Rapport BIPM-76/15

Some relations between asymptotic results for dead-time-distorted processes

Part II: The variances

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1. General introductory remarks

As this second part is a direct sequel to the earlier report dealing with the expectation values [17], we shall go on with the numbering of the equations and references in order to simplify comparisons and minimize repetitions. The basic approach is the same as before: the total measuring time t, which is supposed to be sufficiently long, is subdivided by t' into two complementary sections. It is practical to chosse for this subdivision the arrival of the first event and hence to put $t' = t_1$. This implies that for the second time interval (from t_1 to t) we always have to do with an ordinary renewal process. The type of the first process (from 0 to t_1, with t_1 included) is determined by the choice of the time origin.

The various cases arising from the two types of dead times considered and the three counting processes for the first section will be discussed separately in what follows.

For the general case, the variance for the total measuring interval of length t, which is subdivided by t_1, can be split up into two contributions, namely

$$\sigma_k^2(t) = \int_0^t \varphi(t_1) \left[S_1^2(t_1) + S_2^2(t - t_1) \right] dt_1$$

$$\equiv V_1 + V_2 . \tag{28}$$

Thereby $\varphi(t_1)$ is the interval density for the arrival of the first (registered) event, whereas S_1^2 and S_2^2 are the variances for the number of pulses arriving in the first and second part of the time interval t, respectively. This approach is completely analogous to the one used previously for the expectation values (see section 4 of [17]), as shown more explicitly in the Appendix.

2. Evaluation of S_1^2 and S_2^2

Let us consider a certain counting process (specified by the index p in what follows), which may be of the ordinary (or), equilibrium (eq) or free counter (fr) type. The general relations given below are valid for a non-extended or an extended dead time, unless stated otherwise.

Since t_1 is chosen as the arrival time of the first event, there is always exactly one pulse in the first part of the interval and we have

$$ {}_pS_1^2(t_1) = ({}_p\hat{k}_1 - 1)^2 , \tag{29}$$

where the number ${}_p\hat{k}_1$ of expected events in the interval t_1 depends on the way the time origin has been chosen*. We are now going to derive a relation for the "effective" mean number ${}_p\hat{k}_1$ appearing in (29). A look at Table 3 shows that for any measuring condition p the expectation value is of the form

$$ {}_p\hat{k}(t) = \alpha \cdot t + {}_p\beta . \tag{30}$$

The explicit expressions for α and ${}_p\beta$ are put together in Table 4.

Type of process	τ non-extended	τ extended
any	$\alpha = \lambda\rho$	$\alpha = \rho/y$
ordinary	${}_p\beta = -\lambda x + \frac{1}{2}\lambda^2 x^2$	${}_p\beta = -x/y$
equilibrium	0	0
free counter	$\frac{1}{2}\lambda^2 x^2$	$(y-1-x)/y$

Table 4 - Values of α and ${}_p\beta$ for the expectation value (30), with the abbreviations

$x = \rho\tau$, $\lambda = (1 + x)^{-1}$ and $y = e^x$

(ρ is the count rate of the original Poisson process)

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* The pitfall of taking $k_1 = 1$ and thus $S_1^2 = 0$ should be avoided, of course.

For a given process p, $\hat{k}_1$ can therefore be written as

$$ {}_p\hat{k}_1(t_1) = {}_p\hat{k}(t) - {}_{or}\hat{k}(t - t_1) $$
$$ = \alpha t + {}_p\beta - \left[\alpha (t-t_1) + {}_{or}\beta\right] $$
$$ = \alpha t_1 + {}_p\beta - {}_{or}\beta \, , \qquad (31) $$

assuring thereby that the expectations of the two parts sum up correctly (see also Appendix). Substitution into (29) leads to

$$ {}_pS_1^2(t_1) = (\alpha t_1 + {}_p\beta - {}_{or}\beta - 1)^2 \, , \qquad (32) $$

and the corresponding explicit expressions, derived by means of Table 4, can be found in Table 5.

Type of process	τ non-extended	τ extended
ordinary	$(\lambda\rho \cdot t_1 - 1)^2$	$(\frac{\rho}{y} \cdot t_1 - 1)^2$
equilibrium	$(\lambda\rho \cdot t_1 + \lambda x - \frac{1}{2}\lambda^2 x^2 - 1)^2$	$(\frac{\rho}{y} \cdot t_1 + \frac{x}{y} - 1)^2$
free counter	$(\lambda\rho \cdot t_1 + \lambda x - 1)^2$	$(\frac{\rho}{y} \cdot t_1 - \frac{1}{y})^2$

Table 5 - Values of the quantity ${}_pS_1^2(t_1)$

Let us now consider the quantity S_2^2. For the time interval starting at t_1 the process is always of the ordinary type, thus

$$ {}_pS_2^2(t - t_1) = {}_{or}\sigma_k^2(t - t_1) \, . \qquad (33) $$

For t large enough we may use the asymptotic results known from previous studies, namely

- for a non-extended dead time (eq. 13 of [1]):

$$ S_2^2 \cong \lambda^3 \left[\rho (t-t_1) - \frac{1}{12} \lambda x (12-6x-4x^2-x^3) \right] , \qquad (34) $$

- for an extended dead time (see e.g. [2]):

$$ S_2^2 \cong \frac{1}{y^2} \left[(y-2x) \, \rho \, (t-t_1) - x(y-3x) \right] . \qquad (35) $$

Hence, S_2^2 can always be written in the general form

$$S_2^2 \cong A \cdot t_1 + B \,, \tag{36}$$

with the values A and B given in Table 6.

Type of dead time	A	B
non-extended (n)	$-\rho\lambda^3$	$\lambda^3\left[\rho t - \frac{1}{12}\lambda x\,(12-6x-4x^2-x^3)\right]$
extended (e)	$-\frac{\rho}{y^2}\,(y-2x)$	$\frac{1}{y^2}\left[\rho t\,(y-2x) - x\,(y-3x)\right]$

Table 6 - Explicit expressions for the coefficients A and B in (36)

3. The variance V_1 for the first time interval

According to (28), the contribution V_1 arising from the first interval section (0 to t_1) is given for a process p by

$$_pV_1 = \int_0^t {}_p\varphi(t_1) \cdot {}_pS_1^2(t_1)\, dt_1 \,. \tag{37}$$

For the asymptotic case and with (32), this can also be written as

$$_pV_1 \cong \int_0^\infty {}_p\varphi(t_1) \cdot (\alpha^2 \cdot t_1^2 - 2\alpha\, {}_p\gamma \cdot t_1 + {}_p\gamma^2)\, dt_1 \,, \tag{38}$$

where $_p\gamma = 1 + {}_{or}\beta - {}_p\beta$.

If we denote by

$$_pm_r = \int_0^\infty {}_p\varphi \cdot t_1^r \, dt_1 \tag{39}$$

the moment of order r for a process p, the asymptotic value of V_1 can also be expressed by

$$_pV_1 \cong \alpha^2 \cdot {}_pm_2 - 2\alpha\, {}_p\gamma \cdot {}_pm_1 + {}_p\gamma^2 \,. \tag{40}$$

For the sake of convenience, the explicit expressions for the relevant combination of parameters is given in Table 7. The moments ${}_p m_1$ and ${}_p m_2$ have been calculated previously (Tables 1 and 2 of [6]); the values required here are listed in Table 8.

Type of process	$2\alpha\,{}_p\gamma$	${}_p\gamma^2$
- for a non-extended dead time:		
ordinary	$2\lambda\rho$	1
equilibrium	$2\lambda\rho\,(1-\lambda x+\frac{1}{2}\lambda^2x^2)$	$(1-\lambda x+\frac{1}{2}\lambda^2x^2)^2$
free counter	$2\lambda\rho\,(1-\lambda x)$	$(1-\lambda x)^2$
- for an extended dead time:		
ordinary	$2\rho/y$	1
equilibrium	$2\rho(1-x/y)/y$	$(1-x/y)^2$
free counter	$2\rho/y^2$	$1/y^2$

Table 7 - Explicit expressions for the coefficients in (40)

Type of process	${}_p m_1$	${}_p m_2$
- for a non-extended dead time:		
ordinary	$\frac{1}{\rho}(1+x)$	$\frac{2}{\rho^2}(1+x+x^2/2)$
equilibrium	$\frac{1}{\rho}\frac{1+x+x^2/2}{1+x}$	$\frac{2}{\rho^2}\frac{1+x+x^2/2+x^3/6}{1+x}$
free counter	$\frac{1}{\rho}$	$\frac{2}{\rho^2}$
- for an extended dead time:		
ordinary	$\frac{1}{\rho}\,y$	$\frac{2}{\rho^2}\,y\,(y-x)$
equilibrium	$\frac{1}{\rho}(y-x)$	$\frac{2}{\rho^2}(y^2-2xy+x^2/2)$
free counter	$\frac{1}{\rho}$	$\frac{2}{\rho^2}$

Table 8 - The first two moments for the various measuring conditions

Application of (40) then gives the values of V_1 reported in Table 9.

Type of process	τ non-extended	τ extended
ordinary	λ^2	$1 - 2\,x/y$
equilibrium	$\lambda^2 \left[1 + \frac{1}{12}\,\lambda^2\,x^3\,(4+x)\right]$	$1 - 2\,x/y$
free counter	λ^2	$1/y^2$

Table 9 - Values of the variance V_1 for the different measuring conditions

4. The variance V_2 for the second time interval

For the time interval from t_1 to t, the contribution V_2 is given according to (28) for a process p by

$$ {}_pV_2 = \int_0^t {}_p\varphi(t_1) \cdot {}_pS_2^2\,(t - t_1)\; dt_1 \, . \tag{41}$$

With (36), this can be written for the asymptotic case as

$$ {}_pV_2 \cong \int_0^\infty {}_p\varphi(t_1) \cdot (A\,t_1 + B)\; dt_1 \, . \tag{42}$$

Using again the respective moments, we thus have

$$ {}_pV_2 \cong A \cdot {}_pm_1 + B \, . \tag{43}$$

With the values assembled in Tables 6 and 8, the calculation of V_2 is straightforward (Table 10).

Type of process	τ non-extended	τ extended
ordinary	${}_nB - \lambda^2$	${}_eB - (y - 2x)/y$
equilibrium	${}_nB - \lambda^3(1 + \frac{1}{2}\,\lambda\,x^2)$	${}_eB - (y - 2x)\,(y - x)/y^2$
free counter	${}_nB - \lambda^3$	${}_eB - (y - 2x)/y^2$

Table 10 - Values of the variance V_2 for the different measuring conditions, with ${}_nB$ and ${}_eB$ as given in Table 6

Simple values are obtained for the differences which we can form for V_1, V_2 or their sum $\sigma_k^2(t)$, taken between the various processes. The results, readily obtained from Tables 9 and 10, are compiled in Table 11.

	τ non-extended	τ extended
${}_{eq}V_1 - {}_{or}V_1$	$\frac{1}{12}\lambda^4 x^3 (4+x)$	0
${}_{fr}V_1 - {}_{or}V_1$	0	$(1-y^2+2xy)/y^2$
${}_{fr}V_1 - {}_{eq}V_1$	$\frac{-1}{12}\lambda^4 x^3 (4+x)$	$(1-y^2+2xy)/y^2$
${}_{eq}V_2 - {}_{or}V_2$	$\lambda^3 x (1-\frac{1}{2}\lambda x)$	$(y-2x)\,x/y^2$
${}_{fr}V_2 - {}_{or}V_2$	$\lambda^3 x$	$(y-2x)\,(y-1)/y^2$
${}_{fr}V_2 - {}_{eq}V_2$	$\frac{1}{2}\lambda^4 x^2$	$(y-2x)\,(y-1-x)/y^2$
${}_{eq}\sigma_k^2(t) - {}_{or}\sigma_k^2(t)$	$\frac{1}{12}\lambda^4 x\,(12+6x+4x^2+x^3)$	$(y-2x)\,x/y^2$
${}_{fr}\sigma_k^2(t) - {}_{or}\sigma_k^2(t)$	$\lambda^3 x$	$(1-y+2x)/y^2$
${}_{fr}\sigma_k^2(t) - {}_{eq}\sigma_k^2(t)$	$\frac{1}{12}\lambda^4 x^2(6-4x-x^2)$	$[1-y+x\,(2-y+2x)]/y^2$

Table 11 - Differences in the partial variances V_1 and V_2 or the total variance $\sigma_k^2(t)$ for the different processes

5. Survey of the final results and conclusions

It follows from the information in Table 11 that the origin for the differences in the asymptotic variances for the various processes is complex. In general, there are contributions arising from the first as well as from the second part of the time interval. This is in contrast to the much simpler situation met for the expectation values (compare sections 5 and 6 of [17]) where all the differences could be traced back to a single term (the quantity $\alpha\,\bar{t}_1$ in eq. 11), but not quite unsuspected. It certainly reduces the practical interest of such a decomposition of the total measuring time into two parts. Nevertheless, even in those cases where the difference stems from both V_1 and V_2, it can be exactly evaluated and therefore permits a clear appreciation of its origin.

Finally, we summarize in Table 12 the asymptotic variances $\sigma_k^2(t) = V_1 + V_2$ for the six measuring conditions studied. As it has been done for the expectation values, we indicate the earliest references (known to us) where a specific result is indicated; again, complements will be welcome. Although all the formulae listed have been mentioned somewhere else before, a critical survey for the various experimental conditions may be useful as the relevant results are largely scattered in the literature where slips, misprints and real errors are sufficiently frequent to be a nuisance. The uniform notation adopted will simplify intercomparisons and perhaps encourage further studies.

Type of process	$\sigma_k^2(t)$	earliest references	comment
- For a non-extended dead time:			
ordinary	$\lambda^3 \left[\rho t - \frac{1}{12}\lambda x(12-6x-4x^2-x^3)\right]$	[18] or [1], eq. 13	1, 2
equilibrium	$\lambda^3 \left[\rho t + \frac{1}{6}\lambda x^2(6+4x+x^2)\right]$	[18] or [1], eq. 8	3
free counter	$\lambda^3 \left[\rho t + \frac{1}{12}\lambda x^2(18+4x+x^2)\right]$	[13], p.53, [18] or [1] eq. 18	4, 2, 5
- For an extended dead time:			
ordinary	$\frac{1}{y^2}\left[(y-2x)\rho t - x(y-3x)\right]$	[20], p.284, [13], p.56, [2] p.589	6, 7
equilibrium	$\frac{1}{y^2}\left[(y-2x)\rho t + x^2\right]$	[19], eq.28, [13], p.56, [2] p.590	8
free counter	$\frac{1}{y^2}\left[(y-2x)\rho t+1-y+x(2-y+3x)\right]$	[11]*, eq.46, [10], eq.41, [21]** p. 616	7

Table 12 - Summary of the asymptotic formulae for the variance $\sigma_k^2(t)$ of the number of events in a measuring interval t

* After allowance for an obvious misprint (also appearing in [15])

** After correcting a misprint (acknowledged by the author)

Comments for Table 12:

1. Campbell (eq. 22 of [9]) gives only ${}_{or}\sigma_k^2(t) \cong \lambda^3 \cdot \rho t$.

2. De Lotto et al. (eq. 30 of [10]) claim that ${}_{or}\sigma_k^2(t) = {}_{fr}\sigma_k^2(t) = \lambda^3 \cdot \rho t$.

3. The formula given by Foglio Para et al. (on p. 54 of [13] is incorrect; in our notation it would lead to $\lambda^3 \left[\rho t + \frac{1}{6} \lambda x^2 (6 + 2x + x^2/2) \right]$.

4. Feller (eq. 30 of [11]) gives only ${}_{fr}\sigma_k^2(t) \cong \lambda^3 \cdot \rho t$.

5. The expression $\lambda^3(\rho t + \frac{3}{2} \lambda x^2 + \frac{1}{3} x^3 - \frac{1}{4} \lambda x^4)$ given in [13] is in fact equivalent.

6. The result given by De Lotto et al. (eq. 43 of [10]) is incorrect (wrong sign of first term).

7. Result is exact for $t > 2\tau$.

8. Result is exact for $t > \tau$.

The major part of the present report has been elaborated during my stay at the Centre Médico-Chirurgical de la Porte de Choisy, Paris 13e, in October 1975. As a result of the excellent care received there, in particular from Drs. Moreaux and Bonnet, it has been possible to finish it afterwards. My sincere thanks go to these outstanding physicians and their collaborators.

APPENDIX

Relation of the present approach to the decomposition of $\hat{k}(t)$

In (6) the expectation value was written as

$$\hat{k}(t) = \int_0^t \varphi(t_1)\left[\hat{k}_1(t_1) + \hat{k}_2(t-t_1)\right] dt_1 ,$$

a form which is obviously of the same type as (28) used for the variance. This can be split up into two contributions and we want to show now that they correspond indeed exactly to those denoted in eq. (10) by K_1 and K_2.

Let us begin with the first contribution. For a process p, the asymptotic value is

$${}_pK_1 \cong \int_0^\infty {}_p\varphi(t_1) \cdot {}_p\hat{k}_1(t_1)\; dt_1 .$$

By applying the decomposition (31), this can also be written as

$$\begin{aligned} {}_pK_1 &\cong \int_0^\infty {}_p\varphi(t_1)\,(\alpha t_1 + {}_p\beta - {}_{or}\beta)\; dt_1 \\ &= \alpha \cdot {}_pm_1 + {}_p\beta - {}_{or}\beta \\ &= \alpha \cdot {}_{or}m_1 + (\alpha \cdot {}_pm_1 + {}_p\beta) - (\alpha \cdot {}_{or}m_1 + {}_{or}\beta) = 1 , \end{aligned} \qquad \text{(A16)}$$

since $\alpha \cdot {}_{or}m_1 = 1$ for an ordinary process (cf. Table 2b)* and the sum $\alpha \cdot {}_pm_1 + {}_p\beta$ depends only on the type of the dead time (not on the process p).

For the second contribution, we have in the asymptotic limit for a process p

$$\begin{aligned} {}_pK_2 &\cong \int_0^\infty {}_p\varphi(t_1) \cdot {}_{or}\hat{k}_2\,(t - t_1)\; dt_1 \\ &= \int_0^\infty {}_p\varphi(t_1)\left[\alpha\,(t-t_1) + {}_{or}\beta\right] dt_1 \\ &= \alpha \cdot t + {}_{or}\beta - \alpha \cdot {}_pm_1 = {}_{or}\hat{k}(t) - \alpha \cdot {}_pm_1 . \end{aligned} \qquad \text{(A17)}$$

Since (A16) and (A17) correspond exactly to the decomposition given in (10), we see that the present approach for the variances is indeed well in line with the one used previously for the expectation values.

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* where ${}_pm_1$ was denoted by $\overline{t_1}$

References*

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(November 1976)

* References [1] to [16] are listed in [17] .