***K*<sub></sub>J Figure.** The values are given in order of increasing standard uncertainty starting from the bottom of the figure, but it should be noted that none of the uncertainties include the multiplicative factor 2.325 discussed above. The two vertical dotted lines symmetric about the origin of the bottom scale indicate the relative standard uncertainty assigned by the CIPM in 1988 to a "perfectly" realized volt representation based on the Josephson effect and $K_{\mathrm{J-90}}$. The year given with the laboratory abbreviations is the year the result was published.

As noted above, the NIST-98 and NPL-90 values labeled $K_{\mathrm{J}}^2 R_{\mathrm{K}}$ are moving-coil watt balance results. They are calculated from the value of $h$ deduced from the relation $h = 4/K_{\mathrm{J}}^2 R_{\mathrm{K}}$ and the expression $K_{\mathrm{J}} = (8\alpha/\mu_0 ch)^{1/2}$ given earlier. The $N_{\mathrm{A}}$ P/N/I-03 value of $K_{\mathrm{J}}$ has been discussed in detail above. The NML-89 and PTB-91 values of $K_{\mathrm{J}}$ were measured directly and hence do not require the assumption $K_{\mathrm{J}} = 2e/h$ or the values of any other constants, as previously indicated. The NPL-79 and NIM-95 values of $K_{\mathrm{J}}$ are based on values of $h$ obtained indirectly from measurements of the proton gyromagnetic ratio by the high field method $\Gamma'_{\mathrm{p-90}}(\mathrm{hi})$ [the prime indicates that the protons, p, are in a spherical sample of pure $H_2O$ at 25 °C and the subscript 90 indicates that the value $\Gamma'_{\mathrm{p}}(\mathrm{hi})$ is measured in conventional electrical units, i.e., units based on the JE and QHE and the conventional values $K_{\mathrm{J-90}}$ and $R_{\mathrm{K-90}}$]. Similarly, the NIST-80 value of $K_{\mathrm{J}}$ is based on the value of $h$ obtained indirectly from a measurement of the Faraday constant $\mathcal{F}_{90}$. The equations that relate $h$ to $\Gamma'_{\mathrm{p-90}}(\mathrm{hi})$ and to $\mathcal{F}_{90}$ are

$$h = \frac{c\alpha^2 g_{\mathrm{e}}}{2K_{\mathrm{J-90}}R_{\mathrm{K-90}}R_\infty}\left(\frac{\mu_{\mathrm{e}}}{\mu'_{\mathrm{p}}}\right)^{-1}[\Gamma'_{\mathrm{p-90}}(\mathrm{hi})]^{-1} \tag{4}$$

$$h = \frac{cM_{\mathrm{u}}A_{\mathrm{r}}(\mathrm{e})\alpha^2}{K_{\mathrm{J-90}}R_{\mathrm{K-90}}R_\infty}(\mathcal{F}_{90})^{-1}, \tag{5}$$

where $g_{\mathrm{e}}$ is the electron $g$-factor ($u_{\mathrm{r}} = 3.8 \times 10^{-12}$) and $\mu_{\mathrm{e}}/\mu'_{\mathrm{p}}$ is the electron to shielded proton magnetic moment ratio ($u_{\mathrm{r}} = 1.1 \times 10^{-8}$—note that the prime has the same meaning as before).

***$R_{\mathrm{K}}$* Figure.** The first paragraph of the note on the $K_{\mathrm{J}}$ figure applies to the $R_{\mathrm{K}}$ figure, except that the two vertical dotted lines indicate the relative standard uncertainty assigned in 2000 September by the 22nd CCEM to a "perfectly" realized quantum Hall effect representation of the ohm. (As pointed out above, the 22nd CCEM reduced the original 0.2 μΩ uncertainty, corresponding to a relative standard uncertainty $u_{\mathrm{r}} = 20 \times 10^{-8}$, to 0.1 μΩ, corresponding to a relative standard uncertainty $u_{\mathrm{r}} = 10 \times 10^{-8}$.)

The Uwash-87 result for $R_{\mathrm{K}}$ is obtained via the relation $R_{\mathrm{K}} = \mu_0 c/2\alpha$ given above using the value of the fine-structure constant $\alpha$ implied by the experimental value of the electron magnetic moment anomaly $a_{\mathrm{e}}$ and its theoretical expression calculated from QED. The 2002 Stanford University (Stan-02) result for $R_{\mathrm{K}}$ is obtained from the value of $\alpha$ inferred from the atom-intereferometric measurement of $h/m(^{133}\mathrm{Cs})$ using the relation

$$\alpha = \left[\frac{2R_\infty A_{\mathrm{r}}(^{133}\mathrm{Cs})}{cA_{\mathrm{r}}(\mathrm{e})}\frac{h}{m(^{133}\mathrm{Cs})}\right]^{1/2}, \tag{6}$$

where $A_{\mathrm{r}}(^{133}\mathrm{Cs})$ is the relative atomic mass of the $^{133}\mathrm{Cs}$ atom and $m(^{133}\mathrm{Cs})$ is its mass. The four values labeled $R_{\mathrm{K}}$ are all based on direct calculable capacitor measurements, and hence, as noted above, they do not require the assumption $R_{\mathrm{K}} = h/e^2 = \mu_0 c/2\alpha$ nor the values of any other constants.

The PTB-99/NIMJ-97 value of $R_{\mathrm{K}}$ is calculated from the value of $\alpha$ deduced from the NIMJ-97 $d_{220}$ silicon lattice spacing measurement and the PTB-99 result for $h/m_{\mathrm{n}}d_{220}$ using the relation

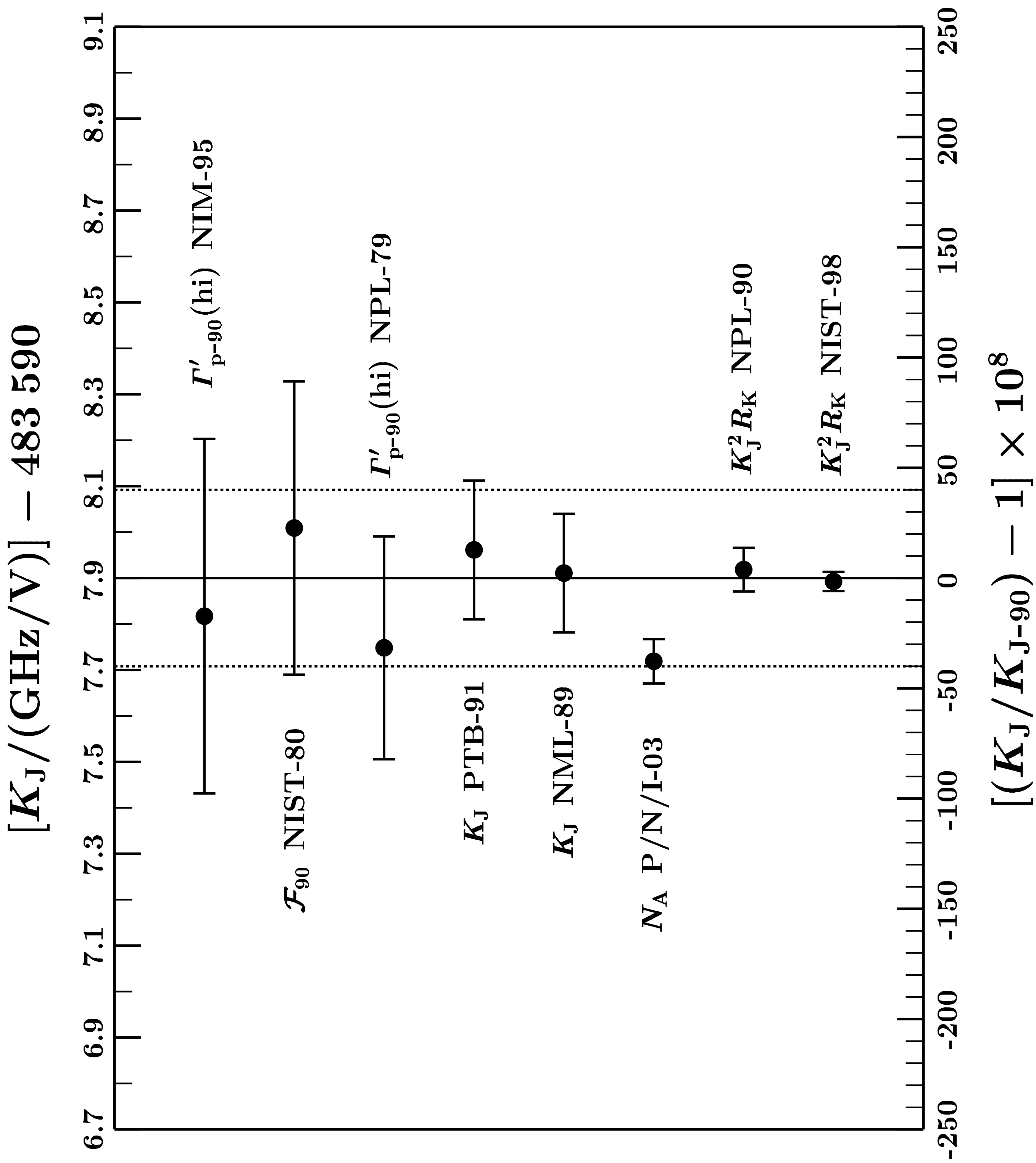
$$\alpha = \left[\frac{2R_\infty A_{\mathrm{r}}(\mathrm{n})d_{220}}{cA_{\mathrm{r}}(\mathrm{e})}\frac{h}{m_{\mathrm{n}}d_{220}}\right]^{1/2}, \tag{7}$$

where $A_r(\mathrm{n})$ is the relative atomic mass of the neutron and $m_n$ is the mass of the neutron. At the time of the 2002 adjustment, the PTB-81 and IMGC-94 values of $d_{220}$ were in question as a result of measurements carried out at IMGC and NIMJ, and thus were omitted from consideration. Subsequent work has shown that the reason for doubting those values was apparently unjustified. It is hoped that measurements underway will shed light on the cause of the difference between the NIMJ-97 value of $d_{220}$ and the PTB-81 and IMGC-97 values. The latter two agree but each yields a value of $R_K$ through Eq. (7) nearly equal to the location of the vertical right-hand dotted line, with uncertainties comparable to that of the PTB-99/NIMJ-97 value of $R_K$.

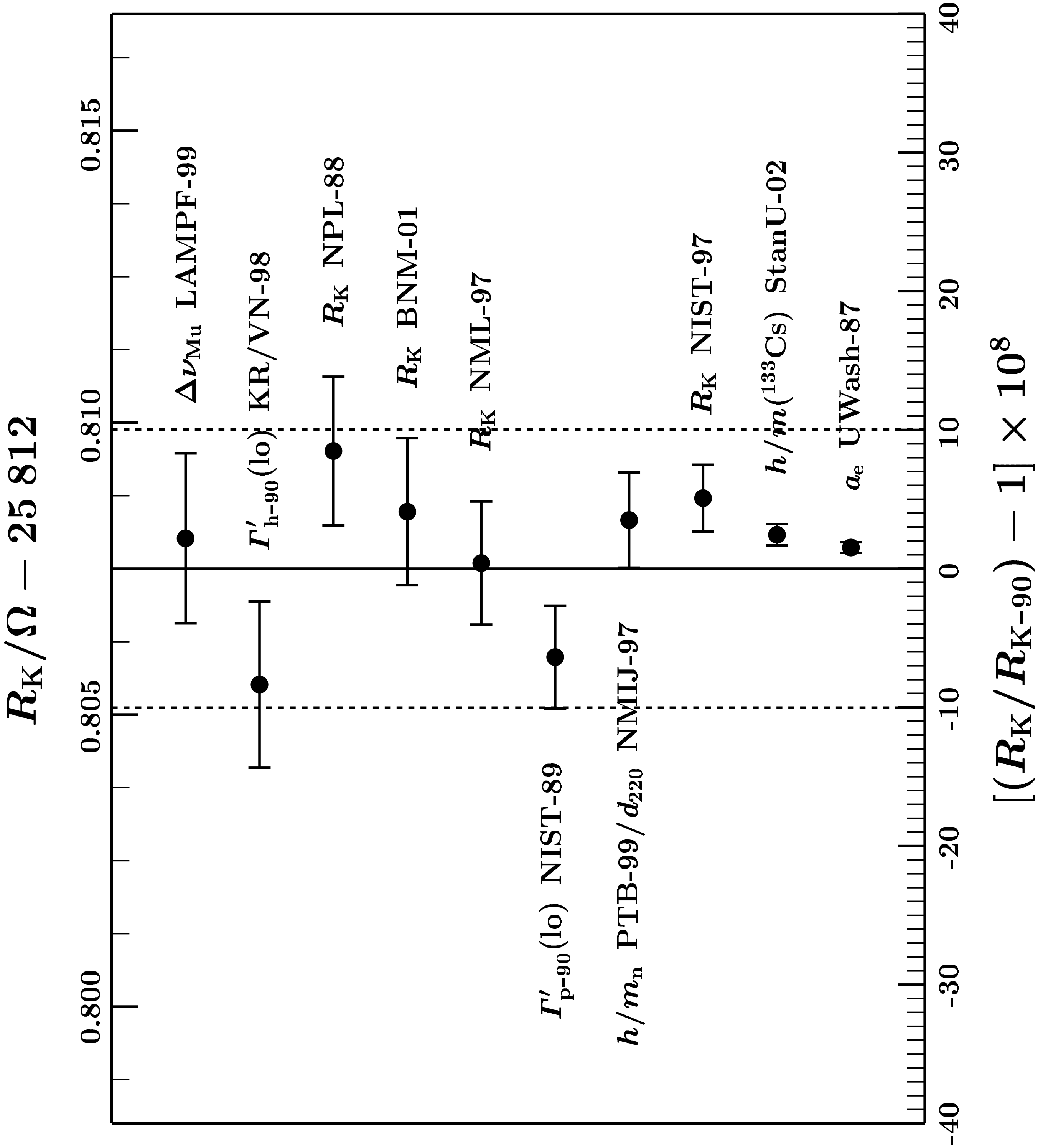
The NIST-89 result for $R_K$ follows from the value of $\alpha$ inferred from the determination of the proton gyromagnetic ratio by the low-field method $\Gamma'_{\mathrm{p-90}}(\mathrm{lo})$. The KRISS/VNIIM 1998 (KR/VN-98) result for $R_K$ follows similarly from the determination of the gyromagnetic ratio of the helion h (nucleus of the $^3$He atom) by the low-field method $\Gamma'_{\mathrm{h-90}}(\mathrm{lo})$. The LAMPF-99 result for $R_K$ is obtained from the value of $\alpha$ deduced from measurements of the muonium ground-state hyperfine splitting and related Zeeman transition frequencies carried out at LAMPF and the theoretical expression for the splitting based on atomic physics and QED, as mentioned above. The expression relating $\alpha$ to $\Gamma'_{\mathrm{p-90}}(\mathrm{lo})$ is

$$\alpha = \left( \frac{4\mu_0 R_\infty \Gamma'_{\mathrm{p-90}}(\mathrm{lo})}{K_{\mathrm{J-90}} R_{\mathrm{K-90}} g_e} \frac{\mu_\mathrm{e}}{\mu'_\mathrm{p}} \right)^{1/3} , \tag{8}$$

and that relating $\alpha$ to $\Gamma'_{\mathrm{h-90}}(\mathrm{lo})$ is the same but p is replaced by h, i.e., the proton is replaced by the helion.



$[K_J/(\mathrm{GHz/V})] - 483\,590$
$[(K_J/K_{J\text{-}90}) - 1] \times 10^8$
$\Gamma'_{p\text{-}90}(\mathrm{hi})$ NIM-95
$\mathcal{F}_{90}$ NIST-80
$\Gamma'_{p\text{-}90}(\mathrm{hi})$ NPL-79
$K_J$ PTB-91
$K_J$ NML-89
$N_A$ P/N/I-03
$K_J^2 R_K$ NPL-90
$K_J^2 R_K$ NIST-98
6.7
6.9
7.1
7.3
7.5
7.7
7.9
8.1
8.3
8.5
8.7
8.9
9.1
-250
-200
-150
-100
-50
0
50
100
150
200
250



$R_K/\Omega - 25\,812$
0.800
0.805
0.810
0.815
$\Delta\nu_{Mu}$ LAMPF-99
$\Gamma'_{h-90}$(lo) KR/VN-98
$R_K$ NPL-88
$R_K$ BNM-01
$R_K$ NML-97
$\Gamma'_{p-90}$(lo) NIST-89
$h/m_n$ PTB-99/$d_{220}$ NMIJ-97
$R_K$ NIST-97
$h/m(^{133}Cs)$ StanU-02
$a_e$ UWash-87
-40
-30
-20
-10
0
10
20
30
40
$[(R_K/R_{K-90}) - 1] \times 10^8$