Rapport BIPM-76/16

Some remarks on dead-time losses of coincidences

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1. Introduction

Dead-time effects for coincident pulses are known to be a very difficult subject. Apart from some trivial cases, no rigorous results are yet known. For all practical applications, approximate solutions are used, the quality of which is difficult to judge. Whereas in general they seem to be sufficiently reliable, their deficiency begins to show up clearly for very high count rates.

Unfortunately, the present small note will not really improve this situation. It may, however, provide some guideline for the credibility of the various approaches which have been suggested.

2. Basis of the model

To fix the notation, the situation is sketched in Fig. 1, where the pulse series with the count rates B_1 and G_1 are Poisson distributed and the dead times τ_β and τ_γ are supposed to be of the non-extended type. Our aim is to evaluate the count rate C_2 for the surviving true coincidences. This is equivalent to the determination of the corresponding transmission factor $T_c \equiv C_2/C_1$, when the rates C_1, b_1 and g_1 of the original process as well as the dead times are given. However, only a very special case (namely for $b_1 = g_1$ and $\tau_\beta = \tau_\gamma$) will be amenable to a solution.

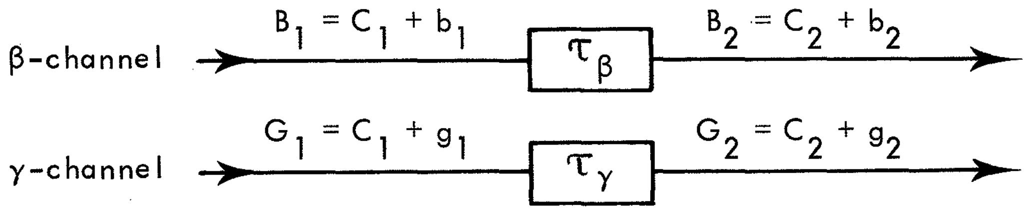

Figure 1 - Schematic arrangement and notation (see text)

Let us first consider the beta channel. Since losses are produced by the dead times of registered pulses, the probability for an original B_1-event to be suppressed is given by the chance that any B_2-event occurs in an interval of length τ_β preceding the B_1-event. Let $t = 0$ be given by the arrival of a registered B_2-pulse (Fig. 2). If the interval density between B_2-events is described by $f_B(t)$, the total density is known to be

$$F_B(t) = \sum_{k=1}^{\infty} \left\{ f_B(t) \right\}^{*k} , \tag{1}$$

and a well-known limit theorem assures that this tends asymptotically towards the corresponding count rate, i.e.

$$\lim_{t \to \infty} F_B(t) = B_2 . \tag{2}$$

As this is a general property of renewal processes, similar results also hold e.g. for pulses of the type G_2, b_2 or C_2.

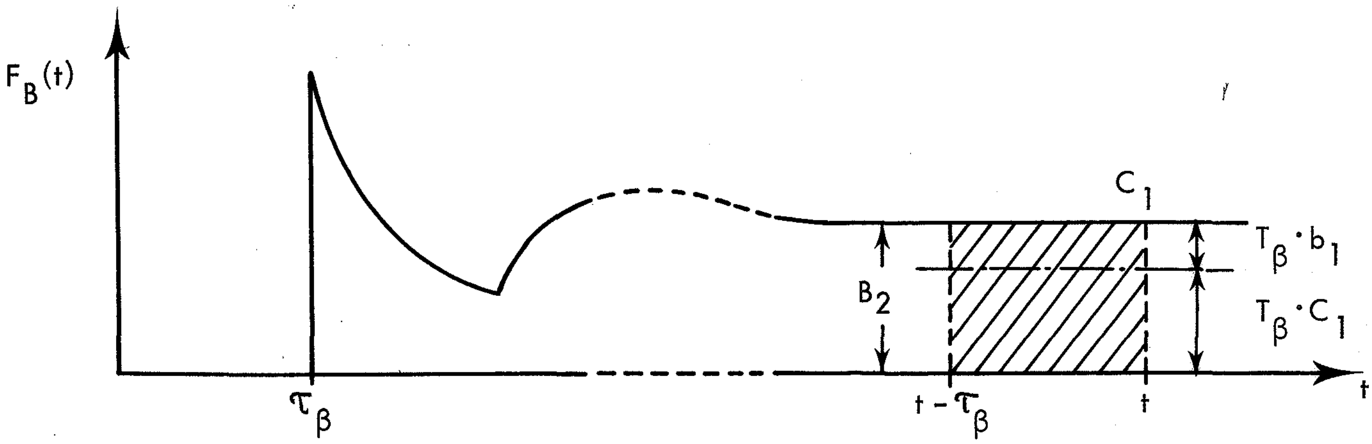

Fig. 2 - Total arrival density for a B_2-pulse in the vicinity of an original C_1-event at t. The origin $t = 0$ is determined by the arrival of a (registered) B_2-event.

If the interval density $f_C(t)$ for the true coincidences C_2 were known, their rate could be determined (at least in principle) by forming the first moment

$$C_2^{-1} = \int_0^{\infty} t \cdot f_C(t) \, dt .$$

However, $f_C(t)$ is unknown for the time being and may be supposed to be very complicated. On the other hand, equation (2) may now offer a novel possibility since

$$C_2 = F_C(\infty) , \tag{2'}$$

provided we succeed in determining the asymptotic total density F_C ; a knowledge of $f_C(t)$ would then not be needed. Let us therefore look at the requirements for a C_1-event to escape suppression.

The probability for a C_1-event, located at t, to get lost in the beta channel is

$$P_\beta(t) = \int_{t-\tau_\beta}^{t} F_B(x)\, dx . \tag{3}$$

Hence, its (local) survival probability becomes

$$T_\beta(t) = 1 - P_\beta(t) = 1 - \int_{t-\tau_\beta}^{t} F_B(x)\, dx . \tag{4}$$

For $t \longrightarrow \infty$ this quantity goes over into the usual transmission factor

$$T_\beta \equiv B_2/B_1 = \lim_{t \longrightarrow \infty} T_\beta(t) = 1 - B_2 \tau_\beta , \tag{5}$$

where use has been made of (2). This may also be written as

$$T_\beta = 1 - (C_1 + b_1)\, T_\beta \tau_\beta , \tag{5'}$$

a form which is equivalent to the ~~better~~-known formula
$T_\beta = (1 + B_1 \tau_\beta)^{-1}$, as is easily verified.

We now try to write (5') as a product, where the first factor contains the effect due to the original C_1-pulses in the form

$$_\beta T_{C_1} = 1 - C_1 T_\beta \tau_\beta . \tag{6}$$

Then

$$T_\beta = {_\beta T_{C_1}} \cdot T_{b_1|C_1} , \tag{7}$$

where the conditional transmission factor is now given by

$$T_{b_1|C_1} = \frac{T_\beta}{{}_\beta T_{C_1}} = \frac{1 - (C_1 + b_1)\, T_\beta\, \tau_\beta}{1 - C_1\, T_\beta\, \tau_\beta} \;. \tag{8}$$

The gamma channel is treated in a completely analogous way.

To proceed further, we have to consider the very special case where

$$b_1 = g_1 \equiv a_1 \qquad \text{and} \qquad \tau_\beta = \tau_\gamma \equiv \tau \;. \tag{9}$$

Then we clearly have also

$$B_1 = G_1 \equiv A_1 \qquad \text{and} \qquad B_2 = G_2 \equiv A_2 \;.$$

This is most unfortunate as it brings us far away from the usual experimental situation, but for the moment we can see no way to avoid it. In this situation (with $T_\beta = T_\gamma \equiv T$) we have

$$T_{a_1|C_1} = T_{b_1|C_1} = T_{g_1|C_1} = \frac{1 - (C_1 + a_1)\, T\tau}{T_{C_1}} = \frac{T}{T_{C_1}} \;, \tag{10a}$$

where

$$T_{C_1} = {}_\beta T_{C_1} = {}_\gamma T_{C_1} = 1 - C_1\, T\, \tau \;. \tag{10b}$$

The transmission factor T_c for the surviving true coincidence can now (and only now) be written in the form of a product and turns out to be

$$T_c = T_{C_1} \cdot T_{b_1|C_1} \cdot T_{g_1|C_1} = T_{C_1} \cdot T^2_{a_1|C_1} = \frac{T^2}{T_{C_1}} \;,$$

hence

$$T_c = \frac{T^2}{1 - C_1\, T\tau} \;, \tag{11}$$

where

$$T = \frac{1}{1 + A_1 \tau} = 1 - A_2\, \tau \;.$$

An alternative, but equivalent form reads

$$T_c = T^2 + C_2\, T\, \tau = T\, (T + C_2 \tau) \;, \tag{12}$$

where the "observable" rate C_2 is used as an independent variable, which may sometimes be preferable.

It is easy to check two limiting cases, namely that

$$\begin{aligned} &- \text{for} \quad C_1 \longrightarrow 0 \ : \quad T_c \longrightarrow T^2 \quad \text{and} \\ &- \text{"} \quad a_1 \longrightarrow 0 \ : \quad T_c \longrightarrow T \ , \end{aligned} \tag{13}$$

as has to be expected.

As mentioned before, our final formulae (11) or (12) are of little practical interest, because the assumption $b_1 = g_1$ (or likewise $B_1 = G_1$), which would suppose equal counting efficiencies ε_β and ε_γ , is too far from reality.

Nevertheless, we think that the result is not quite superfluous all the same, for use of it can be made at least in the following two ways

- numerical checks by Monte Carlo simulation,
- control whether the usual (approximate) formulae give the correct limiting values (13) or not.

This will be done in the next two sections.

3. Some Monte Carlo simulations

If T_c is expressed in terms of the ratios $x_1 = C_1/A_1$ or $x_2 = C_2/A_2$, we easily find the relations

$$T_c = \frac{T^2}{1 - x_1(1-T)} \quad \text{and} \tag{11'}$$

$$T_c = T^2 + x_2\, T\,(1-T) \ . \tag{12'}$$

The last form shows that for T constant, a plot of the transmission factor T_c for coincidences as a function of the ratio C_2/A_2 should give a straight line which starts (for $x_2 = 0$) at T^2 and ends (for $x_2 = 1$) at T. This simple relation (Fig. 3) can be checked by Monte Carlo simulations.

Such calculations have already been performed some years ago. The results - for $A_1 = 50\ 000$, 100 000, 150 000 and 200 000 s^{-1}, always with $\tau = 5\ \mu s$ - can be found in [1] and they show excellent agreement with (11'). In the meantime these simulations have been extended to higher count rates (A_1 from 300 000 to 1 000 000 s^{-1}) with T as low as 1/6. As can be seen from Figs. 4 and 5 (which are based on different simulations), the agreement with formula (11) is still very good. We therefore think that this relation (or likewise (12)), in spite of the perhaps somewhat heuristic way it was obtained, is a rigorous one. Unfortunately, several attempts to extend such a simple probabilistic approach to the more interesting situation where $B_1 \neq G_1$ have failed.

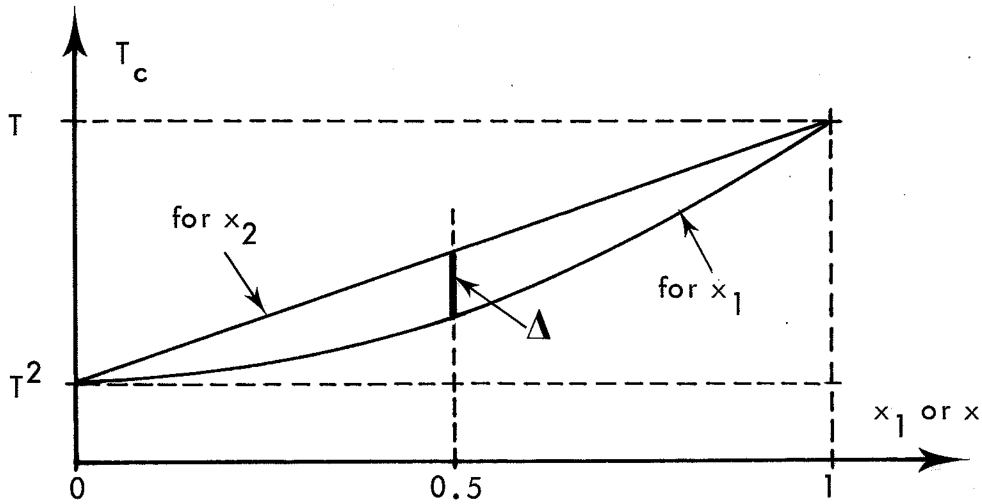

Figure 3 - Schematic graphical plot of T_c when $x_1 = C_1/A_1$ or $x_2 = C_2/A_2$ are used for the abscissa. The difference Δ for $x_1 = 0.5$ and $x_2 = 0.5$, indicated in the figure can be readily shown to be $(T/2)\cdot(1-T)^2/(1+T)$.

4. A comprehensive list of suggested theoretical formulae*

In what follows we want to compare some of the most widely used formulae for the $4\pi\beta$-γ coincidence method with respect to what they predict for the transmission T_c for true coincidences. For this purpose, the resolving time of the coincidence circuit will be assumed to be zero. In order to simplify the comparison, background will be neglected and we suppose equal dead times in both channels. Furthermore, corrections due to the finite life-time of the source or to the decay scheme will not be taken into account. The notation used is the one introduced in Fig. 1. We first list some of the appropriate formulae.

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* This section is an abridged and simplified (by assuming $\tau_\beta = \tau_\gamma$) version of information communicated previously to A. Spernol (letter of February 10, 1971). For a representative collection of formulae now used at various laboratories, see [2].

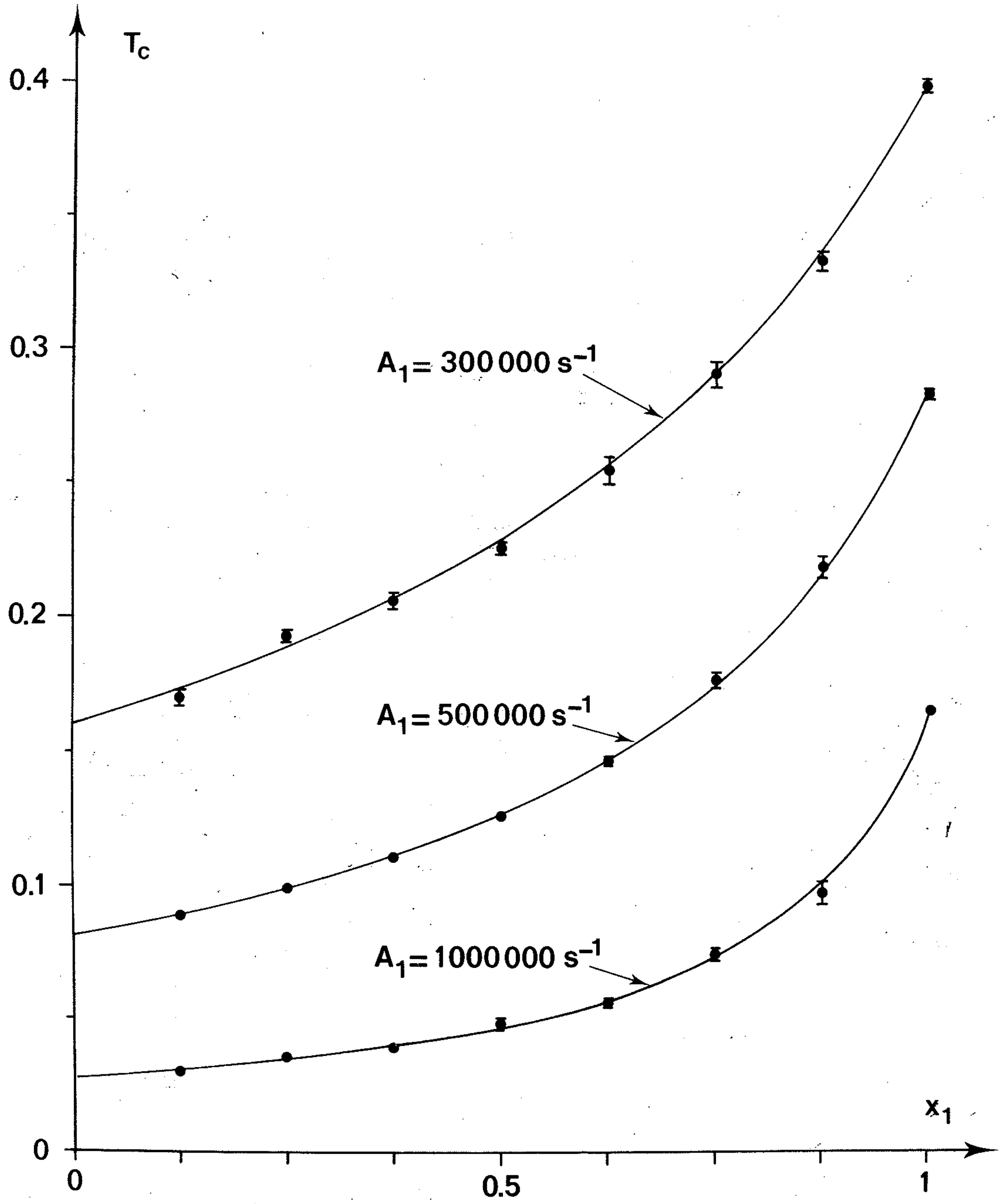

Figure 4 - Result of Monte Carlo simulations for the transmission factor $T_c = C_2/C_1$ as a function of $x_1 = C_1/A_1$. The dead times are always taken as $\tau = 5\ \mu s$. Each experimental point is based on 5 000 registered C_2-events (only 2 500 for $A_1 = 1\,000\,000\ s^{-1}$). The full line is the theoretical expectation according to (11').

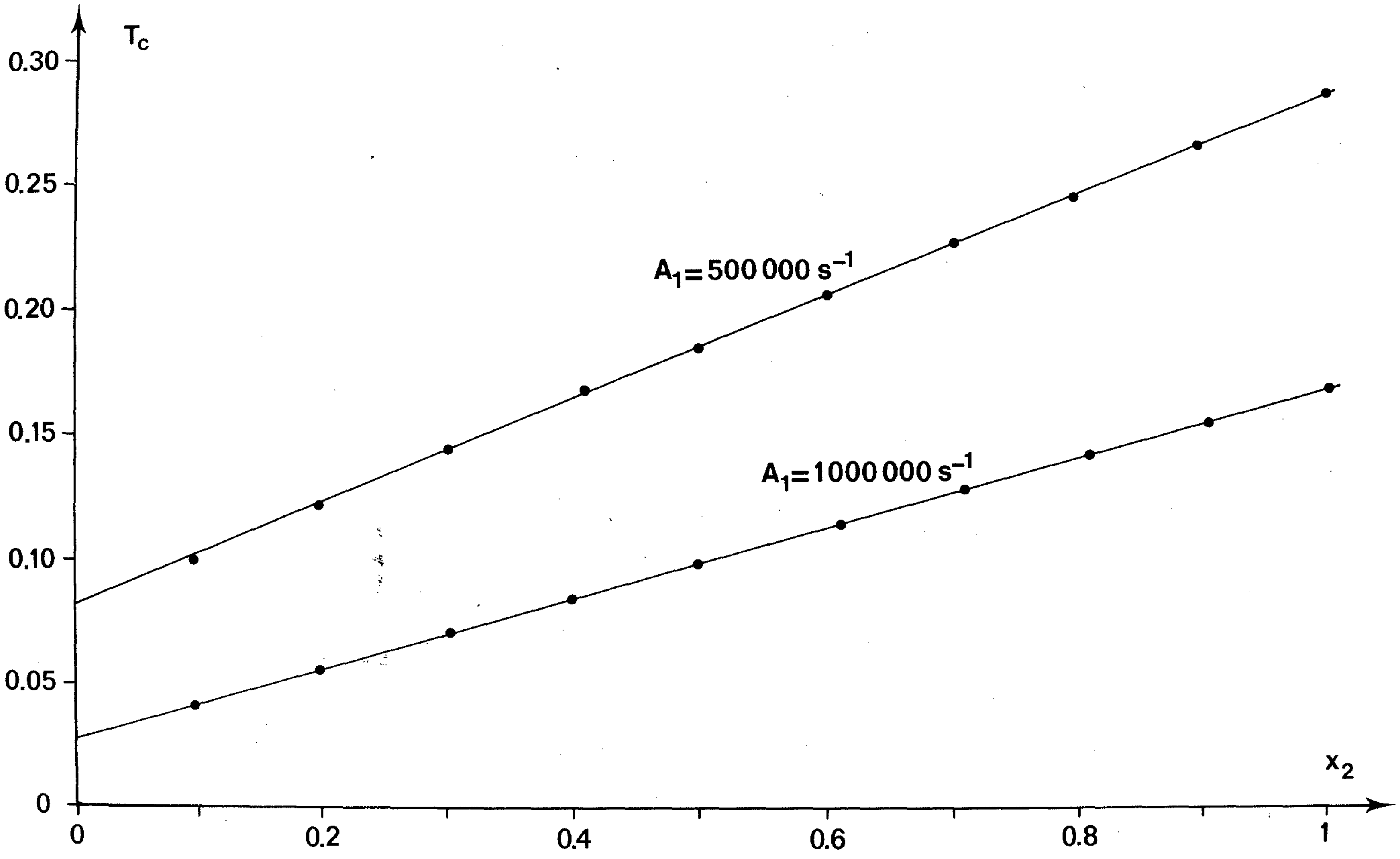

Figure 5 - Similar to Fig. 4, but for the abscissa $x_2 = C_2/A_2$, for which theory predicts a straight line given by (12'). For $A_1 = 500\ 000\ s^{-1}$ each point is based on 10 000 C_2-events (but 25 000 for $A_1 = 1\ 000\ 000\ s^{-1}$). The approximate equidistance of the x_2 values has been obtained by choosing for the input parameter C_1 the values $A_1\, x_2 / [T + x_2 (1 - T)]$, with $x_2 = 0.1, 0.2, \ldots, 1.0$.

a) Formula of Campion - Bryant

In the interpretation of Bryant ([3], eq. (6)), Campion's formula [4] can be written in the form

$$C_1 = \frac{C_2}{1 - (B_2 + G_2 - C_2)\tau} \,, \tag{14}$$

from which we may deduce that

$$T_c = \frac{C_2}{C_1} = \frac{1 - (B_2 + G_2)\tau}{C_1\,(1/C_1 - \tau)} \,.$$

With the usual abbreviations

$$T_\beta = (1 + B_1\tau)^{-1} \qquad \text{and} \qquad T_\gamma = (1 + G_1\tau)^{-1}$$

for the transmission factors in the beta and gamma channels we finally obtain

$$T_c = \frac{1 - (T_\beta B_1 + T_\gamma G_1)\tau}{1 - C_1\tau} \,. \tag{15}$$

b) "Intercomparison" formula (I)

Of the formula mentioned in a) there exist in fact several variants, but the differences only show up for different dead times (there is no well-defined "Campion formula"*). A particularly simple form has for instance been suggested in [5], and essentially identical versions have been proposed for use in a number of international comparisons of activities. They all reduce in our case to

$$\dot{N}_o = \frac{B_2 \cdot G_2}{C_2} \cdot \frac{1}{1 - C_2\tau} \tag{16}$$

for the source rate which is also given by $N_o = B_1 G_1/C_1$.

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* Equation (12) of [4] actually corresponds to the transmission factor $T_c = 1 - (B_1 + G_1 - C_1)\tau$, where for instance T_β is taken as $1 - B_1\tau$, instead of $(1 + B_1\tau)^{-1}$. It is therefore only a first order correction and will not be discussed further in what follows.

Hence

$$\frac{1}{1 - T_c \cdot C_1 \tau} = \frac{T_c}{T_\beta \cdot T_\gamma} ,$$

or

$$T_c^2 \cdot C_1 \tau - T_c + T_\beta T_\gamma = 0 .$$

The solution for T_c is therefore

$$T_c = \frac{1}{2 \cdot C_1 \tau} \left(1 - \sqrt{1 - 4 \cdot T_\beta T_\gamma \cdot C_1 \tau}\right) . \qquad (17)$$

c) "Intercomparison" formula (II)

Another formula occasionally used by participants in intercomparisons leads for our simplified conditions to

$$N_o = \frac{B_2 \cdot G_2}{C_2} (1 + C_2 \tau) . \qquad (18)$$

For the transmission of the true coincidences, this can be readily shown to correspond to

$$T_c = \frac{T_\beta T_\gamma}{1 - T_\beta T_\gamma \cdot C_1 \tau} . \qquad (19)$$

d) Formula of Gandy (I)

A simplified formula for practical use is given by Gandy (eq. (II-38) in [6]), from which it follows that

$$C_1 = C_2 \cdot \frac{1 + (B_1 + G_2)\tau}{1 + C_2 \tau} . \qquad (20)$$

This corresponds to a transmission factor

$$T_c = \frac{1}{1 + (B_1 + G_1 - C_1)\tau} . \qquad (21)$$

e) Formula of Gandy (II)

In his original paper (eq. (18) of [7]) Gandy states a relation which goes up to third-order terms of the dead time. In our notation it reads

$$T_c^{-1} = 1 + (B_1 + G_1 - C_1)\tau + \left[B_1 \cdot G_1 - C_1 (B_1 + G_1)/2\right]\tau^2$$
$$+ C_1 (B_1 - G_1)^2 \tau^3/3 \; . \tag{22}$$

f) Formula of Bryant

Finally, we list the formula given by Bryant (eq. (4) in [3]) which is

$$N_o = \frac{B_2 \cdot G_2}{C_2} \left[1 + \frac{2 C_2 \tau}{2 - (B_2 + G_2)\tau}\right] \; . \tag{23}$$

From this we obtain for the transmission factor

$$T_c = T_\beta T_\gamma \cdot \frac{2 - (B_2 + G_2 - 2 C_2)\tau}{2 - (B_2 + G_2)\tau} \; .$$

After some elementary -but tedious -algebra, an equivalent expression can also be found in terms of the original count rates, namely

$$T_c = T_\beta T_\gamma \cdot \frac{2 - (T_\beta B_1 + T_\gamma G_1)\tau}{2 - (T_\beta B_1 + T_\gamma G_1 + 2 T_\beta T_\gamma C_1)\tau}$$
$$= T_\beta T_\gamma \cdot \frac{2 + (B_1 + G_1)\tau}{2 + (B_1 + G_1 - 2 C_1)\tau} \; . \tag{24}$$

This enumeration does not pretend to be exhaustive; too many different formulae have been suggested in the last 20 years or so for the coincidence method. Nevertheless we hope to have included those types which are most often applied by the users.

5. Comparison of the asymptotic behaviour of T_c

A possible way to decide whether a suggested formula for T_c is "in principle" acceptable or not consists in checking its behaviour for two extreme cases. For doing this in the simplest way, we choose the special case where $B_1 = G_1 \equiv A_1$, whence also $T_\beta = T_\gamma \equiv T$ for equal dead times.

The limiting cases we wish to consider are $C_1 = 0$ and $C_1 = A_1$, respectively, for which we know the correct answers in advance, namely

$$\begin{aligned} T_c &= T^2 \quad &&\text{for} \quad C_1 = 0 \quad \text{and} \\ T_c &= T \quad &&\text{"} \quad C_1 = A_1 \; . \end{aligned} \tag{25}$$

In the first case the absence of common events in both channels guarantees their independence; in the second case, the processes are exactly identical in both channels so that the result is the same as for a single one.
In Table 1 we summarize the results of such a comparison for the six types of formula considered in the previous section.

Table 1 - Some analytical forms of the transmission factor T_c for true coincidences, assuming $B_1 = G_1$ and equal dead times τ (for $\tau_r = 0$).

Formula	T_c	T_c for $C_1 = 0$	T_c for $C_1 = A_1$
Campion-Bryant (15)	$\frac{2T-1}{1-C_1\tau}$	$2T-1$	T
"Intercomparison" (I) (17)	$\frac{1}{2C_1\tau}\left(1-\sqrt{1-4T^2C_1\tau}\right)$	T^2	$\frac{T^2}{1-T}$
"Intercomparison" (II) (19)	$\frac{T^2}{1-T^2C_1\tau}$	T^2	$\frac{T^2}{1-T+T^2}$
Gandy (I) (21)	$\frac{T}{2-T(1+C_1\tau)}$	$\frac{T}{2-T}$	T
Gandy (II) (22)	$\frac{T^2}{1-TC_1\tau}$	T^2	T
Bryant (24)	$\frac{T^2}{1-TC_1\tau}$	T^2	T

It follows from a comparison of the limiting values in Table 1 with (25) that only Gandy's complete formula (i.e. including at least second-order terms) and Bryant's expression for T_c pass our test, whereas for all the other (and more commonly used) formulae one of the asymptotic values is erroneous. On the other hand, it is interesting to note that for the two cases with correct limiting behaviour the expression for T_c is identical to the previously suggested formula (11), giving thereby further support to our claim for its validity.

Although an incorrect limiting value in no way implies that the values derived for practical experimental situations are necessarily doubtful too, the formulae which lead to the last two entries in Table 1 should clearly be preferred and considered as more serious candidates for describing the actual corrections needed, at least for the idealized conditions assumed in the underlying model.

6. Outlook

It is certainly gratifying to see that both a simple probabilistic argument and a check of the asymptotic behaviour lead to an expression for the transmission factor T_c, for the case of equal count rates in both channels, which is in excellent agreement with simulations made in extreme conditions. Nevertheless, this does not guarantee that either Gandy's or Bryant's expressions are rigorous (and certainly not both, as they are not identical). In fact, it is obvious from its derivation that Gandy's result only gives the first (probably two) terms in a series development, and in view of the rapidly increasing degree of complexity there is little hope to go much further. In contrast to this, Bryant's very elegant approach looks more promising. In the framework of his assumptions the results derived seem to be fully consistent, in particular also with regards to the random coincidences. However, some of the implicit additional assumptions (as e.g. the uniform overlapping of dead times) are not fully convincing to us and might be worth a more exacting study.

In this context it may also be appropriate to remember that a rigorous mathematical solution of the coincidence problem, although most welcome, is only one part of the story. The other is that an important assumption made in the basic model is never really fulfilled in any experimental arrangement: the original Poisson process is always more or less disturbed by unavoidable dead times of the detectors or the electronics, and these effects may be important at high count rates. They are usually not well defined and therefore difficult to account for. Perhaps methods based on the empirical interval distributions might be useful for estimating their influence on the final result.

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(December 1976)