Motivation for AMENDMENT 1 to JCGM 100:2008 (GUM:1995 with minor corrections)

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Abstract

An amendment to JCGM 100:2008 is proposed to address an inconsistency concerning the treatment given to measurements whose model is nonlinear. This motivation outlines the rationale behind the amendment.

1 General

JCGM 100:2008 contains an inconsistency concerning the treatment given to measurements whose model is nonlinear. Such cases are discussed in:

— Clause 4.1.4, NOTE (page 9)

— Clause 5.1.2, NOTE (page 19)

— Annex F.2.4.4 Asymmetric distributions of possible values (pages 66 - 68)

— Annex H, Example H.1, Clause H.1.7, Second-order terms (page 85)

— Annex H, Example H.2, Clause H.2.4 Approach 2 (page 88)

— Annex H, Example H.4, Clause H.4.3.2, Approach 2 (pages 97 - 98)

The amendment resolves this inconsistency.

2 Analysis

In the NOTE to clause 4.1.4, an alternative method is suggested to calculate the estimate y with respect to the usual one, $y = f(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_N)$. With the alternative method, y is calculated as

$$y = \overline{Y} = \frac{1}{n}\sum_{k=1}^{n} Y_k = \frac{1}{n}\sum_{k=1}^{n} f(X_{1,k}, X_{2,k}, \ldots, X_{N,k})$$

In the NOTE, it is claimed that

> This way of averaging, ... may be preferable when f is a nonlinear function of the input quantities $X_1, X_2, \ldots, X_N$, ...

Examples 2 and 4 in Annex H demonstrate the use of the alternative method, which is declared to be preferred. In particular, in example 2 expression (H10)

$$y = f(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_N) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial \overline{X}_i \partial \overline{X}_j} u(\overline{X}_i, \overline{X}_j) + \cdots$$

is given to demonstrate the effect of nonlinearity on the measurand estimate.

In the NOTE to clause 5.1.2, the most important terms of next highest order to be added to the expression of $u(y)$ in the case of nonlinear models are given as

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)^2+\frac{\partial f}{\partial x_i}\frac{\partial^3 f}{\partial x_i \partial x_j^2}\right]u^2(x_i)u^2(x_j).$$

The effect of that correction to $u(y)$ is discussed in example H1.

In Annex F.2.4.4, Asymmetric distributions of possible values, the case of the so-called cosine error is discussed in detail. This case occurs, for example, when measuring the fixed vertical height h (the measurand) of a column of liquid in a manometer. The axis of the height-measuring device may deviate from verticality by a small angle β. The measurement model is thus

$$h = l \cos\beta,$$

where l is the indication of the height-measuring device. The model is nonlinear, so that all the relevant considerations apply. Here, the expression given for the estimate,

$$h = \bar{l}\left[1-(1/2)\,u^2(\beta)\right],$$

includes the correction due to nonlinearity in the case that a normal distribution is taken for β.

In that same Annex F.2.4.4, another example is given concerning the determination of a concentration by titration.

In both examples, the case of rectangular rather than normal distribution is discussed.

3 Inconsistency and remedy

The inconsistency mentioned in clause 1 refers to the fact that in the main text, as well as in Example H.1, the need to correct the estimate y in the case of nonlinear models is not even mentioned. In other places (Annex F.2.4.4 and Example H.2) the correction is acknowledged and applied.

To remedy the inconsistency, clause 4.1.4 and Example H.1 need to be amended. The former in order to explicitly introduce the expression for the correction in the case of independent normal distributions, as discussed in that same clause; the latter to examine the amount of the correction and conclude that in that case it is equal to zero.

4 Discussion

It might look surprising and counter-intuitive that the measurand estimate y deviates from the usual expression $y = f(x_1, x_2, \ldots, x_N)$ in some circumstances. Yet, it should be kept in mind that for asymmetric distributions for the measurand, which inevitably arise in the presence of nonlinear models, the expectation may depart considerably from the mode (the most probable value), the median staying somewhere within these two values. Therefore, the mode might be closer to the true value than the expectation. However, the pair of parameters discussed in JCGM 100:2008 are expectation and standard deviation (strictly speaking, variance) of the probability distribution for the measurand. Using these two moments, coverage intervals for the measurand can be constructed. In addition, the pair can be propagated in a comparably straightforward way.

As a last consideration, the amendment not only resolves an internal inconsistency in JGCM 100:2008, it also resolves an inconsistency with JCGM 101:2008. In the latter document, it is said, in NOTE 3 to 7.6:

> $\tilde{y}$ will not in general agree with the model evaluated at the best estimates of the input quantities, since, for a non-linear model $f(\boldsymbol{X})$, $E(Y) = E(f(\boldsymbol{X})) \neq f(E(\boldsymbol{X}))$ ...

where $\tilde{y}$ is the estimate of the measurand Y as provided by the Monte Carlo method described in JCGM 101:2008.

The disagreement described in the NOTE above disappears if the correction described in this amendment is applied, and if the terms of the Taylor expansion of order higher than the second are negligible (see also JCGM GUM 5:2025, example 16). In general, using the corrected formula for the measurand estimate will give a value close to that provided by JCGM 101:2008, which remains the authoritative method in the case of no prior knowledge about the measurand.