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Interval densities for extended dead times

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1. Introduction

Dead times of the extended (or paralyzable) type, where every incoming pulse, no matter whether counted or not, is followed by a time interval of length τ in which no further registration is possible, are not very popular. This may be due to the fact that they always give rise to larger losses than would result from the same dead times of the non-extended type. This is directly connected with their cumulative character, which can give rise to very long periods of total paralysis. Furthermore, their statistical behaviour is in general much more difficult to handle mathematically. Whenever there is a choice between these two types, therefore, as is the case, for example, when a dead time is inserted electronically into a series of pulses, the simpler non-extended type with the smaller correction would probably be preferred.

However, the type of dead time may be imposed by the physical nature of certain parts in the electronics. In particular, automatic pile-up-suppression, as it is commonly used today in window-amplifiers for Ge(Li) detectors, imposes a dead time which is, to a very good approximation, of the extended type. This can give rise to some serious problems [1] .

In order to be able to deal with such cases, a good knowledge of the interval distributions which correspond to extended dead times is indispensable. Such problems, as a rule [2] , can be considered as special cases of the theory of renewal processes which in turn are fully described by the probability density of the time interval between successive events.

A most instructive way to find the interval density (or its Laplace transform) for the special, but very important case where the original sequence of events forms a Poisson process has been described by Feller [2]. *) This process is known to have some very particular features. Especially noteworthy among them is the independence of the expected interval from the time elapsed since the last event. As a result of this complete "lack of memory", the time origin in a Poisson process can be chosen at will. This property, which is unique for the Poisson process, has been essentially used in Feller's elegant derivation. It is not evident, however, whether the reasoning along his lines can be generalized to permit other applications.

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*) His reasoning may also be found in [3] , partly with the same misprints in the formulae, where a more general type for the dead time is also treated.

2. Another way to arrive at interval densities

An attempt will be made here to arrive at an expression which allows the calculation of interval densities after imposing an extended dead time (see fig. 1). Our problem thus consists of finding the resulting interval density ${}_2f(t)$.

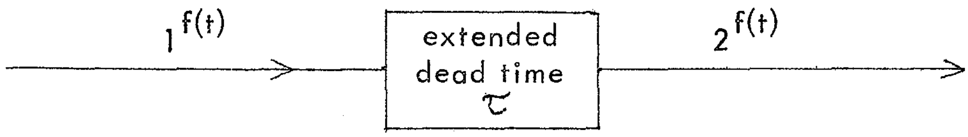

Fig. 1 - Notation for the interval densities

We denote by

${}_1f(t)$ the interval density in the original renewal process, and by

${}_2f(t)$ the interval density in the process resulting after insertion of an extended dead time τ .

For both sequences which are - as a result of the independence of the successive events - of the renewal type, we may write for the density of a k-fold interval

$$f_k(t) = \left\{ f(t) \right\}^{*k} , \qquad k = 1, 2, \ldots, \tag{1}$$

and for the total density

$$D(t) = \sum_{k=1}^{\infty} f_k(t) . \tag{2}$$

We now assume that at the time $t = 0$ a pulse has been registered. Let us determine the interval density to the next event.

According to the definition of an extended dead time, the occurrence of any output pulse for $t > \tau$ requires two conditions to be fulfilled simultaneously, namely

- that there has been an original pulse at time t , and
- that there has been no original pulse within the preceding interval from $t-\tau$ to t, otherwise t would have been overlapped by its dead time.

Because $t-\tau$ cannot be negative, we replace it by the new quantity

$$t_o \equiv \mathrm{Max}\,(t-\tau,\ 0)$$

which assures, since $t \geqslant 0$, that the whole interval considered is after $t=0$.

This now leads us directly to the fundamental equation

$$_2D(t) = U(t_o) \cdot {_1D(t)} \cdot {_1W_o(t_o, t)} , \tag{3}$$

where

$_1W_o(t_o, t)$ is the probability that there is no pulse in the original sequence between t_o and t, when an event has occurred at $t = 0$, and

$U(t_o) \equiv \begin{cases} 0 & \text{for } t_o < 0 \\ 1 & \text{" } t_o > 0 \end{cases}$ is the unit step function.

Since we know in advance that $_2f_k(t)$ will vanish for $t < \tau$, and therefore also $_2D(t)$, it is in general sufficient to consider the region $t > \tau$, where $t_o = t - \tau$.

The desired density $_2f(t)$ can only be extracted in a simple way from (3) if $_1W_o(t_o, t)$ is independent of t, which in general is not the case. The explicit calculation of $_1W_o$ will be discussed in the next chapter.

The problem of determining $_2f(t)$ is particularly simple for the important special case where the original sequence forms a Poisson process. We then have

$$_1f(t) = U(t) \cdot e^{-\rho t} , \tag{4}$$

and thus

$$_1f_k(t) = U(t) \cdot \frac{\rho(\rho t)^{k-1}}{(k-1)!} \cdot e^{-\rho t} , \tag{5}$$

therefore

$$_1D(t) = U(t) \cdot \rho \cdot e^{-\rho t} \cdot \sum_{k=0}^{\infty} \frac{(\rho t)^k}{k!} = U(t) \cdot \rho . \tag{6}$$

Since the time origin can be arbitrarily chosen in this case, we get for $t > \tau$

$$_1W_o(t_o, t) = {_1W_o(t-\tau, t)} = {_1W_o(0, \tau)} = e^{-\rho\tau} . \tag{7}$$

According to (3), the total output density for an original Poisson process is therefore with (6) and (7)

$$_2D(t) = U(t-\tau) \cdot \rho \cdot e^{-\rho\tau} . \tag{8}$$

But for the transformed total densities (2), the following relation always holds

$$\tilde{D}(s) = \sum_{k=1}^{\infty} \tilde{f}_k(s) = \sum_{k=1}^{\infty} \tilde{f}^k(s) = \frac{\tilde{f}(s)}{1 - \tilde{f}(s)} , \tag{9}$$

from which we conclude that

$$\tilde{f}(s) = \frac{\tilde{D}(s)}{1 + \tilde{D}(s)} \quad . \tag{10}$$

But since for a Poisson process, according to (8), we have

$$_2\tilde{D}(s) = \frac{\rho}{s} \cdot e^{-(s+\rho)\tau} \quad ,$$

the transform of the desired output density is now determined from (10) as

$$_2\tilde{f}(s) = \frac{\rho}{\rho + \rho/_2\tilde{D}(s)} = \frac{\rho}{\rho + s \cdot e^{(s+\rho)\tau}} \quad . \tag{11}$$

This is identical with the result derived by Feller ([2], eq. 43) on the basis of his considerations concerning the total length of the effective dead times resulting from their cumulative nature.

3. Calculation of $_1W_o(t_o\ ,\ t)$

For the general case, the probability for no event in the interval under study is a bit harder to determine. In close analogy with the considerations discussed earlier in the context of non-extended dead times [4], we assume that exactly k pulses of the original series have arrived in the interval from 0 to $t-\tau$ (see fig. 2).

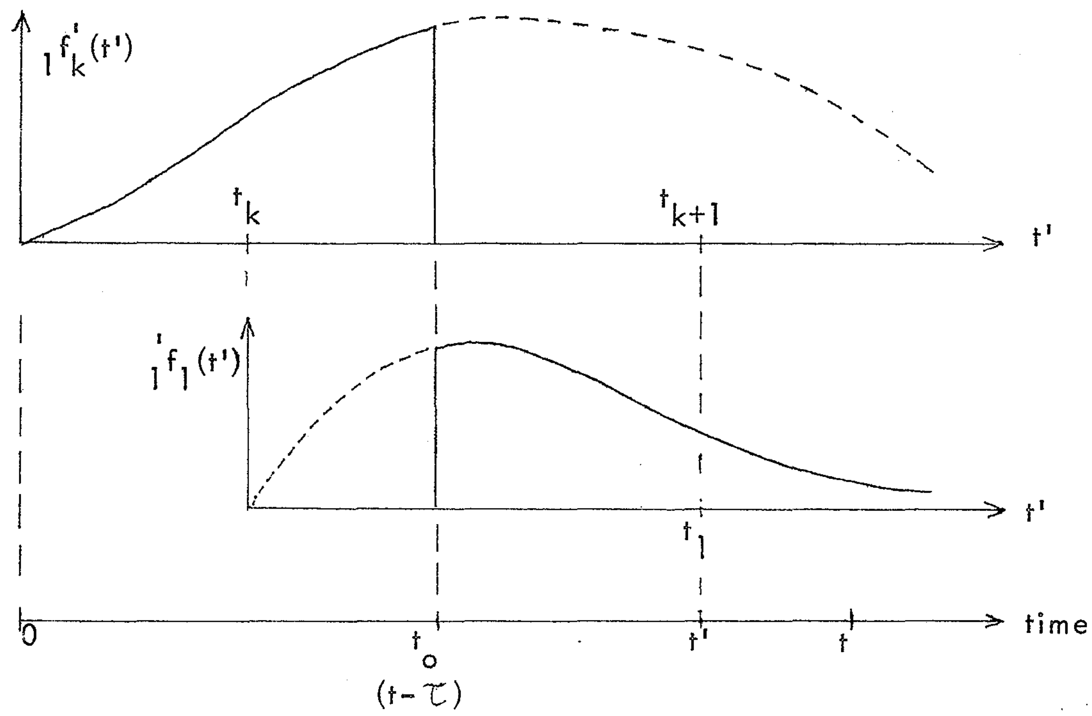

Fig. 2 - Schematic representation of the densities needed for deriving $_1W_o$

The probability density for the first pulse arriving after pulse number k is then given by

$$p_k(t') = \int_0^{\infty} {}_1f'_k(\alpha) \cdot {}_1'f_1(t' - \alpha)\, d\alpha \equiv {}_1f'_k(t') * {}_1'f_1(t') \quad , \tag{12}$$

where $\quad {}_1f'_k(t') = U(\tau - t') \cdot {}_1f_k(t') \quad$ and

$$ {}_1'f_1(t') = U(t' - [\tau - t_k]) \cdot {}_1f_1(t') \quad , \text{ for } t' > t_o \quad ,$$

with t_o as defined in (3).

Thus (12) may be written as

$$p_k(t') = \int_0^{t_o} {}_1f_k(\alpha) \cdot {}_1f_1(t' - \alpha)\, d\alpha \quad , \tag{13}$$

and the probability for finding this pulse within the interval (t_o, t) is given by

$$P_k(t) = \int_{t_o}^{t} p_k(t')\, dt' \quad . \tag{14}$$

Therefore, the required probability for no event within (t_o, t) is finally obtained as

$${}_1W_o(t_o, t) = 1 - \sum_{k=0}^{\infty} P_k(t) \quad . \tag{15}$$

An easy check for formula (15) is provided by the <u>Poisson process</u>. In this case, we expect ${}_1W_o$ to depend only on the length τ of the interval, but not on its position t (provided that $t > \tau$).

We now have

$${}_1f_k(t) = U(t) \cdot \frac{\rho(\rho t)^{k-1}}{(k-1)!} \cdot e^{-\rho t} \quad ,$$

and with (13) therefore

$$p_k(t') = \int_0^{t_o} \frac{(\rho\alpha)^{k-1}}{(k-1)!} \cdot e^{-\rho\alpha} \cdot \rho \cdot e^{-\rho(t'-\alpha)}\, d\alpha$$

$$= \frac{\rho^{k+1} \cdot e^{-\rho t'}}{(k-1)!} \cdot \int_0^{t_o} \alpha^{k-1}\, d\alpha = \frac{\rho(\rho t_o)^k}{k!} \cdot e^{-\rho t'} \quad .$$

This yields with (14), for the conditional probability that pulse k+1 arrives within (t_o, t)

$$P_k(t) = \frac{\rho(\rho t_o)^k}{k!} \cdot \int_{t_o}^{t} e^{-\rho t'}\, dt'$$

$$= \frac{(\rho t_o)^k}{k!} \cdot (e^{-\rho t_o} - e^{-\rho t}) ,$$

and accordingly the probability for any pulse in (t_o, t) is given by

$$\sum_{k=0}^{\infty} P_k(t) = (e^{-\rho t_o} - e^{-\rho t}) \cdot \sum_{k=0}^{\infty} \frac{(\rho t_o)^k}{k!}$$

$$= 1 - e^{-\rho(t-t_o)} .$$

For a Poisson process, the required probability for no event within (t_o, t) is thus given according to (15) by the relation

$${}_1W_o(t_o, t) = e^{-\rho(t-t_o)} , \tag{16}$$

where $t - t_o = \begin{cases} t & \text{for } 0 < t < \tau \\ \tau & \text{" } \; t > \tau . \end{cases}$

As we expected, (16) is for $t > \tau$ equal to the probability that there is no event in the time interval from 0 to τ in a Poisson process.

4. Interval densities for an original Poisson process

The aim of this chapter is to arrive at explicit formulae for the single and the multiple interval densities for the special case where the original process has been of the Poisson type. From a practical point of view, this is the most important application. The problem essentially consists of finding the originals of some Laplace transforms.

Since we shall always deal with the resulting distribution, the index "2" will now be dropped.

a) Single intervals - The transform $\tilde{f}(s)$ of the interval density, which results when a Poisson process with rate ρ has been modified by the insertion of an extended dead time τ, has been found earlier to be given by

$$\tilde{f}(s) = \frac{\rho \cdot e^{-(\rho+s)\tau}}{s + \rho \cdot e^{-(\rho+s)\tau}} . \tag{11'}$$

In order to find the corresponding original $f(t)$, we now interpret $\tilde{f}(s)$ as the sum of an infinite geometric series, i.e.

$$\tilde{f}(s) = x_o \cdot \sum_{j=0}^{\infty} x^j = \frac{x_o}{1 - x} \quad . \tag{17}$$

A comparison with (11') leads to

$$x = -\frac{\rho}{s} \cdot e^{-(\rho + s)\tau} \tag{18}$$

and $x_o = \frac{\rho}{s} \cdot e^{-(\rho+s)\tau} = -x$,

which permits us to write now

$$\tilde{f}(s) = -\sum_{j=1}^{\infty} x^j \quad . \tag{19}$$

Let us first determine the original corresponding to a single term x^j in (19):

$$\mathcal{L}^{-1}\left\{x^j\right\} = \mathcal{L}^{-1}\left\{(-\rho \cdot e^{-\rho\tau})^j \cdot \frac{e^{-j\tau s}}{s^j}\right\} \quad .$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s^j}\right\} = U(t) \cdot \frac{t^{j-1}}{(j-1)!} \quad , \qquad \text{for} \quad j = 1, 2, \ldots ,$$

the shift rule

$$\mathcal{L}^{-1}\left\{\tilde{f}(s) \cdot e^{-\alpha s}\right\} = U(t-\alpha) \cdot f(t-\alpha) \quad , \text{ for } \alpha \geqslant 0,$$

leads to the relation

$$\mathcal{L}^{-1}\left\{x^j\right\} = U(t-j\tau) \cdot (-\rho \cdot e^{-\rho\tau})^j \cdot \frac{(t-j\tau)^{j-1}}{(j-1)!} \quad . \tag{20}$$

This gives for the original density with (19)

$$f(t) = -\mathcal{L}^{-1}\left\{\sum_{j=1}^{\infty} x^j\right\} = -\sum_j U(t-j\tau) \cdot (-\rho \cdot e^{-\rho\tau})^j \cdot \frac{(t-j\tau)^{j-1}}{(j-1)!}$$

$$= \rho \sum_{j=1}^{\infty} U(T_j) \cdot \frac{(-T_j)^{j-1}}{(j-1)!} \cdot e^{-j\rho\tau}, \tag{21}$$

where $T_j \equiv \rho(t - j\tau)$.

The notation adopted here is as close as possible to that used previously (e.g. in [5]). The step function U ensures that $j\tau$ does not exceed t and thus in effect reduces the sum to a finite number of terms.

Let us briefly look at the behaviour of f(t). For the different time intervals, the contributions may be split up as follows

$$f(t) = \begin{cases} 0 & \text{for} \quad 0 < t \leq \tau \\ \alpha_1 & \quad " \quad \tau < t \leq 2\tau \\ \alpha_1 + \alpha_2 & \quad " \quad 2\tau < t \leq 3\tau \\ \alpha_1 + \alpha_2 + \alpha_3 & \quad " \quad 3\tau < t \leq 4\tau \\ \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 & \quad " \quad 4\tau < t \leq 5\tau \\ \quad \ldots & \quad \ldots \end{cases}$$

with
$$\alpha_1 = \rho \cdot \exp(-\rho\tau)$$
$$\alpha_2 = -\rho^2 \cdot (t-2\tau) \cdot \exp(-2\rho\tau)$$
$$\alpha_3 = \frac{1}{2}\rho^3 \cdot (t-3\tau)^2 \cdot \exp(-3\rho\tau)$$
$$\alpha_4 = -\frac{1}{6}\rho^4 \cdot (t-4\tau)^3 \cdot \exp(-4\rho\tau)$$
$$\ldots$$

We therefore expect f(t) to be constant within the interval $\tau < t \leq 2\tau$ and to decrease linearly between 2τ and 3τ to a fraction $1 - \rho\tau \cdot \exp(-\rho\tau) \cong 1 - \rho\tau$ of its initial value.

This theoretical shape of the interval density f(t) is very well confirmed by direct experimental measurements, as can be seen from fig. 3a.

b) Multiple intervals - As has been mentioned in chapter 2 already, the "history" of the process prior to the last interval has no influence whatsoever on the occurrence of the later events. Therefore, the contributions to multiple intervals are independent, and the density for a k-fold interval is given by the k-fold self-convolution of f, as stated in (1). For the transform, this results in

$$\tilde{f}_k(s) = \tilde{f}^k(s) \quad , \qquad \text{for} \quad k \geq 1 \quad ,$$

where f_1 is identified with f.

For our original Poisson process, this leads with (17) to

$$\tilde{f}_k(s) = \left(\frac{-x}{1-x}\right)^k \quad . \tag{22}$$

Using the relation (compare e.g. [6], p. 10)

$$\frac{1}{(1-x)^k} = \sum_{j=0}^{\infty} \binom{k+j-1}{j} \cdot x^j \quad , \qquad \text{for} \quad x^2 < 1 \quad , \tag{23}$$

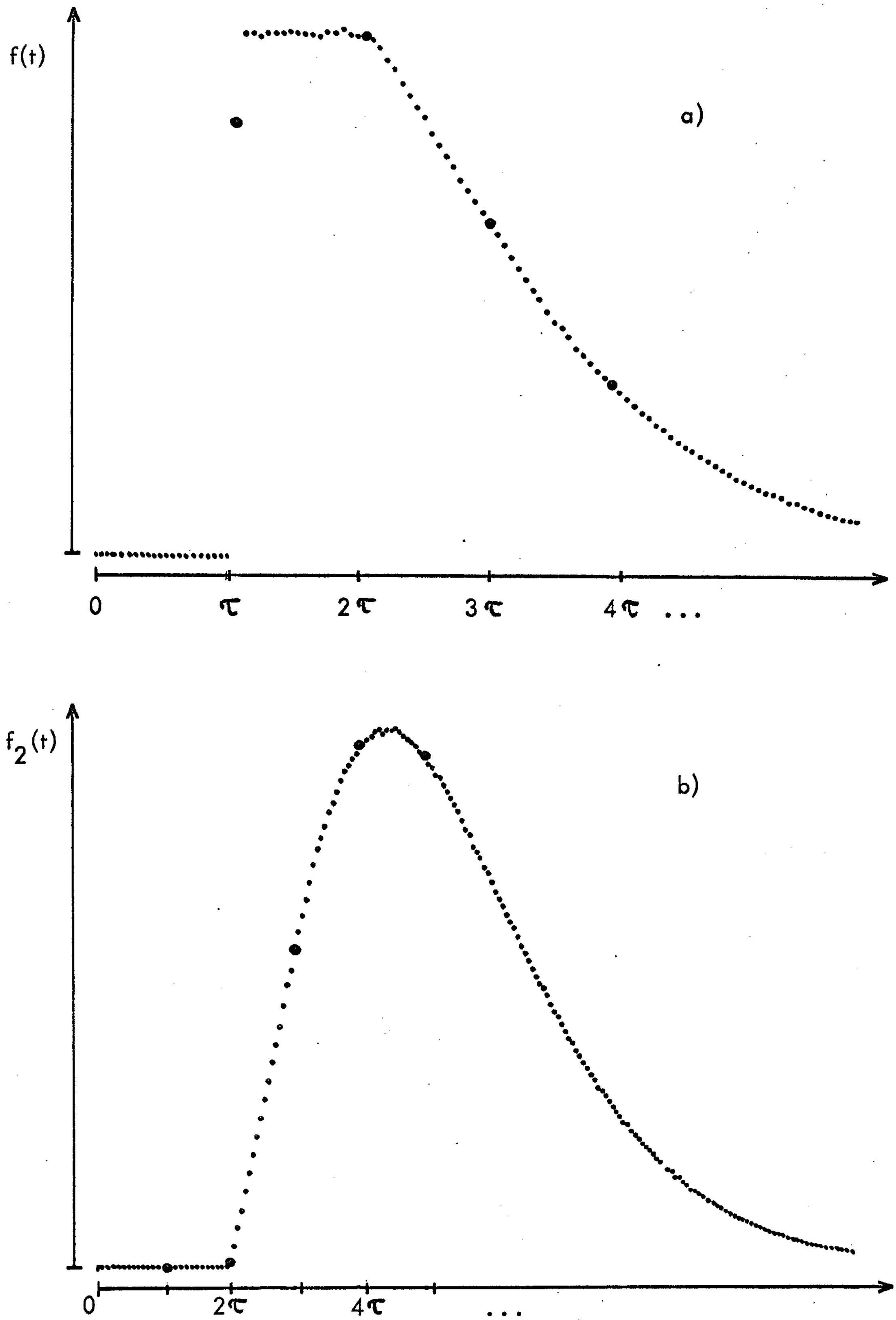

Fig. 3 - Experimental determination of interval densities for an extended dead time (for $\rho \cong 2\,000\ s^{-1}$, $\tau \cong 400\,\mu s$).

The accentuated points lie at $t = \tau,\ 2\tau,\ 3\tau,\ \ldots$

a) Single interval density (measuring time 5 h)

b) Double interval density (measuring time 12 h).

this can be written as

$$\tilde{f}_k(s) = (-1)^k \sum_{j=0}^{\infty} \binom{k+j-1}{j} \cdot x^{k+j} \quad . \tag{24}$$

But according to (20), we have the correspondence

$$\mathcal{L}^{-1}\left\{x^{k+j}\right\} = U(t-\left[k+j\right]\tau) \cdot (-\rho \cdot e^{-\rho\tau})^{k+j} \cdot \frac{(t-\left[k+j\right]\tau)^{k+j-1}}{(k+j-1)\,!}$$

$$= U(T_{k+j}) \cdot (-1)^{k+j} \cdot \rho \cdot \frac{T_{k+j}^{k+j-1}}{(k+j-1)\,!} \cdot e^{-(k+j)\rho\tau} \quad . \tag{25}$$

Therefore, (24) now turns out to be

$$f_k(t) = \sum_{j=0}^{\infty} (-1)^j \cdot \binom{k+j-1}{j} \cdot U(T_{k+j}) \cdot \rho \cdot \frac{T_{k+j}^{k+j-1}}{(k+j-1)!} \cdot e^{-(k+j)\rho\tau} .$$

Finally, with the new summation index $j' = k+j$ and by applying the well-known identity

$$\binom{n}{k} \equiv \binom{n}{n-k}$$

for the binomial coefficients, the density for the k-fold interval can also be written in the form (the prime in j' is dropped again)

$$f_k(t) = \rho \cdot (-1)^{k-1} \sum_{j=k}^{J} \cdot \binom{j-1}{k-1} \cdot \frac{(-T_j)^{j-1}}{(j-1)!} \cdot e^{-j\rho\tau} , \tag{26}$$

where $k = 1, 2, 3, \ldots,$ and $J = \left[\left[t/\tau\right]\right]$ is the largest integer below t/τ . For $J < k$, the density $f_k(t)$ should be taken as zero. It is easy to verify that $k=1$ brings us back to (21).

In particular, for the density of <u>double intervals</u>, (26) gives readily

$$f_2(t) = \begin{cases} 0 & \text{for} \quad 0 < t \leq 2\tau \\ \beta_2 & \text{"} \quad 2\tau < t \leq 3\tau \\ \beta_2 + \beta_3 & \text{"} \quad 3\tau < t \leq 4\tau \\ \beta_2 + \beta_3 + \beta_4 & \text{"} \quad 4\tau < t \leq 5\tau \\ \beta_2 + \beta_3 + \beta_4 + \beta_5 & \text{"} \quad 5\tau < t \leq 6\tau \\ \ldots & \qquad \ldots \end{cases}$$

$$
\begin{aligned}
\text{with}\quad \beta_2 &= \rho^2 \cdot (t-2\tau) \cdot \exp(-2\rho\tau) \\
\beta_3 &= -\rho^3 \cdot (t-3\tau)^2 \cdot \exp(-3\rho\tau) \\
\beta_4 &= \frac{1}{2}\rho^4 \cdot (t-4\tau)^3 \cdot \exp(-4\rho\tau) \\
\beta_5 &= -\frac{1}{6}\rho^5 \cdot (t-5\tau)^4 \cdot \exp(-5\rho\tau) \\
&\ldots
\end{aligned}
$$

This density also, which rises linearly in the interval $2\tau \leq t \leq 3\tau$, is well verified experimentally (fig. 3b).

In general, the k-fold interval density (26) is described by a polynomial in t, the degree of which depends on the range of t itself. In particular, for the first time interval after the dead time of total length $k\tau$, i.e. in the range $k\tau < t \leq (k+1)\tau$, the probability density is given by

$$
f_k(t) = \rho^k \cdot \frac{(t-k\tau)^{k-1}}{(k-1)!} \cdot e^{-k\rho\tau} \quad . \tag{27}
$$

If the interval chosen is characterized by the quantity J as defined above, the degree of the polynomial in this interval is determined by

$$
G = \begin{cases} 0 & \text{for} \quad J < k \\ J-1 & \text{"} \quad J \geq k \end{cases} \quad . \tag{28}
$$

5. Determination of the moments

In what follows, we shall restrict ourselves to determining the first few moments. Nevertheless, the method of calculating them by means of their transforms is quite general and moments of higher order may, if needed, be obtained along the same lines.

We realize, of course, that on the grounds on (1) it would be sufficient to determine the moments for a single interval, since their combination for multiple convolutions is well known. For checking purposes, however, we prefer to derive here the relevant expressions directly for the multiple intervals.*)

As is well known, the ordinary moments $m_r(t)$ of a random variable t can be obtained directly from the transformed density $\tilde{f}(s)$ by differentiation, since

$$
m_r(t) = (-1)^r \cdot \left.\frac{d^r\tilde{f}(s)}{ds^r}\right|_{s=0} \quad . \tag{29}
$$

Instead of the sum (19), we derive from (22) another form which is equivalent, but more useful for the present purpose, namely

*) Those readers who are only interested in a simple derivation may always assume k=1 in what follows, avoiding thereby some of the more troublesome transformations.

$$\tilde{f}_k(s) = \left(\frac{-x}{1-x}\right)^k = \left(\frac{1}{1-\frac{1}{x}}\right)^k = \left(\sum_{j=0}^{\infty} x^{-j}\right)^k \quad . \tag{30}$$

Inserting x, as defined by (18), leads to

$$\tilde{f}_k(s) = \left[\sum_{j=0}^{\infty} \left(\frac{-s}{\rho}\right)^j \cdot e^{j(\rho+s)\tau}\right]^k$$

$$= \left[\sum_{j=0}^{\infty} (-R)^j \cdot s^j \cdot \left(1 + sj\tau + s^2 \frac{j^2\tau^2}{2} + \ldots\right)\right]^k , \tag{31}$$

where $R \equiv \frac{1}{\rho} \cdot e^{\rho\tau}$.

If we restrict ourselves to the moments up to order r=3, a power series expansion of (31) as far as s^3 is sufficient. This first leads us to

$$\tilde{f}_k(s) = \Big[1 - Rs \cdot (1 + s\tau + s^2\tau^2/2 + \ldots)$$

$$+ (Rs)^2 \cdot (1 + 2s\tau + \ldots) - (Rs)^3 \cdot (1 + \ldots)\Big]^k$$

$$= \left[1 - s \cdot R + s^2 \cdot R(R-\tau) - s^3 \cdot R(R^2 - 2R\tau + \tau^2/2) + \ldots\right]^k \quad .$$

Applying the results (A5) derived in the Appendix with

$$a_1 = -R$$

$$a_2 = R \cdot (R - \tau)$$

$$a_3 = -R \cdot (R^2 - 2R\tau + \tau^2/2) \quad ,$$

the transform can be reduced to the following power series in s:

$$\tilde{f}_k(s) = 1 - s \cdot kR + s^2 \cdot kR\left(\frac{k+1}{2}R - \tau\right)$$

$$- s^3 \cdot kR\left[\frac{(k-1)(k-2)}{6}R^2 + (k-1)R(R-\tau) + \left(R^2 - 2R\tau + \frac{\tau^2}{2}\right)\right] .$$

From this and (29), we finally obtain for the moments

$$\begin{aligned} m_o(t) &= 1 \\ m_1(t) &= k \cdot R \\ m_2(t) &= kR \cdot \left[(k+1)R - 2\tau\right] \\ m_3(t) &= kR \cdot \left[(k^2+3k+2)R^2 - 6(k+1)R\tau + 3\tau^2\right] \quad . \end{aligned} \tag{32}$$

For the central moments, this gives

$$\begin{aligned} \mu_2(t) &\equiv m_2(t) - m_1^2(t) = kR\ (R - 2\tau) \\ \mu_3(t) &\equiv m_3(t) - 3\ m_1(t) \cdot m_2(t) + 2\ m_1^3(t) \\ &= kR \cdot (2 \cdot R^2 - 6 \cdot R\tau + 3 \cdot \tau^2) \quad . \end{aligned} \tag{33}$$

This result for the multiple intervals confirms that the central moments up to third order are additive for convolutions, as we would have expected (cf. e.g. [7], p. 88).

A comparison with the corresponding interval distribution for a non-extended dead time is quite instructive. For an original Poisson process with rate ρ, the single interval distribution is known to be [4] a simple shifted exponential

$$\Big[f(t)\Big]_{non} = U(t - \tau) \cdot \rho \cdot e^{-\rho(t-\tau)} \quad . \tag{34}$$

The respective results for some moments and combinations of them for the two types of dead times are collected in table 2.

type of dead time τ	m_1	μ_2	μ_3	μ_2/m_1^2	μ_3^2/μ_2^3
non-extended	$\frac{1}{\rho} + \tau$	$1/\rho^2$	$2/\rho^3$	$(1+\rho\tau)^{-2}$	4
extended	R	$R(R-2\tau)$	$R(2R^2-6R\tau+3\tau^2)$	$1 - \frac{2\tau}{R}$	$\frac{(2R^2-6R\tau+3\tau^2)^2}{R(R-2\tau)^3}$

Table 2 - Comparison of some characteristic data of the interval distributions, with $R = \frac{1}{\rho} \exp(\rho\tau)$

It is interesting to note that although the first three moments given above are all larger in the extended than in the non-extended case, the opposite is true for the relative variance μ_2/m_1^2 and for the skewness μ_3^2/μ_2^3. By elementary, but sometimes lengthy series expansions, the following results for the differences may be derived

$$\begin{aligned} (m_1)_{ext} - (m_1)_{non} &\cong x^2/2\rho \\ (\mu_2)_{ext} - (\mu_2)_{non} &\cong x^3/(3\rho^2) \\ (\mu_3)_{ext} - (\mu_3)_{non} &\cong x^4/(4\rho^3) \\ (\mu_2/m_1^2)_{ext} - (\mu_2/m_1^2)_{non} &\cong -x^2 \\ (\mu_3^2/\mu_2^3)_{ext} - (\mu_3^2/\mu_2^3)_{non} &\cong -4x^3 \quad , \end{aligned} \tag{35}$$

where only the first non-vanishing term in a series expansion for the dimensionless quantity $x \equiv \rho\tau$ is given.

In particular, of course, the expressions for $m_1(t)$ also lead us to the well-known correction formulae for the count rates. Since the output count rate r is given by

$$r = 1/m_1 \quad ,$$

we obtain

- for a non-extended dead time τ :

$$r_{non} = \frac{1}{1/\rho + \tau} = \frac{\rho}{1 + \rho\tau} \quad , \tag{36}$$

- for an extended dead time τ :

$$r_{ext} = 1/R = \rho \cdot e^{-\rho\tau} . \tag{37}$$

APPENDIX

1. Coefficients for an expanded multinomial

We consider the polynomial (k = 0, 1, 2, ...)

$$\begin{aligned} P(x) &= (1 + a_1 x + a_2 x^2 + \dots + a_i x^i + \dots)^k \\ &= 1 + A_1 x + A_2 x^2 + \dots + A_n x^n + \dots \quad . \end{aligned} \tag{A1}$$

The problem now consists of determining the coefficients A_n of the resulting power series in x. In the reference manuals we have readily at hand here, no general expression of the form

$$A_n = f(a_1, a_2, \dots, a_n ; k)$$

could be found, relating the new with the old coefficients a_i, which are supposed to be known.

Since the problem is seemingly of an elementary nature and may easily turn up again in other circumstances, we decided to determine explicit expressions for the first few coefficients A_n.

For this purpose, let us compare (A1) with the general multinomial

$$Q = (y_o + y_1 + y_2 + \dots + y_m)^k$$

$$= \sum_{(r_i)} \binom{k}{r_1 r_1 \dots r_m} \prod_{i=0}^{m} y_i^{r_i} \quad , \tag{A2}$$

where the sum has to be extended over all combinations (r_i) for which the condition

$$\sum_i r_i = k$$

is satisfied and where the multinomial coefficients are given by

$$\binom{k}{r_1 r_2 \ldots r_m} \equiv \frac{k!}{\prod_{i=0}^{m} r_i!} \quad .$$

By choosing especially

$$y_i = a_i \cdot x^i \quad ,$$

P(x) can be identified with Q and is then given by

$$P(x) = \sum_{(r_i)} \binom{k}{r_o r_1 \ldots} \cdot \prod_i a_i^{r_i} \cdot x^{i \cdot r_i} \quad . \tag{A3}$$

The coefficients in (A1) are obtained by collecting all combinations of r_i (subject to the condition $\sum r_i = k$) which lead to the same power of x. The exponent of x is given by $\sum i \cdot r_i$ as can be seen from (A3).

This leads us to consider the combinations enumerated in table A1.

$\sum j \cdot r_j$	r_o	r_1	r_2	r_3	r_4	r_5	...	$\binom{k}{r_o r_1 \ldots}$
0	k							1
1	k-1	1						k
2	k-2	2	0					k(k-1)/2
	k-1	0	1					k
3	k-3	3	0	0				k(k-1)(k-2)/6
	k-2	1	1	0				k(k-1)
	k-1	0	0	1				k
4	k-4	4	0	0	0			k(k-1) (k-2) (k-3)/24
	k-3	2	1	0	0			k(k-1) (k-2)/2
	k-2	1	0	1	0			k(k-1)
	k-2	0	2	0	0			k(k-1)/2
	k-1	0	0	0	1			k
5	k-5	5	0	0	0	0		k(k-1) (k-2) (k-3) (k-4)/120
	k-4	3	1	0	0	0		k(k-1) (k-2) (k-3)/6
	k-3	2	0	1	0	0		k(k-1) (k-2)/2
	k-3	1	2	0	0	0		k(k-1) (k-2)/2
	k-2	1	0	0	1	0		k(k-1)
	k-2	0	1	1	0	0		k(k-1)
	k-1	0	0	0	0	1		k
		...						...

Table A1 – Possible combinations of r_i which result in the same power of x in (A3), together with the corresponding multinomial coefficients

By means of table A1 and equation (A3), the coefficients A_n are now easily found by the sum

$$A_n = \sum_{(r_i)} \binom{k}{r_o r_1 \ldots} \cdot \prod_i a_i^{r_i} \quad , \tag{A4}$$

where $\sum_j j \cdot r_j = n$ is constant.

The result for the first five coefficients is therefore

$$\begin{aligned}
A_1 &= k \cdot a_1 \quad , \\
A_2 &= \binom{k}{2} \cdot a_1^2 + k \cdot a_2 \quad , \\
A_3 &= \binom{k}{3} \cdot a_1^3 + 2\binom{k}{2} \cdot a_1 a_2 + k \cdot a_3 \quad , \\
A_4 &= \binom{k}{4} \cdot a_1^4 + 3\binom{k}{3} \cdot a_1^2 a_2 + 2\binom{k}{2} \cdot a_1 a_3 + \binom{k}{2} \cdot a_2^2 + k \cdot a_1^4 \quad , \\
A_5 &= \binom{k}{5} \cdot a_1^5 + 4\binom{k}{4} \cdot a_1^3 a_2 + 3\binom{k}{3} \cdot a_1^2 a_3 + 3\binom{k}{3} \cdot a_1 a_2^2 \\
&\quad + 2\binom{k}{2} \cdot a_1 a_4 + 2\binom{k}{2} \cdot a_2 a_3 + k \cdot a_5 \quad .
\end{aligned} \tag{A5}$$

A simple law for the formation of the coefficients does not appear yet and is in fact not very likely to exist at all.

2. On the position of the maximum density

For multiple intervals ($k \geqslant 2$), the question may arise of determining the position of the maximum value for $f_k(t)$. Unfortunately, the answer is not quite straightforward.

Our main aim is to show in this section that the maximum can never appear before the third time interval of width τ after the end of the dead time. In other words, the maximum can only occur for t in the range $(k+2)\tau < t \leqslant (k+3)\tau$, or later.

Let us consider the first interval after the dead time. The corresponding density $f_k(t)$ for the k-fold interval has been given in (27). For determining the maximum, we consider its derivative

$$f_k'(t) = \frac{\rho^k}{(k-2)!} \cdot (t - k\tau)^{k-2} \cdot e^{-k\rho\tau}. \tag{A6}$$

Whereas for $k=2$ we have $f_k' = \rho^2 \cdot \exp(-2\rho\tau)$, which is always positive, the case $k > 2$ yields for $f_k'(t) = 0$ the condition $t = k\tau$. But this is the beginning of the interval in question where f starts from zero. Therefore, no maximum can possibly exist in this range.

For the second interval, the density (26) is given by

$$f_k(t) = \rho^k \cdot \frac{(t-k\tau)^{k-1}}{(k-1)!} \cdot e^{-k\rho\tau} - \rho^{k+1} \cdot k \cdot \frac{(t-[k+1]\tau)^k}{k!} \cdot e^{-(k+1)\rho\tau}.$$

This leads to the derivative

$$f_k'(t) = \frac{\rho^k}{(k-2)!} \cdot e^{-k\rho\tau} \left\{ (t-k\tau)^{k-2} - \frac{k}{k-1} \cdot \rho \cdot (t-[k+1]\tau)^{k-1} \cdot e^{-\rho\tau} \right\}, \tag{A7}$$

with $(k+1)\tau < t \leqslant (k+2)\tau$.

For an extremum, the curly bracket must vanish. By putting

$$\rho\tau \equiv x \quad \text{and}$$

$$\rho(t - [k+1]\tau) \equiv y,$$

where now $0 < y \leqslant x$,

we obtain the condition for an extremum in the form

$$(y+x)^{k-2} = \frac{k}{k-1} \cdot y^{k-1} \cdot e^{-x}, \tag{A8}$$

or $$h(y) \equiv \frac{1}{y} \cdot \left(1 + \frac{x}{y}\right)^{k-2} = \frac{k}{k-1} \cdot e^{-x}.$$

We remark that $h(y)$ is a monotonically decreasing function of y. If we can prove that $h(y)$ always exceeds the constant $(k/k-1)\exp(-x)$, it is evident that the equation (A8) has no solution. Indeed, for y=x we obtain

$$h(x) = \frac{1}{x} \cdot 2^{k-2} = \frac{k}{k-1} \cdot e^{-x} ,$$

or $$e^{x} = \frac{k}{k-1} \cdot \frac{x}{2^{k-2}} \leqslant 2x , \qquad \text{for } k \geqslant 2 .$$

But this is not possible, since $e^{x} > 2x$ for any value of x. From this it follows that, even in the second interval after the dead time, there can never be a maximum for the interval density $f_k(t)$.

For the following intervals, the general analysis becomes much more involved. Therefore, we shall have to confine ourselves to some simple and rather incomplete results, although it is just in these regions that the maxima are actually located.

For the <u>third</u> interval, i.e. in the range $(k+2)\tau < t \leqslant (k+3)\tau$, the density for a <u>k-fold</u> interval is given according to (26) by

$$f_k(t) = \rho^{k} \cdot \frac{(t-k\tau)^{k-1}}{(k-1)!} \cdot e^{-k\rho\tau} - \rho^{k+1} \cdot k \cdot \frac{(t-\left[k+1\right]\tau)^{k}}{k!} \cdot e^{-(k+1)\rho\tau}$$
$$+ \rho^{k+2} \cdot \frac{k(k+1)}{2} \cdot \frac{(t-\left[k+2\right]\tau)^{k+1}}{(k+1)!} \cdot e^{-(k+2)\rho\tau} ,$$

which yields for the derivative

$$f_k'(t) = \frac{\rho^{k}}{(k-2)!} \cdot e^{-k\rho\tau} \cdot \Big\{ (t-k\tau)^{k-2} - \frac{\rho \cdot k}{k-1} \cdot (t-\left[k+1\right]\tau)^{k-1} \cdot e^{-\rho\tau}$$
$$+ \frac{\rho^{2} \cdot (k+1)}{2(k-1)} \cdot (t-\left[k+2\right]\tau)^{k} \cdot e^{-2\rho\tau} \Big\} . \qquad (A8)$$

With the abbreviation $\rho(t-\left[k+2\right]\tau) \equiv y$ and by putting $f_k'(t) = 0$, we arrive after some rearrangements at the following condition for the position of the extrema (for $k \geqslant 2$)

$$\frac{k}{k-1} \cdot (y+x)^{k-1} \cdot e^{-x} - (y+2x)^{k-2} = \frac{k+1}{2(k-1)} \cdot y^{k} \cdot e^{-2x} . \qquad (A9)$$

Since for the interval considered, y is again bound by $0 < y \leqslant x$, a simple way of getting some insight into the solution may be to determine the values x for the limits $y=0$ and $y=x$.

For <u>y = 0</u> we are led to

$$\frac{k}{k-1} \cdot x^{k-1} \cdot e^{-x} - (2x)^{k-2} = 0 ,$$

or $$x \cdot e^{-x} = \frac{k-1}{k} \cdot 2^{k-2} , \qquad (A10)$$

but this question has <u>no</u> solution for $k \geqslant 2$.

For $y = x$ we obtain finally

$$(x \cdot e^{-x})^2 = \frac{2}{k+1} \left[k \cdot 2^{k-1} \cdot x \cdot e^{-x} - (k-1) \cdot 3^{k-2} \right] . \qquad \text{(A11)}$$

For the special case $k=2$, this leads with $z \equiv x \cdot e^{-x}$ to the equation

$$z^2 = \frac{2}{3} \cdot (4z - 1) .$$

Whereas the solution 2.387 does not correspond to any value of x and has therefore to be rejected, the value $z = 0.2792$ allows for the two solutions $x \cong 0.429$ and 1.937, respectively. These two values indicate the limits for $\rho\tau$ to ensure that the maximum of $f_2(t)$ occurs in the range $4\tau < t \leq 5\tau$. This is the case for our experiment (fig. 4) where $\rho\tau \cong 0.8$.

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