Defining the SI Second via Option 2: Challenges and Opportunities

Jérôme Lodewyck, Tetsuya Ido

Introduction

Option 2: defining the second with multiple transitions

1. **Definition** based on weights w_i
   - N: normalisation constant for continuity
   - N and w_i univocally fix the definition
   - (1 relation) + ($n-1$ independent ratios) $\equiv$ n frequencies

$$\prod_{k=1}^{n} \nu_k^{w_k} \equiv N \text{ Hz}$$

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$$N_i = N \prod_{k=1}^{n} \rho_{i,k}^{w_k}$$

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   - calculable from the global fit of frequency ratios $\rho_{i,j}$
   - initially independent of the choice of weights

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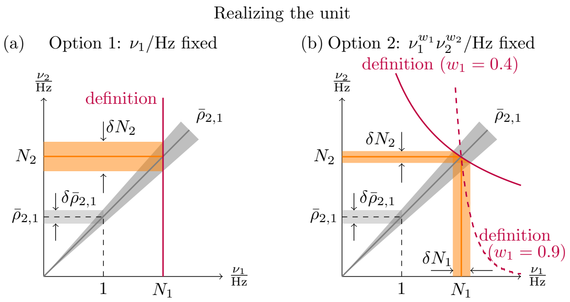
   - initially independent of the choice of weights
   - by construction, recommended frequency always average to N:
   - uncertainty on the recommended frequency δN_i (akin uncertainty on SRS)

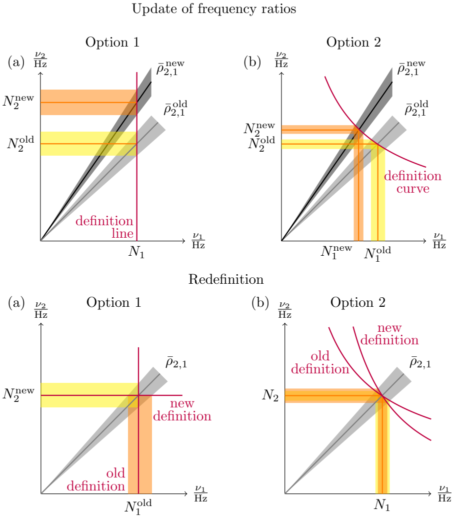
$$\prod_{k=1}^{n} N_k^{w_k} = N, \quad \forall \rho_{i,j}.$$

$$\frac{\nu_i}{\text{Hz}} = N_i \pm \delta N_i$$

Understanding option 2: graphical representation



- The definition is represented by an exactly known curve in the $\{\nu_i/\text{Hz}\}$ space:
  - Straight line for option 1
  - Curve with finite slope for option 2
- Frequency ratios are represented by a slope with an uncertainty
- Recommended frequencies N_i for the realization are at the intersection point
- Option 2 "balances" the uncertainty on N_i over all transitions.



Update of frequency ratios

- The definition is constant
- New recommended frequencies at the intersection of new frequency ratios and the definition

Future redefinitions

- Recommended frequencies are constant
- The definition pivots around the intersection
- "Option 2b": the weights could be updated on a regular basis directly by the CIPM to follow the progress in optical frequency standards

Choice of weights

Uncertainty on recommended frequencies

- Option 1: $\delta N_{i_0} = 0$ for PFS, while $\delta N_i^2 \simeq u_i^2 + u_{i_0}^2$ for SRS
- Option 2: balance δN_i over all transitions

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Choice of weights

- $\frac{\delta N_i}{N_i}$ small compared to the clock uncertainty u_i for all i.
- $\Rightarrow w_i \propto \frac{1}{u_i^2}$: *most precise clocks must have a larger weights*

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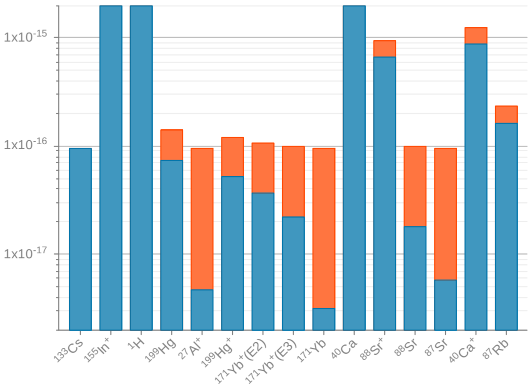
Deriving u_i

- Least squares over residues: $s_{j,k} = \frac{u_j^2 + u_k^2}{u_{j,k}^2} - 1$, with $u_{j,k}$ coming from the global fit.
- Uncertainties representative of clocks involved in frequency ratio measurements
- Need at least two connections.

Example: 2021 CIPM fit of freq. ratios

Uncertainties ($/10^{-18}$) in the **realization** of the unit $\longrightarrow$ For the **current SI second** (^{133}Cs)

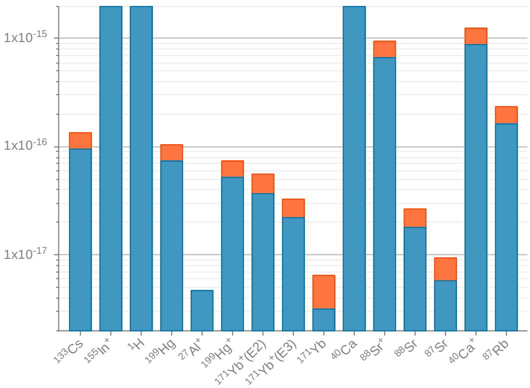
Species	u_i	N_i	$\delta N_i / N_i$
^{133}Cs	95	9192631770.0000000	0
^{155}In$^+$	2161	1267402452901041.283	2163
^{1}H	3001	1233030706593513.654	3000
^{199}Hg	74	1128575290808154.319	121
^{27}Al$^+$	4.7	1121015393207859.159	96
^{199}Hg$^+$	53	1064721609899146.964	109
^{171}Yb$^+$E2	37	688358979309308.239	102
^{171}Yb$^+$E3	22	642121496772645.119	97
^{171}Yb	3.2	518295836590863.630	96
^{40}Ca	6276	455986240494138.191	6276
^{88}Sr$^+$	669	444779044095486.342	667
^{88}Sr	18	429228066418007.006	98
^{87}Sr	5.8	429228004229872.992	96
^{40}Ca$^+$	884	411042129776400.360	885
^{87}Rb	163	6834682610.9043126	172



Example: 2021 CIPM fit of freq. ratios

Uncertainties ($/10^{-18}$) in the **realization** of the unit → For **option 1** with $^{27}Al^{+}$

Species	u_i	N_i	$\delta N_i / N_i$
^{133}Cs	95	9192631770.0000000	96
$^{155}In^{+}$	2161	1267402452901041.283	2162
^{1}H	3001	1233030706593513.654	3002
^{199}Hg	74	1128575290808154.319	74
$^{27}Al^{+}$	4.7	1121015393207859.159	0
$^{199}Hg^{+}$	53	1064721609899146.964	52
$^{171}Yb^{+}E2$	37	688358979309308.239	42
$^{171}Yb^{+}E3$	22	642121496772645.119	24
^{171}Yb	3.2	518295836590863.630	5.6
^{40}Ca	6276	455986240494138.191	6277
$^{88}Sr^{+}$	669	444779044095486.342	672
^{88}Sr	18	429228066418007.006	19
^{87}Sr	5.8	429228004229872.992	7.5
$^{40}Ca^{+}$	884	411042129776400.360	885
^{87}Rb	163	6834682610.9043126	166



Uncertainties ($/10^{-18}$) in the **realization** of the unit $\longrightarrow$ For **option 1** with ^{171}Yb

Species	u_i	N_i	$\delta N_i / N_i$
^{133}Cs	95	9192631770.0000000	96
^{155}In$^+$	2161	1267402452901041.283	2162
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^{171}Yb$^+$E2	37	688358979309308.239	41
^{171}Yb$^+$E3	22	642121496772645.119	24
^{171}Yb	3.2	518295836590863.630	0
^{40}Ca	6276	455986240494138.191	6277
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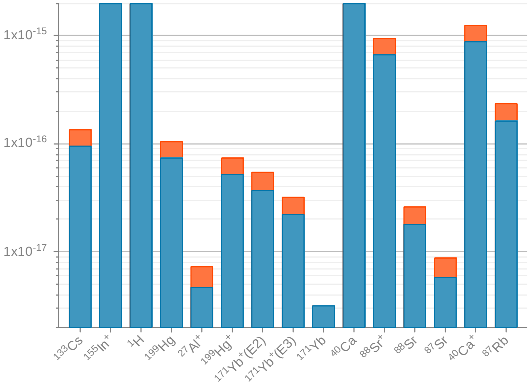

EXAMPLE: 2021 CIPM FIT OF FREQ. RATIOS

Uncertainties ($/10^{-18}$) in the **realization** of the unit → For **option 1** with ^{87}Sr

Species	u_i	N_i	$\delta N_i / N_i$
^{133}Cs	95	9192631770.0000000	96
^{155}In$^+$	2161	1267402452901041.283	2162
^{1}H	3001	1233030706593513.654	3002
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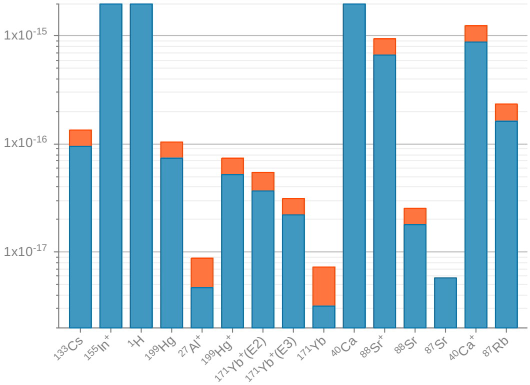
^{40}Ca	6276	455986240494138.191	6277
^{88}Sr$^+$	669	444779044095486.342	672
^{88}Sr	18	429228066418007.006	18
^{87}Sr	5.8	429228004229872.992	0
^{40}Ca$^+$	884	411042129776400.360	885
^{87}Rb	163	6834682610.9043126	166



Uncertainties ($/10^{-18}$) in the **realization** of the unit $\longrightarrow$ For **option 2**

Species	u_i	N_i	w_i	$\delta N_i / N_i$
^{133}Cs	95	9192631770.0000000	0.001	96
^{155}In$^+$	2161	1267402452901041.283	0.000	2162
^{1}H	3001	1233030706593513.654	0.000	3002
^{199}Hg	74	1128575290808154.319	0.001	74
^{27}Al$^+$	4.7	1121015393207859.159	0.249	4.1
^{199}Hg$^+$	53	1064721609899146.964	0.002	52
^{171}Yb$^+$E2	37	688358979309308.239	0.004	41
^{171}Yb$^+$E3	22	642121496772645.119	0.011	23
^{171}Yb	3.2	518295836590863.630	0.549	2.2
^{40}Ca	6276	455986240494138.191	0.000	6277
^{88}Sr$^+$	669	444779044095486.342	0.000	672
^{88}Sr	18	429228066418007.006	0.017	18
^{87}Sr	5.8	429228004229872.992	0.166	5.1
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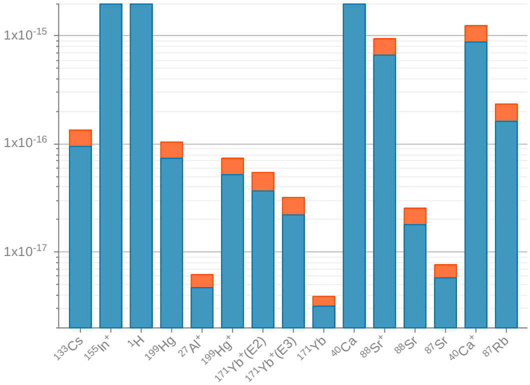

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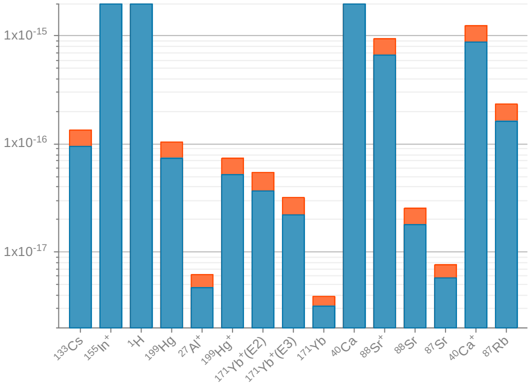
^{87}Rb	163	6834682610.9043126	0.000	166
		Combined ^{87}Sr and ^{27}Al$^+$		2.7
		Combined ^{171}Yb and ^{27}Al$^+$		1.4
		Combined ^{87}Sr and ^{171}Yb		1.5
		Combined ^{87}Sr, ^{171}Yb, and ^{27}Al$^+$		0.5



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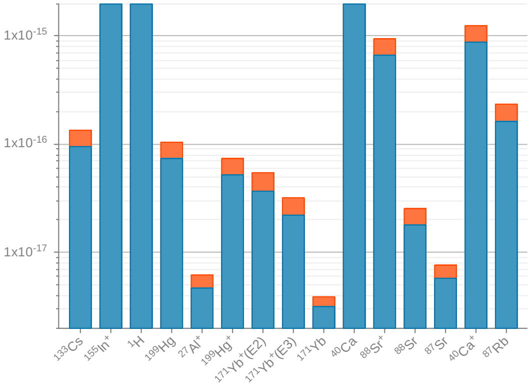
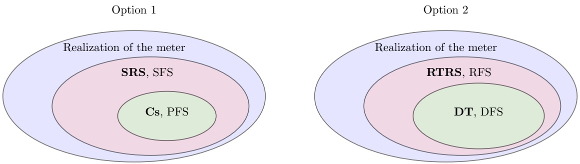


- 2021: ^{87}Sr, ^{171}Yb, ^{27}Al$^+$, (^{88}Sr, ^{171}Yb$^+$)
- 2025: significant contributions of ^{171}Yb$^+$, ^{88}Sr$^+$, ^{155}In$^+$

Terminology

Translating the concepts of primary and secondary frequency standards



3 categories of frequency standards

Option 1		**Option 2**
Transitions for the realization of the meter	=	Transitions for the realization of the meter
Secondary Representations of the Second	=	Recommended Transitions for the Realization of the Second
Cs transition / transition "x"	<	Defining transitions

Pros & cons

Pros

- **Easier consensus**: balanced weights, chosen with quantitative criteria
- **Promoting variety of frequency standards**: encourage improvement of various FS and freq. ratios *vs.* focus of resources on a single species.
- **Ready to be implemented**: relies on the global fit of frequency ratios
- **Mitigate risks**: mistakes and obsolescence of FS have less impact on the realization uncertainty, albeit more likely to occur.
  option 2: higher probability and low impact vs *option1: lower probability and high impact*
- *(Option 2b)* **Adapt to the evolution of optical frequency standards:** weights are updated based on a quantitative indicator.

Pros & cons

Cons (and anwers!)

- **Added realization uncertainty with the PFS of option 1 alone** ; but:
  - $\delta N_i / N_i < u_i$, and improving with new freq. ratio measurements.
  - Lower realization uncertainty than option 1 for SRS, and combinations thereof (e.g. TAI)
  - N_i derived from redundant freq. ratio measurements.
- **Industrial need** ; but: performance/market driven, current development of Rb, optical clocks, independent of the definition.
- **Difficult to explain** ; but: concept of average (without maths) easy to understand.
- **Physical interpretation of N** ; but the value of N is an historical accident
- *(Option 2b)* **Different from the current formulation of the definitions of units**

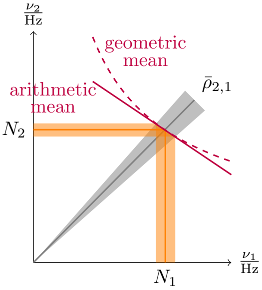
Alternatives formulations

Other means

- Arithmetic mean (C. Calosso, N. Nemitz)
- Harmonic mean (U. Sterr)

All these means are practically equivalent, but:

- The geometric mean is formally the "right" choice
- Other means can be more understandable



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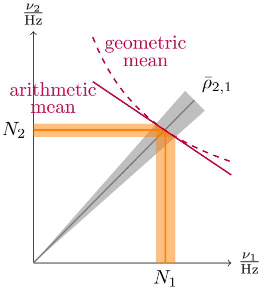

Continuity constant N

$$\prod_{k=1}^{n} N_k^{w_k} = N$$

$$\forall \rho_{i,j}$$

- Replacing N with $\{N_i\}$ would give physical significance to the continuity constant (C. Calosso, N. Nemitz)
- But hide the fact that continuity implies one degree of freedom, and may bring confusion as the N_i vary.

Conclusion

Option 2:

- Definition involves several transitions based on their performances.
- Realization based on recommended frequencies ($\Leftrightarrow$ current SRS)
- Fair balancing of uncertainties over all transitions
- Readily implementable with current tools (fit of frequency ratios, steering of TAI, ...)

References

- *On a definition of the SI second with a set of optical clock transitions* Jérôme Lodewyck, Metrologia **56** 055009 (2019)
- *A definition of the SI second based on several optical transitions* Jérôme Lodewyck, J. Phys.: Conf. Ser. **2889** 012026 (2024)
- *Properties of a definition of the SI second based on several optical transitions* Jérôme Lodewyck, and Tetsuya Ido arXiv:2503.13278
- *An Accessible Formulation for Defining the SI Second Based on Multiple Atomic Transitions* Claudio Calosso, Nils Nemitz arXiv:2503.01778