Are counting losses due to input or output events ?

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The question raised in the title seems to belong to those queries which have no clear answer because they are ill posed. Indeed, pulses do not produce losses; rather, it is the dead times following them which may be responsible for such an effect. Is it meaningful, however, to then ask whether the dead times are due to input or output events, knowing that the latter cannot exist without the former ones? Have we perhaps fallen, quite naively, into one of the traps set by questions of the chicken-and-egg type, the resolution of which, if possible at all, has no influence whatsoever on observable quantities?

Some readers may be quick to remind us that the association of dead times with events is a matter which depends on the type considered. Indeed, non-extended dead times are known to follow the output events whereas extended ones are initiated (or renewed) by each incoming pulse. This is true by definition, but it does not imply, as far as the losses are concerned, that all these "dead periods" also produce (independently) counting losses, for there will often be significant overlap.

The problem discussed here has turned up in the context of a unified treatment of counting losses produced by a generalized dead time (with parameters τ and θ). There then arises for practical purposes the need to make a decision, within the adopted model, as to which pulses the losses are best thought to be associated with.

Some practical examples may help to clarify the situation. The simplest case corresponds to a dead time of the non-extended type where the losses are associated with the output pulses. For the sake of simplicity, let us assume in all that follows a Poisson input, with count rate ρ. Since the density of the pulses then eliminated is ρ, the loss per output event is $\rho\tau$, which then leads for the loss in count rate to

$$L = R\,\rho\tau\,, \tag{1}$$

where R is the output rate.

By means of the count-rate balance, which simply says that

$$R = \rho - L\,, \tag{2}$$

we can readily confirm that the output rate is now indeed given by

$$R = \frac{\rho}{1 + \rho\tau}\,. \tag{3}$$

Can we also describe the losses by associating a non-extended dead time with the input events? In principle this could certainly be done, but we then have to apply instead of τ a shorter "effective" dead time τ_{in}. As the losses are now

$$L = \rho \cdot \rho \tau_{in} , \tag{4}$$

a comparison with (1) and (3) shows that we have to put

$$\tau_{in} = \frac{R}{\rho} \tau = \frac{\tau}{1 + \rho\tau} . \tag{5}$$

This effective dead time τ_{in}, when used in (4) and (2), leads to the correct output rate R given by (3), but the underlying physics is less obvious. The quantity τ_{in} is of a rather artificial nature and cannot be measured directly. Since incoming pulses may follow each other by a time interval shorter than τ_{in}, superposition of effective dead times will inevitably occur. This shows that τ_{in} is only an average value; the effective length of such overlapping periods may be much longer and depends on the exact time distribution of the pulse arrivals.

Let us now consider a dead time of the extended type (and nominal length τ). It is usual (and justified) to associate it with the incoming events. However, this does not allow us to conclude that the losses thereby produced can again be described by a formula analogous to (1), with R replaced by ρ. Rather, we are obliged to introduce also for this situation an effective dead time τ_{in} such that

$$L = \rho^2 \tau_{in} ,$$

as in (4). Using the fact that the output rate for this case is known to be

$$R = \rho e^{-\rho\tau} = \rho - L , \tag{6}$$

we can see readily that now

$$\tau_{in} = \frac{1}{\rho} (1 - e^{-\rho\tau}) . \tag{7}$$

Likewise, the association of an extended dead time with the output pulses requires that

$$\tau_{out} = \frac{1}{\rho} (e^{\rho\tau} - 1) . \tag{8}$$

Finally, we may also consider the case of a generalized dead time, where the output rate is given by the Takács formula

$$R = \frac{\theta\rho}{e^{\theta\rho\tau} + \theta - 1} . \tag{9}$$

By the same approach as used above we can again determine the effective dead times, thought to be associated with either the input or the output pulses. The result is

$$\tau_{in} = \frac{1}{\rho} \left(\frac{e^{\theta\rho\tau} - 1}{e^{\theta\rho\tau} + \theta - 1} \right) < \tau \tag{10}$$

and

$$\tau_{out} = \frac{1}{\theta\rho} (e^{\theta\rho\tau} - 1) > \tau \; . \tag{11}$$

The previous formulae can be recovered from (10) and (11) as the special cases corresponding to $\theta = 0$ or 1.

What can we conclude from all this? At first sight not too much, for it is apparently always possible - at least in a formal way - to associate dead times which produce counting losses either with the input or with the output pulses. Yet, one will notice two advantages resulting from the second choice. The first is a minor and somewhat formal one, namely the fact that in this case we are normally led to simpler expressions for the effective dead time. Of more relevance is the observation made before that the association with output events corresponds to a well-defined physical situation where the dead times do not overlap and the corresponding losses are therefore independent. The effective length is normally not constant (except for $\theta = 0$, the non-extended case), but this is clearly in the nature of the processes considered and cannnot be avoided. In addition, all periods of paralysis are initiated by output pulses. For the general case it therefore seems quite natural to associate dead times with them. From the experimental point of view it is relevant to note that this corresponds to a situation where all the quantities of interest can be measured directly. In all cases, the minimum duration of the paralysis is equal to τ, i.e. the nominal value of the dead time.

A rather different and probably more useful way to look at this problem of association can be obtained if we consider the output rate R as an unknown quantity. In this case, (9) or similar expressions corresponding to more complicated experimental situations (e.g. series arrangements of dead times) are not available, but should be derived. Then a possible starting point is always given by (2), with

$$L = R \rho' \tau_{out} , \tag{12}$$

where ρ' is a quantity which may (or may not) be identical with the initial pulse density ρ, depending on the situation considered. For an undistorted Poisson input we always have $\rho' = \rho$. Obviously, the determination of R now calls for an independent evaluation of τ_{out}, and this has to be based on a detailed analysis of the process by which "normal" dead times of length τ superimpose to produce the effective paralysis time τ_{out}.

It is one of the many services rendered by William Feller to counting statistics to have recognized and carefully studied exactly this problem already forty years ago in a paper which has become a landmark in this field. He assumed an incoming Poisson process and analyzed in detail (for an extended dead time) the duration of paralysis, which he recognized as the outcome of a special renewal process. From the Laplace transform of the probability density for the "effective" dead time, the corresponding mean value τ_{out} is readily obtained. It is most fortunate that this approach can be easily extended to the case of a generalized dead time, as will be shown in a separate report.

Let us quickly come back to the initial question and try to sum up the insight gained. The association of counting losses with pulses is to some extent an arbitrary one. However, we have seen that both physical insight and mathematical simplicity favour the idea to have the losses coupled to the output events. Yet, the best support for this choice is provided by the availability of the "Feller mechanism" which allows us in an elegant way to determine τ_{out}, from which the output rate R can then be easily obtained by use of the count-rate balance. No equivalent general procedure is known for an association with the input pulses.

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