On the precision of the modulo-counting technique

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1. Introduction

In its original and simplest form, the modulo technique is a method which allows to distinguish between events accurring pairwise or individually in a random process [1]. This possibility essentially stems from the fact that the quality of a number k of registered counts of being even or odd remains unchanged when any number of pairs are superimposed. By a generalization of this simple idea to multiplets ($K > 2$), one can also measure experimentally the frequencies $\pi(J \mid K)$ with which, for a given measuring interval δ, the number k of counts belongs to a certain residue class J (mod K), with $J = 0, 1, \ldots, K-1$.

If for the original sequence a specific type of stochastic process is assumed, which may be characterized by the probability to observe exactly k events in a given time interval δ, then the corresponding count rate ρ can be deduced from a measurement of $\pi(J \mid K)$, at least in principle. Practical difficulties which may arise are, for example, that a meaningful determination of ρ is not possible for $\rho\delta \gtrsim K$ or that occasionally there may be several possible solutions. However, these problems can be avoided by an appropriate choice of the experimental conditions.

In the important case of a Poisson process, the probability for observing exactly k events is known to be

$$P_\mu(k) = \frac{\mu^k}{k!} \cdot e^{-\mu} , \qquad \text{with } \mu = \rho \cdot \delta . \tag{1}$$

The modulo probabilities

$$W(J \mid K) \equiv \text{Prob}\{k = J \ (\text{mod } K)\} \tag{2}$$

have been determined previously for this process. They can be expressed as (eq. 12 in [2])

$$W(J \mid K) = \frac{1}{K} \sum_{i=0}^{K-1} x_i^{-J} \cdot e^{\mu(x_i - 1)} , \tag{3}$$

where x_i are the K roots of the equation $x^K = 1$.

For the special case $K = 2$ we therefore have

$$W(J \mid 2) = \frac{1}{2}\left[1 + (-1)^J \cdot e^{-2\mu}\right], \qquad J = 0, 1, \tag{4}$$

since here $x_o = 1$ and $x_1 = -1$.

2. Precision in the determination of ρ

In order to decide on the presence or absence of multiple pulses, the count rate ρ_{tot}, obtained by direct normal counting, is compared to the value ρ which results from a comparison of the experimental modulo-counting frequency $\pi(J \mid K)$ with the corresponding theoretical value as given in (3).

The precision in the measurement as well as the limit of detection of afterpulses are essentially given by the uncertainty of ρ which, in turn, depends on the accuracy in the determination of $\pi(J \mid K)$. Therefore, an important first step consists in determining the statistical precision $s(\rho)$ with which the count rate ρ can be obtained in this somewhat indirect way.

The relation between the slope of the curve $W(J \mid K)$, the experimental uncertainty $s(\pi)$ and the precision $s(\rho)$ of the corresponding count rate is given (for small errors) by

$$\left|\frac{d}{d\mu} W(J \mid K)\right| = \frac{s(\pi)}{\delta \cdot s(\rho)}, \tag{5}$$

since the time interval (or delay in the correlation measurement) δ can be assumed to be known with negligible error. By using eq. (20) of [2], the error in $\pi(J \mid K)$ is seen to be given by

$$s(\pi) = \frac{K}{\nu \cdot t} \cdot s(n_J), \tag{6}$$

for fixed values of K, ν and t. The corresponding experimental arrangement is represented in Fig. 5 of [2], where n_J is the number of test pulses which have passed the two gates during the measuring period t, for an oscillator of frequency ν. For K = 2 the scheme also corresponds to the set-up B in Fig. 1.

In order to estimate the statistical uncertainty $s(n_J)$, we first need to have some idea about the distribution of n_J. If ρ is very large compared to ν, there will be many events arriving during $1/\nu$, i.e. the states x_k describing

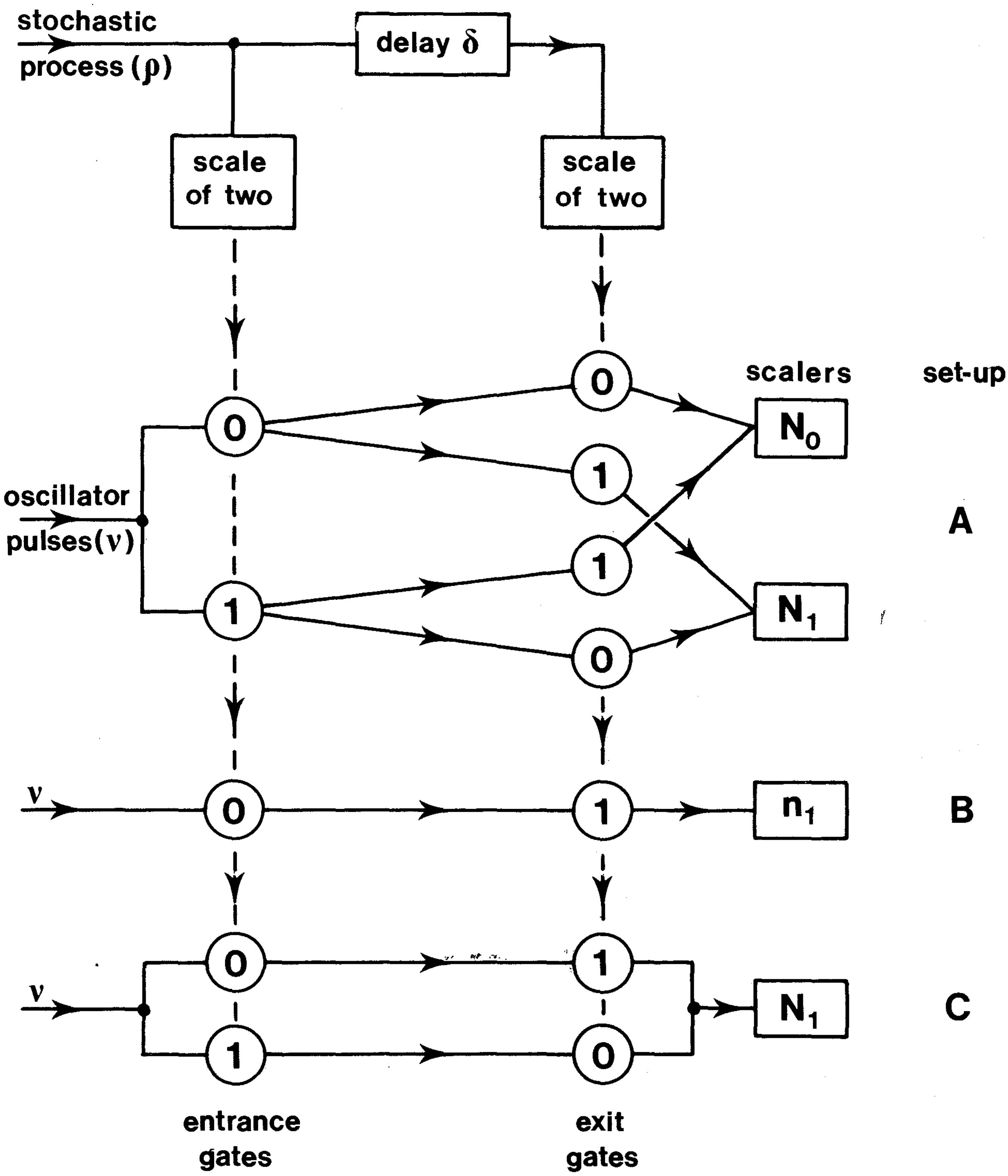

Fig. 1 – Schematic representation of three possible experimental set-ups for the modulo-counting technique, shown for the case $K = 2$. They are called

A : complete arrangement,
B : simplified arrangement (for $J = 1$),
C : symmetrical arrangement (for $J = 1$).

A gate is open only if the scale of two acting upon it is in the position (0 or 1) indicated by the number encircled. In reality, the delay δ will be produced by shift registers which are between the entrance and exit gates.

the random process will change a large number of times between subsequent test pulses. Correlation effects can then be ignored and the results of the periodic samplings become independent. In this case n_J follows approximately the binomial law, i.e.

$$\mathrm{Prob}\,(n_J) = \binom{N}{n_J} p^{n_J} \cdot (1-p)^{N-n_J} , \tag{7}$$

where $N = \nu \cdot t$ and

$$p = E\left\{\frac{n_J}{N}\right\} = \frac{W(J\,I\,K)}{K} .$$

The variance of n_J is then given by

$$s^2(n_J) = N \cdot p\,(1-p) . \tag{8}$$

Therefore

$$s(n_J) = \frac{1}{K} \sqrt{N \cdot W\,(K-W)} ,$$

and with (6)

$$s(\pi) = \sqrt{\frac{W\,(K-W)}{N}} . \tag{9}$$

This relation may be inserted into (5) and then leads to

$$s(\rho) = s(\pi) \cdot \frac{\rho}{\mu \cdot \left|\frac{dW}{d\mu}\right|} = \frac{\rho}{\mu} \sqrt{\frac{W\,(K-W)}{N}} \left\{ \left|\frac{dW}{d\mu}\right| \right\}^{-1} , \tag{10}$$

where the derivative of W can be readily determined from (3) as

$$\frac{dW}{d\mu} = \frac{1}{K} \sum_{i=0}^{K-1} (x_i - 1) \cdot x_i^{-J} \cdot e^{\mu(x_i - 1)} . \tag{11}$$

Thus, for given values of K and J, all the quantities on the right-hand side of (10) are explicitly known and can therefore be inserted. In order to choose the "best" measuring conditions, we may demand that $s(\rho)$ attains a minimum. The corresponding value of μ (or ρ) can then be found as the solution of the equation

$$\frac{d}{d\mu} s(\rho) = 0 . \tag{12}$$

Since, according to (10), $s^2(\rho)$ is inversely proportional to $N = \nu \cdot t$, the statistical uncertainty in ρ can not only be reduced by extending the measuring time t, but also by increasing the oscillator frequency ν, and the question then arises as to which point the latter will be profitable. There can obviously be little or no gain in information if two test pulses follow each other so closely that there is practically no chance for an event of the stochastic process to have arrived in the meantime. Besides, the distribution of n_J could then no longer be described by the binomial law. The question is therefore: which is the smallest interval for the test pulses which still guarantees (at least to some reasonable degree of confidence) the independence of successive controls? A look at a number of graphical plots of $W(J\,|\,K)$ - some of which have been given in previous reports - shows that as a rule of thumb we may assume that (for any J)

$$W(J\,|\,K) \cong \frac{1}{K} \qquad \text{if} \quad \mu = \rho\delta \gtrsim K . \tag{13}$$

From this it may be concluded that

$$\nu_{max} \cong \rho / K , \tag{14}$$

but it will normally be safe to have the oscillator run at a frequency somewhat below this limit to assure independence of the checks.

3. Measurements taken with the "simplified" arrangement

Let us now perform some more explicit calculations for the important special case $K = 2$. We start with (10) and assume a simplified experimental arrangement of the type sketched in Fig. 4 or 5 of ref. [2]. For convenience it is also reproduced as case B in Fig. 1. Either from (11) or directly from (4) we obtain

$$\frac{d}{d\mu} W(J\,|\,2) = (-1)^{J+1} \cdot e^{-2\mu} , \tag{15}$$

where $J = 0$ or 1.

Inserting (4) and (15) into (10) yields

$$s(\rho) = \frac{\rho}{\mu\sqrt{N}} \cdot \frac{\sqrt{\frac{1}{2}\left[1 + (-1)^J \cdot e^{-2\mu}\right]\left[2 - \frac{1}{2}\left\{1 + (-1)^J \cdot e^{-2\mu}\right\}\right]}}{e^{-2\mu}} .$$

After some rearrangements, this may also be brought into the form

$$s(\rho) = \frac{\rho}{2\mu\sqrt{N}} \sqrt{3\, e^{4\mu} + 2\,(-1)^J \cdot e^{2\mu} - 1} . \tag{16}$$

In order to determine the numerical value of μ permitting the smallest uncertainty in ρ, for given measuring conditions (characterized by the parameters ρ , N and J), we form the derivative of $s(\rho)$. By denoting

$$\left[3\, e^{4\mu} + 2\,(-1)^J \cdot e^{2\mu} - 1\right]^{1/2} \equiv U ,$$

we find

$$\frac{dU}{d\mu} = \frac{2}{U}\left[3\, e^{2\mu} + (-1)^J\right] e^{2\mu}$$

and hence

$$\frac{d}{d\mu}\, s(\rho) \sim \frac{2}{U}\left[3\, e^{2\mu} + (-1)^J\right] e^{2\mu} \cdot \mu - U. \tag{17}$$

The condition (12) therefore leads to

$$2\mu\; e^{2\mu}\left[3\, e^{2\mu} + (-1)^J\right] = U^2 ,$$

or

$$3\,(1 - 2\mu) \cdot e^{4\mu} + 2\,(-1)^J \cdot (1 - \mu)\, e^{2\mu} = 1 . \tag{18}$$

The numerical solutions of equation (18) can be found to be

$$\begin{aligned} \mu &\cong 0.534 \quad \text{for} \quad J = 0 , \\ \mu &\cong 0.353 \quad \text{"} \quad J = 1 . \end{aligned} \tag{19}$$

If these values are inserted into (16), we find for the minimum uncertainties in ρ

$$\begin{aligned} &- \text{for } J = 0 : \quad s_{min}(\rho) \cong 5.15\ \rho/\sqrt{N} \\ &- \text{"}\ \ J = 1 : \quad s_{min}(\rho) \cong 3.82\ \rho/\sqrt{N} . \end{aligned} \tag{20}$$

This shows that the choice J = 1 is to be preferred for this arrangement with K = 2 and that the delay should be such that $\mu = \rho\,\delta \cong 0.35$. It can be seen from Fig. 2, however, that the choice is not very critical.

4. Possible merits of a "complete" or "symmetrical" measurement

A closer inspection of the experimental arrangement reveals that the results obtained in section 3 can, at least in principle, be improved by a more symmetrical set-up. As a matter of fact, the expected number of registered pulses in a "complete" measurement of the type sketched in Fig. 3 of ref. [2] is $E\{N_J\} = K \cdot E\{n_J\}$. The corresponding arrangement for K = 2 is also shown as case A in Fig. 1. Although for high values of K this would require a prohibitive increase in the complexity of the electronics (the number of gates being equal to $K\,(K+1)$, for instance), this inconvenience might well be tolerated for small values, and in particular for $K = 2$. Let us check, therefore, whether the gain in statistical precision would justify such a modification.

In this case we have

$$p = E\left\{\frac{N_J}{N}\right\} = W(J\,I\,K) \tag{21}$$

and (9) now becomes

$$s^2(\pi) = \frac{1}{N}\,W\,(1 - W)\,. \tag{22}$$

In contrast to (9), this formula is invariant against an exchange of W and $1 - W$. As can be seen from (4), this corresponds for $K = 2$ to a change from $J = 0$ to $J = 1$, and it explains why in the arrangements A and C the choice of J has no influence in this case on the precision with which the count rate can be determined. Thus for $W = 1/2$, for instance, the variance would be reduced by a factor of K^2 by the complete arrangement.

In view of (4) and (15), (10) now leads to

$$s(\rho) = \frac{\rho}{\mu\sqrt{N}}\left\{\frac{1}{2}\left[1 + (-1)^J \cdot e^{-2\mu}\right]\left[1 - \frac{1}{2}(-1)^J \cdot e^{-2\mu}\right]\right\}^{\frac{1}{2}} \cdot e^{2\mu}.$$

After some rearrangements, this may be brought into the simple form

$$s(\rho) = \frac{\rho}{2\mu\sqrt{N}}\sqrt{e^{4\mu} - 1}\,. \tag{23}$$

By applying again condition (12) we find

$$e^{4\mu}\,(1 - 2\mu) = 1\,, \tag{24}$$

the numerical solution of which is

$$\mu \cong 0.398. \tag{25}$$

By using this value in (23) we get for the minimum uncertainty in the complete arrangement A, for both values of J,

$$s_{min}(\rho) \cong 2.49\,\rho\,/\sqrt{N}\,. \tag{26}$$

Since this is only two thirds of the smaller value reached in (20) by the simpler arrangement B, the improvement obtained by the complete measurement seems worth trying. Fig. 2 shows that the exact choice of μ is even less critical than previously. With increasing μ, the ratio of the variances obtained on the basis of (16) and (23) tends towards the value 3.

In practice, the symmetrical arrangement C of Fig. 1 will be preferred which requires only two shift registers, instead of four in set-up A. Since $N_o + N_1 = \nu t$, this implies no loss of information. In what follows we shall assume that the arrangement C will be used.

If we apply (14), i.e. $\nu = \nu_{max} \cong \rho/2$, the minimum uncertainty (26) becomes

$$s_{min}(\rho) \cong \frac{2.5\,\rho}{\sqrt{\rho\, t/2}} \cong 3.5\sqrt{\frac{\rho}{t}}\ . \tag{27}$$

If this result is compared with the usual uncertainty resulting from direct counting (assuming Poisson statistics for $k = \rho_{tot} \cdot t$), namely

$$s(\rho_{tot}) = \frac{s(k)}{t} = \frac{\sqrt{k}}{t} = \sqrt{\frac{\rho_{tot}}{t}}\ , \tag{28}$$

we see that the measurement of the count rate by the modulo technique is always about 3 or 4 times less precise than the direct counting method, which is assumed to be applied in parallel, hence for the same measuring time t. The modulo technique is thus about 10 times less efficient in determining count rates, i.e. it would take 10 times longer to arrive at the same statistical uncertainty. This is hardly surprising in view of the indirect way the measurement is performed and the limited information used. However, the interesting point is that the results obtained by the two methods have not to be identical, in general, and that the difference will allow us to determine the pair rate.

5. Uncertainty in the pair rate

Let us now consider the important case of an original Poisson process (of rate ρ), which is modified in such a way that occasionally a pulse appears as a doublet (or pair) instead of a single event. The corresponding probability will be denoted by θ. For a sufficiently long measuring time, we would then observe the mean rates

$$\begin{aligned} &\rho_p = \rho \cdot \theta && \text{for pairs} \\ \text{and}\quad &\rho_s = \rho\,(1-\theta) && \text{" singles,} \end{aligned} \tag{29}$$

while the total pulse rate is obviously

$$\rho_{tot} = \rho_s + 2 \cdot \rho_p\ . \tag{30}$$

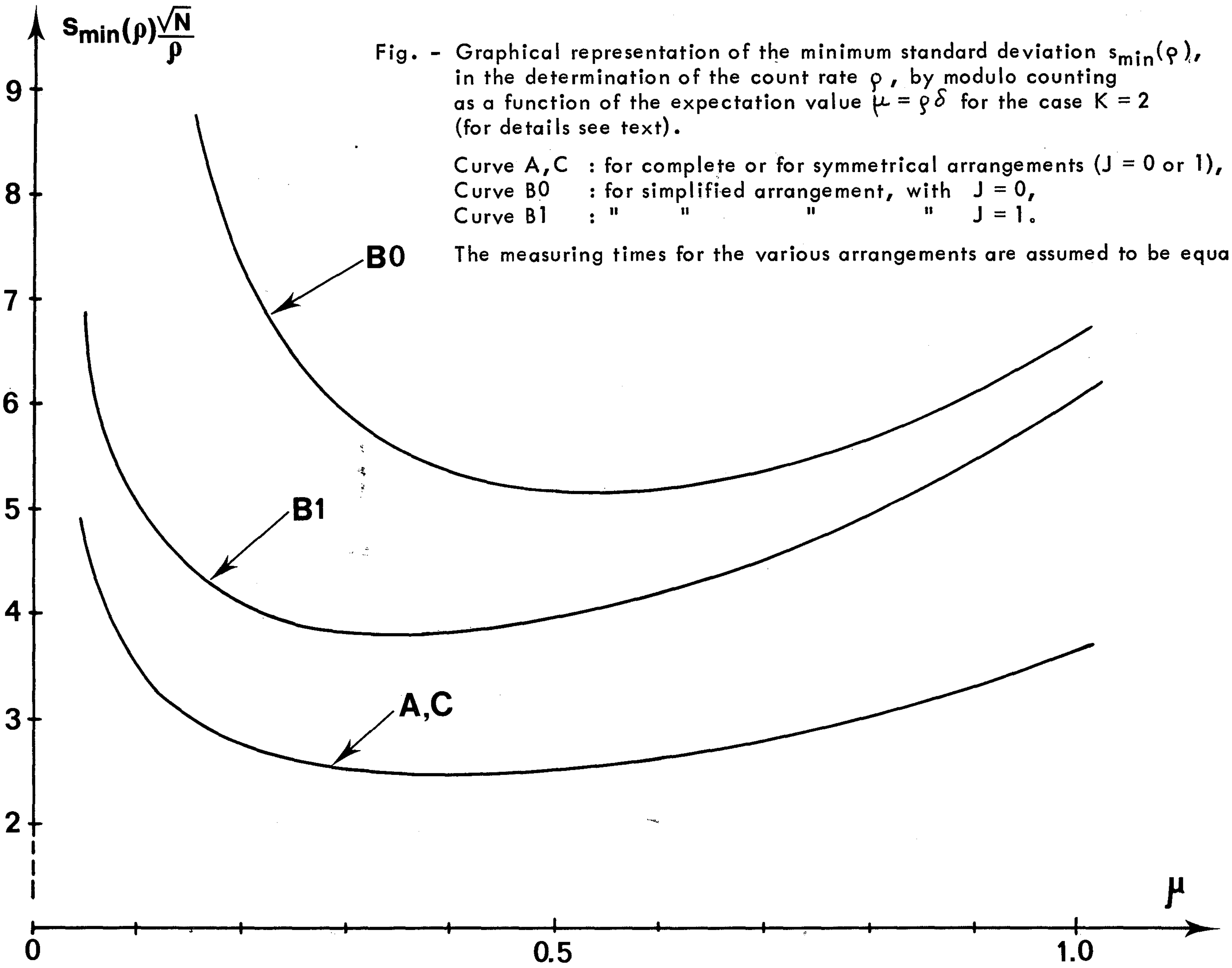

Fig. - Graphical representation of the minimum standard deviation $s_{min}(\rho)$, in the determination of the count rate ρ, by modulo counting as a function of the expectation value $\mu = \rho\delta$ for the case $K = 2$ (for details see text).

Curve A,C : for complete or for symmetrical arrangements ($J = 0$ or 1),
Curve B0 : for simplified arrangement, with $J = 0$,
Curve B1 : " " " " $J = 1$.

The measuring times for the various arrangements are assumed to be equal.

In general, however, the apparent pair rate ρ_2 as well as the apparent singles rate ρ_1 are not only a function of the length of the measuring interval δ, but they will also depend on the interval distribution between the primary and the secondary pulses forming a pair. Since we are dealing with a stationary process, the relation

$$\rho_{tot} = \rho_1(\delta) + 2 \cdot \rho_2(\delta) \qquad (30')$$

still holds for any value of δ. It is important to realize that counting by means of the modulo-two technique gives directly $\rho_1(\delta)$, remaining insensitive to $\rho_2(\delta)$.

Only two simple cases for the pair distribution will be considered here. As it has been shown previously ([1], [3]), the rate of the single pulses is given

- for an exponential parent-daughter distribution, with mean T, by

$$\rho_1(\delta) = \rho_s + 2\ \rho_p \left(\frac{1 - e^{-\delta/T}}{\delta/T}\right),$$

which can also be written with (28) as

$$\rho_1(\delta) = \rho_{tot} - 2\ \rho_p \left(1 - \frac{1 - e^{-\delta/T}}{\delta/T}\right); \qquad (31)$$

- for a constant parent-daughter interval T by

$$\rho_1(\delta) = \begin{cases} \rho_s + 2\ \rho_p & \text{if } \delta \leqslant T, \\ \rho_s + 2\ \rho_p \cdot \frac{T}{\delta} & \text{" } \delta \geqslant T \end{cases}$$

$$= \rho_{tot} - 2\ \rho_p \cdot \text{Max}\ (0,\ 1 - T/\delta). \qquad (32)$$

In this case, there is no possibility of distinguishing between paired and single events for $\delta \ll T$.

The expression for the pair rate is therefore of the general form

$$\rho_p = \frac{\rho_{tot} - \rho_1}{2 \cdot Q}, \qquad (33)$$

with

$$Q = \begin{cases} 1 - \frac{T}{\delta}\left(1 - e^{-\delta/T}\right) & \text{for an exponential interval} \\ \text{Max}\ (0,\ 1 - T/\delta) & \text{" a constant interval.} \end{cases}$$

If the distribution of the pair intervals is not well known, we can normally arrange (by lowering the count rate) that $\delta \gg T$, and in this case we always have $Q \cong 1$.

To achieve the high accuracy required in these measurements, corrections due to a possible dead time have to be taken into account for ρ_{tot} and ρ_1. The effect of dead time on the apparent pair rate ρ_1 can be deduced from earlier discussions in [4] and [3]. Since these corrections have only a bearing on the best estimate of ρ_p, but not on its standard deviation, we shall give no further details here. In what follows it will be tacitly assumed that the appropriate corrections have already been applied.

To determine the uncertainty of ρ_p, it seems acceptable to neglect not only the errors stemming from ν and Q, but - in view of (27) and (28) - also from ρ_{tot}. The main contribution is thus due to the experimental measurement of $\pi(J\,12)$. From this quantity, as shown before, we can determine ρ, which in the present notation should now be called ρ_1, since the modulo-two counting is necessarily "blind" for pairs.

Hence, we get from (33) for the statistical uncertainty of the pair rate

$$s(\rho_p) = \frac{1}{2\,Q} \cdot s(\rho_1)\,, \tag{34}$$

and therefore, by applying (23), for the symmetrical arrangement C

$$s(\rho_p) = \frac{\rho}{4\,Q\,\mu\sqrt{N}} \sqrt{e^{4\mu} - 1}\ . \tag{35}$$

A similar, although somewhat less favorable, result would be obtained by means of (16) for the simplified arrangement B.

Using (26), the minimum uncertainty for the pair rate is seen from (34) to be given approximately by

$$s_{min}(\rho_p) \cong \frac{1}{2\,Q} \frac{\rho}{\sqrt{N}}\ 2.5 \cong \frac{\rho}{\sqrt{\nu\,t}}\,, \tag{36}$$

for Q not too far from unity.

For small relative pair rates - and this is the case of most practical interest -, the probability θ for afterpulsing is about

$$\theta = \frac{\rho_p}{\rho} \cong \frac{\rho_p}{\rho_{tot}}\ . \tag{37}$$

By assuming with $\nu \cong \rho_{tot}/2$, according to (14), again the optimum conditions to be realized, we finally arrive for the uncertainty of the afterpulsing probability θ at the estimate

$$s_{min}(\theta) \cong \frac{s_{min}(\rho_p)}{\rho_{tot}} \cong \sqrt{\frac{2}{\rho_{tot} \cdot t}} \; . \tag{38}$$

Obviously, this value also corresponds roughly to the limit of detection for afterpulses.

Let us have a quick look at the numerical values implied by our final result (38). For a count rate of $\rho_{tot} = 2\ 000\ s^{-1}$ and a measuring time of $t = 1\ 000\ s$, the smallest detectable pair contribution has to be of the order

$$\theta_{min} \cong s_{min}(\theta) \cong 10^{-3} \; . \tag{39}$$

Since according to (38) the sensitivity of the modulo method increases with the count rate, we may try to improve the limit by choosing e.g. $\rho_{tot} = 20\ 000\ s^{-1}$ and $t = 10\ 000\ s$ ($\cong 3$ h). In this extreme case, it thus seems possible to measure θ with a precision of up to

$$s_{min}(\theta) \cong 10^{-4} \; . \tag{40}$$

Thereby, we should keep in mind that augmenting the count rate may sometimes be in conflict with the requirement that $\delta \gg T$, since the optimum value of $\mu = \rho\delta \cong 0.4$ can then no longer be realized for a given value of T. This limitation does not exist, however, if the interval density for the partners of a pair can be assumed to be known (e.g. exponential).

On the other hand, the limited accuracy of the dead-time corrections, which are of the order of 15% in our second example (for a dead time of about 4 μs), will certainly make it very difficult to achieve (40). It is fairly obvious that an estimation of the uncertainty for the count rate of multiple afterpulses (K > 2) could be made along similar lines, although the computational problems might become rapidly worse. In the absence of a real experimental need, however, we feel that such an exercise would be premature.

6. Final remarks

The above results for the best measuring conditions have a direct and important bearing on the practical usefulness of the modulo technique, which now seems to be capable of a somewhat higher sensitivity than had been assumed previously. The results of some earlier experimental measurements (by means of an arrangement of the type B) are certainly not in conflict with the estimates of uncertainties given in this study, but the conditions were too far from optimum to permit a definite conclusion. Various series of new measurements, specifically planned for this purpose, will be needed to check the predictions in more detail.

This report would certainly not have been written so quickly - if at all - without the kind, but repeated solicitation of Dr. P.J. Campion (NPL), and he should be thanked for his insistance.

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