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A general test for detecting dead-time distortions in a Poisson process

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1. Introduction

In checking an empirical distribution of counted events which are supposed to follow a Poisson law, one is often faced with the problem of testing whether this assumption is really justified or not. As is well known, such a decision may be based on the result of a chi-square test, for example, where the squares of the differences between the theoretical and the experimental frequencies are used.

It also happens, however, that one knows in advance something about the nature of a distortion. In this case one can test for its presence with a method which is more specific than a general purpose chi-square test. In particular, this is true in the common situation where for some reason or other successive events must be separated by a minimum time-interval in order to be counted individually. This may be due to the finite resolving time (or dead time) of the instrument with which the observations have been performed.

2. A closer look at the problem

We shall now assume that the original sequence of events can be taken as a Poisson process. The probability of observing exactly k events in a time interval t is then given by

$$P_\mu(k) = \frac{\mu^k}{k!} \cdot e^{-\mu} , \tag{1}$$

with the expectation $\mu = \rho \cdot t$, where ρ is the count rate of the process.

Let us now study the dead-time modified frequency distribution $W_\mu(t)$. Since here we are interested only in the case of a small disturbance - the detection of an evident change is not a problem -, we do not have to apply the exact formulae for the probability distributions [1]; approximations will be sufficient and more practical. For $\tau < t$ they can be written in the form of a power series

$$W_\mu(k) \cong P_\mu(k) \sum_{i=0}^{\infty} C_i \cdot \left(\frac{\tau}{t}\right)^i , \tag{2}$$

where the C_i's are coefficients depending on μ and k. For $\tau = 0$ we must have $W = P$, therefore $C_o = 1$.

In addition, the C_i's will generally depend on the type of dead time since this also influences the number of losses. However, because this distinction is known to change only the second order and higher terms in the dead-time corrections, the coefficient C_1 is type-independent. Furthermore, for the small distortions we are considering here, the higher orders are negligible. For the present problem we may therefore write

$$W_\mu(k) \cong P_\mu(k) \left\{1 + C_1 \cdot \frac{\tau}{t}\right\} , \tag{3}$$

where C_1 can be easily determined to be [2]

$$C_1 = k(\rho t - k + 1) . \tag{4}$$

Since μ, the true mean number of counts in t, and $\varkappa$, the experimental mean of k, are related to first order in $\rho\tau$ by

$$\mu \cong \varkappa(1 + \rho\tau) , \tag{5}$$

we can also write equation (1) in the form ($\rho\tau \ll 1$)

$$\begin{aligned} P_\mu(k) &\cong \frac{1}{k!}\left[\varkappa(1+\rho\tau)\right]^k \cdot e^{-\varkappa} \cdot e^{-\varkappa\cdot\rho\tau} \\ &= P_\varkappa(k) \cdot (1 + k\cdot\rho\tau) \cdot (1 - \varkappa\cdot\rho\tau) \\ &= P_\varkappa(k)\left[1 + \rho\tau(k-\varkappa)\right] . \end{aligned} \tag{6}$$

When combined with (3) and (4), this yields

$$\begin{aligned} W_\mu(k) &\cong P_\varkappa(k)\left[1 + \rho\tau(k-\varkappa)\right]\left\{1 + \frac{\tau}{t}k(\rho t - k + 1)\right\} \\ &= P_\varkappa(k)\left\{1 + \frac{\tau}{t}\left[\rho t(k-\varkappa) + k(\rho t - k + 1)\right]\right\} . \end{aligned}$$

Finally, by applying (5) we obtain (always to first order)

$$W_\mu(k) \cong P_\varkappa(k)\left\{1 + \frac{\tau}{t}\left[k - (k-\varkappa)^2\right]\right\} . \tag{7}$$

It is easy to verify that this approximate probability distribution is still normalized in the sense that

$$\sum_{k=0}^{\infty} W_\mu(k) = 1 .$$

Again, this obviously neglects the fact that k cannot exceed the value $t/\tau + 1$, but since $\tau \ll t$, this is consistent with our previous assumption of small dead-time losses.

The form of (7) shows that the points where the original and the modified distributions cross do not depend on the value of τ, since $W_{\mu}(k) = P_{\varkappa}(k)$ requires that

$$k - (k - \varkappa)^2 = 0 \quad ,$$

having the solutions

$$k_{\mp} = \varkappa + \frac{1}{2} \mp \sqrt{\varkappa + \frac{1}{4}} \quad . \tag{8}$$

These two values are the limits of the range of the integers k in which the frequency of occurrence is expected to be augmented by the presence of a dead time. Thereby the hypothetical original frequency distribution, with which the empirical values are compared, is assumed to be Poissonian with an average value equal to the experimentally observed mean. Here again, therefore, the numerical value of the dead time is not required; it is sufficient to know that it is small.

The interesting feature of this approach lies in the fact that (8) allows us to predict the sign of the eventual deviations that an empirical frequency distribution may show with respect to a calculated Poisson law. By means of a simple sign test it is then possible to decide whether the prediction is verified or not by the observations, since in the absence of a definite dead-time distortion the deviations would be positive or negative with equal probability.

3. Application

As a practical example we apply these results to the famous experiment of Rutherford and Geiger [3] where the number k of scintillations, produced by the alpha particles from ^{210}Po on a screen of ZnS, were counted within time intervals of 7.5 s each. Table 1 reproduces their data in a form suitable for our purpose, where F(k) are the experimental frequencies for exactly k events, and N their sum. For comparison, the Poisson probabilities $P_{\varkappa}(k)$ are also calculated, where the empirical mean is taken as*

$$\varkappa = \frac{\sum k \cdot F(k)}{\sum F(k)} = \frac{10\,097}{2\,608} = 3.871\,55 \quad .$$

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* For the total number of observed alphas, the number 10 094 is often found in the literature (e.g. [4] and [5]), which is at variance with the original data [3]. In the notation of Table 1, the discrepancy is due to a replacement of F(13) = F(14) = 1 by F(12) = 2 .

k	$F(k)$	$G(k) = N \cdot P_{\mathcal{H}}(k)$	sign of difference $F-G$ actual	predicted
0	57	54	+	-
1	203	210	-	-
2	383	407	-	-
k_- . . .	. . .	. . .	. . .	. . .
3	525	525	(zero)	+
4	532	508	+	+
5	408	394	+	+
6	273	254	+	+
k_+ . . .	. . .	. . .	. . .	. . .
7	139	141	-	-
8	45	68	-	-
9	27	29	-	-
$\geqslant 10$	16	17	-	-
	$N = 2\ 608$	(2 607)		

Table 1: Evidence for a dead-time distortion in the Rutherford-Geiger data [3]. The theoretical frequency G is rounded to the nearest integer. Only for k=0 the signs of the actual and predicted differences do not agree.

According to (8), the "critical" values of k are thus given by

$$k_- \cong 2.3 \qquad \text{and} \qquad k_+ \cong 6.4 \quad .$$

It is evident from Table 1 that the prediction of the sign of the deviation from a pure Poisson distribution, as based on this model, is quite successful: the signs turn out to be correct in 9 out of 10 cases.

A simple sign test, based on the binomial distribution with probability 0.5, now shows that the chance R for such a good (or an even better) agreement to happen by chance is only

$$R = \left(\frac{1}{2}\right)^{10} \left[\binom{10}{10} + \binom{10}{9}\right] = 2^{-10} \cdot 11 \cong 1.1\% \ .$$

This seems to be a reasonable confidence level to permit the conclusion that the observed deviations cannot be random. On the contrary, the data are clearly distorted by a dead time, although they are often presented in textbooks on mathematical statistics as an illustration for the Poisson process. As a matter of fact, a test based on the chi-square distribution, when applied to these data [4], yields a fair agreement with a pure Poisson distribution ($R \cong 17\%$), but even here the situation is hardly "very satisfactory" [6]. This shows only that the comparison is not specific enough, which is not too surprising in our case since any information about the sign of the deviations is thereby lost.

Finally, to reinforce the argument, one can also try to determine the dead time τ involved in this experiment. In doing so, a value of

$$\tau = (0.05 \pm 0.03)\ \mathrm{s}$$

is obtained, which is certainly an acceptable value for the resolving time of Dr. Geiger's eye - assuming that it was he, as the younger, who actually made the observations.

4. Additional remarks

It may appear as somewhat unsatisfactory in what has been said above that agreement or disagreement between the predicted and observed signs do not take into account the magnitude of the differences nor the statistical uncertainties in the measurements. On the other hand, this independence of any additional assumption about the distributions involved is obviously one of the main advantages of a sign test.

In our case, however, the essential point is only that the signs of the differences are taken into account at all - but not necessarily by simple counting, as in a normal sign test. One could thus easily imagine other statistics, as e.g. the simple variate

$$Q = \sum_{k=0}^{K} S_k \cdot \frac{F(k) - G(k)}{\sqrt{G(k)}} \quad , \tag{9}$$

where S_k is $+1$ (-1) if the corresponding predicted difference is positive (negative).

For $K \gg 1$ and random deviations, as would be the case for a perfect fit, the predicted sign S_k is not correlated with the actual difference $F - G$, and Q tends towards a Gaussian with mean zero and variance K. On the other hand, if the differences are significant and if their sign is correctly described by S_k, most of the contributions will be positive and exceed unity. This fact will then easily show up under any test for normality of Q or its components, since a fair approximation of Q to a Gaussian may already be expected for, say, $K \gtrsim 5$.

In the case of the Rutherford-Geiger data, we obtain the numerical value $Q \cong 7.9$. Since $K = 10$, this is about 2.5 times the expected standard deviation. A one-sided test based on the normal distribution would therefore lead to a significance level of about 99.5%, confirming our earlier conclusion that the observations differ significantly from a Poisson distribution.

We may add that a similar claim has been made earlier by Pacilio [7], but he gave no proof and arrived at a distribution which was later found to be wrong [8] .

References

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