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How to detect a decay distortion in Poissonian data

by Jörg W. Müller

Bureau International des Poids et Mesures, F-92312 Sèvres Cedex

Abstract

If the time of counting pulses from a radioactive source is not negligible compared with the half life, the observed distribution of counts deviates from the Poisson law. We show that the real repartition and the adjusted Poisson distribution cross at two values which can be determined in advance. This allows us to decide on the presence of a distortion. Recent observations confirm this approach.

1. Introduction

In a Poisson process, the mean count rate ρ_0 is assumed to be independent of time. In this case, the probability of observing exactly k events in a time interval t is given by

$$P_{\mu_0}(k) = \frac{\mu_0^k}{k!} e^{-\mu_0}, \qquad (1)$$

with $\mu_0 = \rho_0$ t.

On the other hand, a radioactive source, since it is decaying, necessarily has a finite lifetime. Strictly speaking, therefore, pulses originating from a radioactive source cannot be distributed exactly according to the Poisson law (1).

We now consider a radioactive source which decays in time. In the absence of a background, the simplest realistic assumption for the temporal behaviour of the count rate is the "exponential model", i.e.

$$\rho(t) = \rho_0 e^{-\lambda t}, \qquad (2)$$

where the decay constant λ is related to the half life $T_{1/2}$ by the expression $T_{1/2} = \ln 2/\lambda$.

Let us now assume that for such a decaying source we count repeatedly the number of events observed during short time intervals of length t. If n consecutive intervals are measured, the total period of observation is T = n t, with n » 1. As a result of the decay of the source during T, the observed statistical distribution of the number of counts k will not be correctly described by the Poisson law (1). Instead, a new distribution W(k) has to be expected which, since it can be considered as a superposition of distributions with different mean values, will appear to be "broader".

A convenient parameter for describing the distortion produced by the decay is

$$\theta = \lambda T . \tag{3}$$

Obviously we expect that

$$\lim_{\theta \to 0} W(k) = P_{\mu_0}(k) .$$

The derivation of an exact formula for W(k) has been the subject of a previous paper [1] in which an analytical expression is presented in the form of the product

$$W(k) = P_{\mu_0}(k) \; C(k, \mu_0, \theta) . \tag{4}$$

The correction factor C which appears in this formula involves (for $k = 0$) the exponential integral function or (for $k \geq 1$) the incomplete gamma function. Its numerical evaluation assumes that both μ_0 and θ are known.

In the present study our aim is different: we would like to know how a measured distribution W(k), which is only slightly distorted by decay, can be distinguished from an unperturbed Poisson distribution. For this purpose we try to find an expression of the form

$$W(k) \cong P_{\bar{k}}(k) \; F(k, \bar{k}, \theta) , \tag{5}$$

where the new correction factor F, applicable for $\theta \ll 1$, i.e. for small decay distortions, should depend only on experimentally known quantities, such as the measured mean value $\bar{k}$.

2. Some rearrangements

In a formal way, the task consists in transforming (4) into (5), limiting ourselves to the first non-vanishing order in the parameter θ. This involves several steps.

First we express μ_0 in terms of $\bar{k}$ and θ. As it has been shown in [1] that the experimental mean value $\bar{k}$ for the number of pulses observed in t is given by

$$\bar{k} \cong E(k) = \mu_0 \left(\frac{1 - e^{-\theta}}{\theta}\right) , \tag{6}$$

a simple development to second order in θ yields

$$\bar{k} = \mu_0 \left(1 - \frac{1}{2}\theta + \frac{1}{6}\theta^2 \mp ...\right) ,$$

from which we obtain

$$\mu_0 = \bar{k}\,(1 + \tfrac{1}{2}\,\theta + \tfrac{1}{12}\,\theta^2 + \ldots)\,. \tag{7}$$

An important part of what we try to achieve has already been done in [2].

In particular, eq. A10 in the Appendix gives a second-order approximation to the modified Poisson distribution in the form

$$W(k) \;\cong\; P_{\mu_0}(k)\,\left\{1 - \tfrac{1}{2}\,(k - \mu_0)\,\theta + \tfrac{1}{6}\,[(k - \mu_0)^2 - \mu_0]\,\theta^2\right\}. \tag{8}$$

The expression in the curly brackets is an approximation of the correction factor C appearing in (4). Substituting (7) in (1) gives for the first factor of (8)

$$P_{\mu_0}(k) = \frac{1}{k!}\,\bar{k}^{k}\,(1 + \tfrac{1}{2}\,\theta + \tfrac{1}{12}\,\theta^2 + \ldots)^k \exp\left[-\bar{k}\,(1 + \tfrac{1}{2}\,\theta + \tfrac{1}{12}\,\theta^2 + \ldots)\right]. \tag{9}$$

This expression raises the problem of evaluating the k-th power of a series S of the form

$$S = 1 + a_1\,x + a_2\,x^2 + \ldots\,. \tag{10a}$$

Fortunately, the coefficients b_j appearing in

$$S^k = 1 + b_1\,x + b_2\,x^2 + \ldots \tag{10b}$$

are known from a previous study [3] to be given by

$$b_j = \sum_{n=1}^{j} \binom{k}{n}\,{}_n c_j\,, \tag{11a}$$

with coefficients ${}_n c_j$ which have been explicitly listed. In the present case this yields

$$b_1 = \binom{k}{1}\,{}_1 c_1 = k\,a_1$$

and (11b)

$$b_2 = \binom{k}{1}\,{}_1 c_2 + \binom{k}{2}\,{}_2 c_2 = k\,a_2 + \tfrac{1}{2}\,k\,(k-1)\,a_1^2\,.$$

Since $a_1 = 1/2$ and $a_2 = 1/12$, this gives for (9) the approximation

$$P_{\mu_0}(k) \;\cong\; \frac{1}{k!}\,\bar{k}^{k}\left\{1 + \tfrac{1}{2}\,k\theta + k\left[\tfrac{1}{12} + \tfrac{1}{8}\,(k-1)\right]\theta^2\right\}\left[1 - \tfrac{1}{2}\,\bar{k}\,\theta + \tfrac{1}{24}\,\bar{k}\,(3\bar{k} - 2)\,\theta^2\right] e^{-\bar{k}}$$

or, after performing the multiplication,

$$P_{\mu_0}(k) \cong P_{\bar{k}}(k) \left\{ 1 + \frac{1}{2}(k - \bar{k})\,\theta + \frac{1}{8}\left[(k - \bar{k})^2 - \frac{1}{3}(k + 2\bar{k}) \right] \theta^2 \right\} . \tag{12}$$

We now have to transform the expression which appears in the curly brackets of (8). With (7) we first find

$$\{...\} = 1 - \frac{1}{2}(k - \bar{k})\,\theta + \frac{1}{6}\left[(k - \bar{k})^2 + \frac{1}{2}\bar{k} \right] \theta^2 . \tag{13}$$

After multiplication of the expressions appearing in (13) and (12) we finally obtain for (8), always up to second order in θ, the result

$$W(k) \cong P_{\bar{k}}(k) \left\{ \begin{array}{l} 1 + \frac{1}{2}(k - \bar{k})\,\theta + \frac{1}{8}\left[(k - \bar{k})^2 - \frac{1}{3}(k + 2\bar{k}) \right] \theta^2 \\ \quad - \frac{1}{2}(k - \bar{k})\,\theta - \frac{1}{4}(k - \bar{k})^2\,\theta^2 \\ \qquad\qquad + \frac{1}{6}\left[(k - \bar{k})^2 + \frac{1}{2}\bar{k} \right] \theta^2 \end{array} \right\} .$$

This can be easily brought into the simple form

$$W(k) \cong P_{\bar{k}}(k) \left\{ 1 + \frac{1}{24}\left[(k - \bar{k})^2 - k \right] \theta^2 \right\} . \tag{14}$$

This relation is of the type required by (5) and is therefore well suited to our purpose; it is the main result of the present study.

3. Discussion and application of the result

In discussing (14) it is worth noting that the deviation of W(k) from the undistorted Poisson law, based on the observed mean value $\bar{k}$, is of second order in $\theta = \lambda T$. This is a major difference from the previous similar situation met with a dead time [4] where the distortion was of first order in τ/t, the ratio of dead time to measuring period. In addition, the two effects differ in their sign.

The functions W(k) and $P_{\bar{k}}(k)$ - thought of as continuous curves - cross twice. This occurs, according to (14), when $k = (k - \bar{k})^2$ and leads to exactly the same solutions for k as in the case of the dead-time distortion, namely to

$$k_{\pm} = \bar{k} + \frac{1}{2} \pm \sqrt{\bar{k} + \frac{1}{4}} . \tag{15}$$

In consequence, the question of whether decay has caused a distortion may be resolved, as in [4, 5], by a sign test, since the regions with positive or negative deviations are separated by the crossing points k_- and k_+ which can be determined in advance by means of (15). The value of θ does not have to be known.

For the time being, few experimental data are available for checking eq. (15). However, in a recent paper [6] such decay-distorted Poisson distributions have been measured. The aim of this work was to study the agreement of the experimental distributions with those expected on the basis of the general formula for W(k) given in [1]. One of the curves shown in this paper, namely Fig. 3 (with $\bar{k} \cong 20.0$), may be taken as a useful check. It is reproduced here for convenience in Fig. 1. The observed intersections can be located at 15.5 ± 0.5 and 25.0 ± 0.5. This compares favourably with the values calculated from (15), which are

$$k_- \cong 16 \quad \text{and} \quad k_+ \cong 25 .$$

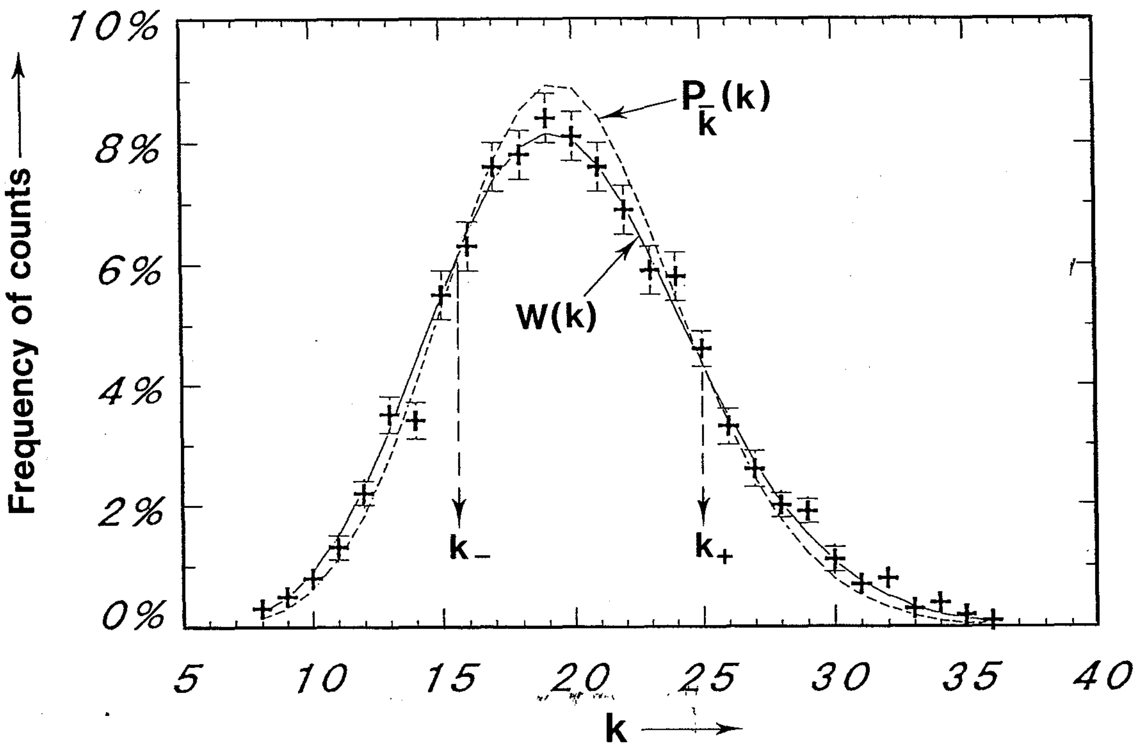

Fig. 1 - Experimental distribution W(k) of the counts, compared with the corresponding Poisson distribution $P_{\bar{k}}(k)$, for $\bar{k} = 20$. The data are taken from [6]. The chosen ratio $T/T_{1/2} = 0.5$ corresponds to $t \cong 0.35$.

We plan to perform similar measurements at the BIPM. This should allow us to determine the position of the crossing points with higher precision and for a number of mean values $\bar{k}$.

Although the perturbations produced by decay or by a dead time have hardly anything in common, they both lead to relation (15). One may suspect, therefore, that there exists a general reason for producing this type of effect.

References

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