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Some relations between asymptotic results for dead-time-distorted processes

Part I: The expectation values

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1. The three types of counting processes

In the past few years, a number of rigorous formulae have been assembled for characterizing the statistics of an original Poisson process which has been distorted by the insertion of a dead time (see e.g. [1] and [2]). Apart from the probability $W_k(t)$ of registering exactly k counts in a given time interval t, the quantities one is most directly interested in from an experimental point of view are the first two moments, i.e. the expectation value $\hat{k}(t)$ for the number of events observed within t and the corresponding variance $\sigma_k^2(t)$.

Unfortunately, these quantities depend not only on the original rate ρ, the length of the measuring interval t and the duration of the dead time τ and its type, but also on the choice of the time origin. Table 1 summarizes the main features of the three types of renewal processes considered in what follows.

Type of process	Density of the first event for a non-extended (n) dead time	Density of the first event for an extended (e) dead time
ordinary process (or)	${}_nf(t)$	${}_ef(t)$
equilibrium process (eq)	${}_ng(t)$	${}_eg(t)$
free counter process (fr)	${}_nh(t)$	${}_eh(t)$

Table 1 - Classification of counting processes and abbreviations used for their interval densities

Whereas in an ordinary process $t = 0$ coincides with a registered (but not counted) event, the time origin for an equilibrium (or stationary) process is chosen at random and can thus also fall within a dead time. On the other hand, in a free counter process the device is immediately ready to accept the first incoming event; hence this process necessarily starts at a moment where no dead time is present.

For multiple intervals, i.e. for the arrival time of event number $k > 1$, the densities are given by the following convolutions:

$$f_k(t) = \{f(t)\}^{*k} , \tag{1a}$$

$$g_k(t) = g(t) * f_{k-1}(t) , \tag{1b}$$

$$h_k(t) = h(t) * f_{k-1}(t) . \tag{1c}$$

If we confine ourselves to an original Poisson process, as will be the case in what follows, it can be shown that the following relations exist

- for a non-extended dead time:

$$_n h_k(t) = {}_n f_k(t + \tau) , \tag{2}$$

- for an extended dead time:

$$_e g_k(t) = {}_e f_k(t + \tau) . \tag{3}$$

In order to make equations more readable, only those indices (for specifying the process and/or the dead time) will be used in later sections which are supposed to be helpful in a given context.

2. Introduction

Apart from some applications in the field of correlation counting (for a short review see [3]), where the effective individual measuring time may well be comparable to the length of the dead times involved, the usual situation is certainly that in which the time t of a measurement exceeds by far the nominal value τ of the dead time as well as the mean interval $1/\rho$ between pulses of the original sequence. Although in the case of a distorted Poisson process the limiting condition $t/\tau \rightarrow \infty$ would be sufficient, we prefer for practical purposes to add the requirement that also $\rho t \rightarrow \infty$, as this will permit us to neglect terms which include a factor $\exp(-\rho t)$.

The important point for us is that - independently of the problem of finding the weakest conditions - the exact formulae for $\hat{k}(t)$ and $\sigma_k^2(t)$, which are often rather involved, may be greatly simplified for $t \rightarrow \infty$. The so-called asymptotic results have been given in [1] for the case of a non-extended dead time; here a unified treatment will produce the expressions for both types of dead time.

Therefore, the purpose of the present study is twofold. On the one hand, it should provide us with a deeper insight into the mechanism of these processes. On the other hand, we shall arrive at some new forms of asymptotic results not commonly known, in particular those pertaining to an extended dead time. In addition, the novel approach permits independent checking of earlier results (some of which had been at variance with previous claims). In view of the usually quite cumbersome arithmetic involved, such controls are certainly most welcome.

In this first part all the relations concerning the asymptotic expectation values will be discussed; the second part will do the same for the variances. A more elegant treatment of these problems, based on some general asymptotic results for renewal processes of the type first derived by Smith [4] must be postponed for the moment since the corresponding formulae for a modified process are not yet readily available. We hope to be able to fill this gap in a near future.

3. General remarks

Let us consider a counting process with a total measuring time t which has been arbitrarily split up into two parts. In general, then, it is unfortunately not true that the expectations (M) as well as the variances (V) are simply additive for the partial intervals t_1' and t_2'. The reason for this is that the corresponding relations

$$M(t = t_1' + t_2') = M(t_1') + M(t_2') \tag{4a}$$

and

$$V(t = t_1' + t_2') = V(t_1') + V(t_2') , \tag{4b}$$

taken as functional equations, would imply solutions of the form

$$M(t) = \alpha_1 \cdot t \qquad \text{and} \qquad V(t) = \alpha_2 \cdot t . \tag{5a, b}$$

We know, however, that this proportionality to time is fulfilled neither by the exact nor by the asymptotic relations [1]. Even for the equilibrium process only (5a) holds, but not (5b). It follows that for the level of approximation we are interested in, the relations (4) suggested by Goldanskii et al. [5] are only applicable to an undisturbed Poisson process, where, in the absence of dead times, both α_1 and α_2 are equal to the original count rate ρ.

The physical reason for the failure of (4) is, of course, that the two parts are no longer independent. The dead times following the registered counts have an aftereffect by coupling subsequent intervals which then become statistically correlated, at least to some extent.

Although equations as those in (4) can therefore not be expected to give a valid description, it would still be very nice to have a simple additive system of a somewhat similar type at hand. This can indeed be arrived at if we allow the functions M in (4) (and likewise V) to be different, although well defined.

4. Evaluation of the expectation values

In the general case, where the total measuring interval of length t is subdivided by t' into two complementary sections, the expectation for the number k of events in t can be written as

$$\hat{k}(t) = \int_0^t \varphi(t') \cdot \left[\hat{k}_1(t') + \hat{k}_2(t-t') \right] dt' \quad , \tag{6}$$

where

$\varphi(t')$ = interval density for the arrival of an "event" at t' (as for instance $f(t')$ or $g(t')$ when registering a pulse at t'), and

k_1, k_2 = number of counts in the first (or second) section.

If $t' \equiv t_i$ denotes the arrival time of event number i in a renewal process starting at $t = 0$, then obviously $\hat{k}_1(t_i) = i$ and we get

$$\hat{k}(t) = i \int_0^t \varphi(t_i)\, dt_i + \int_0^t \varphi(t_i) \cdot \hat{k}_2(t-t_i)\, dt_i \ .$$

Since we are only interested in asymptotic relations, i.e. in the limiting case of very long intervals t, the integrations may be extended to infinity, which will simplify the calculations considerably. Thus

$$\hat{k}(t) \cong i + \int_0^\infty \varphi(t_i) \cdot \hat{k}_2(t-t_i)\, dt_i \ . \tag{7}$$

In the approximation used here the expectations $\hat{k}_2(t)$ are always of the form

$$\hat{k}_2(t) = \alpha \cdot t + \beta \quad ,$$

where α and β depend on the type of the process and of the dead time chosen. More precisely, α is the (equilibrium) count rate at the output. For an original Poisson process with rate ρ, we therefore have

$$\alpha = \begin{cases} \dfrac{\rho}{1+\rho\tau} \equiv \rho\lambda & \text{if } \tau \text{ is non-extended} \\ \rho \cdot e^{-\rho\tau} \equiv \rho/y & \text{" } \tau \text{ " extended ;} \end{cases} \tag{8}$$

the "shifts" β are not really needed for the moment and will be determined later (cf. Table 3). Therefore, (7) can also be written as

$$\begin{aligned} \hat{k}(t) &\cong j + (\alpha t + \beta) \int_0^\infty \varphi(t_j)\, dt_j - \alpha \int_0^\infty \varphi(t_j) \cdot t_j\, dt_j \\ &= j + \alpha t + \beta - \alpha \bar{t}_j \\ &= j + \hat{k}_2(t) - \alpha \bar{t}_j \ , \end{aligned} \tag{9}$$

where $\bar{t}_j$ is the mean arrival time for event number j in the first process (starting at $t = 0$).

If we denote the contributions from the two sections by K_1 and K_2, we can formally say that

$$\hat{k}(t) = K_1 + K_2 \ , \tag{10}$$

with

$$K_1 = j \quad \text{and}$$

$$K_2 \cong \hat{k}_2(t) - \alpha \bar{t}_j \ .$$

For $j = 1, 2, \ldots,$ t' is the arrival time of a registered pulse and the second part is therefore an ordinary process, i.e. $\hat{k}_2(t) = {}_{or}\hat{k}(t)$.

Obviously, the case of most practical use of (9) is for $j = 1$, i.e.

$$\hat{k}(t) \cong {}_{or}\hat{k}(t) + 1 - \alpha \bar{t}_1 \ . \tag{11}$$

In order to simplify the application of (11), the mean values $\bar{t}_1$ as well as the quantities $\alpha \bar{t}_1$ are listed in Tables 2a and 2b, which are taken in part from [6].

These results for the subdivision of the expectation value $\hat{k}(t)$ may appear to be fairly trivial. If they are presented here in some detail all the same, this may be justified by the fact that they can serve as a model for the somewhat similar but more complicated situations we shall meet later in the evaluation of the asymptotic variances.

	dead time non-extended	extended
ordinary process	$1/(\lambda\rho)$	y/ρ
equilibrium "	$(1 + \lambda x^2/2)/\rho$	$(y - x)/\rho$
free counter "	$1/\rho$	$1/\rho$

Table 2a - The mean arrival times $\bar{t}_1$ for the first event of a Poisson process (original count rate ρ), distorted by a dead time τ. The abbreviations used are
$x = \rho\tau$, $\lambda = \frac{1}{1+\rho\tau}$ and $y = e^{\rho\tau}$.

	dead time non-extended	extended
ordinary process	1	1
equilibrium "	$\lambda(1 + \lambda x^2/2)$	$1 - x/y$
free counter "	λ	$1/y$

Table 2b - The quantities $\alpha\bar{t}_1$ appearing in (11), with abbreviations as in Table 2a .

5. Expectation values for a non-extended dead time

Let us first consider an ordinary renewal process of duration t. The time origin is thus given by the arrival of an event which is followed by a non-extended dead time. A particularly simple decomposition can now be obtained by choosing the end of this dead time (although this is not an arrival time) for subdividing the time interval, hence $t' = \tau$. In this case, the first part falls completely within the dead time and therefore contains no registered event, whereas the second is a free counter process of duration $t - \tau$. Hence we can write for the expectations

$$_{or}\hat{k}(t) = K_1 + {}_{fr}\hat{k}(t - \tau),$$

$$\text{i.e.} \quad _{fr}\hat{k}(t) = {}_{or}\hat{k}(t + \tau), \tag{12}$$

since $K_1 = 0$.

It is amusing to see that this result could also have been obtained from (9) for $j = 0$. With $t_o = \tau$ and by specifying $\hat{k}(t) = {}_{or}\hat{k}(t)$ and $\hat{k}_2(t) = {}_{fr}\hat{k}(t)$, we arrive indeed at

$$ {}_{or}\hat{k}(t) = {}_{fr}\hat{k}_2(t) - \alpha\tau $$
$$ = \alpha(t - \tau) + \beta = {}_{fr}\hat{k}(t - \tau) \,. \qquad (12') $$

This corresponds to the case where we take the end of the first dead time as the "arrival" of a fictitious event number zero. However, since the point $t' = t_o$ merely serves to subdivide the interval t, the acceptance of (12') does not presume any belief in the existence of ghosts; it only supposes τ to be non-extending.

A more systematic approach may be based on (11) which can be written in the form of the difference

$$ \hat{k}(t) - {}_{or}\hat{k}(t) \cong 1 - \alpha\bar{t}_1 \,. \qquad (11') $$

By specifying the process considered we find readily, using the information of Table 2b,

- for an equilibrium process:

$$ {}_{eq}\hat{k}(t) - {}_{or}\hat{k}(t) \cong 1 - \lambda\,(1 + \frac{1}{2}\lambda x^2) = \lambda x\,(1 - \frac{1}{2}\lambda x) \,; \qquad (13) $$

- for a free counter process:

$$ {}_{fr}\hat{k}(t) - {}_{or}\hat{k}(t) \cong 1 - \lambda = \lambda x \,. \qquad (14) $$

Finally, the difference between (14) and (13) yields

$$ {}_{fr}\hat{k}(t) - {}_{eq}\hat{k}(t) \cong \frac{1}{2}\lambda^2 x^2 \,. \qquad (15) $$

Since it is known from general considerations that for an equilibrium process the relation

$$ {}_{eq}\hat{k}(t) = \lambda\rho t \qquad (16) $$

holds exactly, we can easily pass from the differences given above to the full values for the asymptotic expectations of k, namely

$$ {}_{or}\hat{k}(t) \cong \lambda\,(\rho t - x + \frac{1}{2}\lambda x^2) \quad \text{and} \qquad (17) $$
$$ {}_{fr}\hat{k}(t) \cong \lambda\,(\rho t + \frac{1}{2}\lambda x^2) \,. \qquad (18) $$

The relations (16) to (18) are in agreement with the corresponding values as stated in [1]. Likewise (12) is readily seen to be consistent with (17) and (18).

It should be mentioned, perhaps, that (17) and (18) may be written in a number of equivalent ways; the choice among them is largely a matter of personal preference. Possible alternative forms which contain only λ in the corrective terms are for example

$$ {}_{or}\hat{k}(t) \cong \lambda \rho t - \frac{1}{2}(1 - \lambda^2) \ , \tag{17'} $$

$$ {}_{fr}\hat{k}(t) \cong \lambda \rho t - \lambda + \frac{1}{2}(1 + \lambda^2) \ . \tag{18'} $$

6. Expectation values for an extended dead time

For an extended dead time, the reasonings remain essentially the same as before. Only in the event that we are interested in knowing the correspondance to our previous relation (12), we have to take into account that the effective dead time is no longer constant since it can now be extended. If its mean duration is denoted by τ_{eff}, the formula corresponding to (12) now reads for an extended dead time

$$ {}_{fr}\hat{k}(t) = {}_{or}\hat{k}(t + \tau_{eff}) \ . \tag{19} $$

The unknown quantity τ_{eff} can be determined in several ways; some of them will be discussed later (see (27) and the Appendix).

Again, the differences are determined by (11'). With the values from Table 2b we obtain

- for an equilibrium process:

$$ {}_{eq}\hat{k}(t) - {}_{or}\hat{k}(t) \cong x/y \ ; \tag{20} $$

- for a free counter process:

$$ {}_{fr}\hat{k}(t) - {}_{or}\hat{k}(t) \cong (y - 1)/y \ . \tag{21} $$

This yields for the difference

$$ {}_{fr}\hat{k}(t) - {}_{eq}\hat{k}(t) \cong (y - x - 1)/y \ . \tag{22} $$

Since for the equilibrium process the expectation is known to be

$$ {}_{eq}\hat{k}(t) = \rho t/y \ , \tag{23} $$

one readily gets for the other cases

$$ {}_{or}\hat{k}(t) \cong (\rho t - x)/y \ , \tag{24} $$

$$ {}_{fr}\hat{k}(t) \cong (\rho t - x + y - 1)/y \ . \tag{25} $$

We note that (24) and (25) are rigorous solutions for any $t > \tau$ since the total renewal density is known to reach its final value at $t = \tau$ for an extended dead time (compare [7]). This is confirmed by the result of direct calculations [2]).

Let us now quickly come back to (19). A comparison between (24) and (25) allows us to write

$$_{fr}\hat{k}(t) = {}_{or}\hat{k}[t + (y-1)/\rho] , \qquad (26)$$

from which we deduce the value

$$\tau_{eff} = (y - 1)/\rho \qquad (27)$$

for the mean length of the effective dead time. This indirect way of reasoning is in agreement with the result (A8) obtained in the Appendix by a more general method.

7. Survey of the results

In Table 3 the main asymptotic relations derived in this report are put together. As there exists quite an extended (and occasionally somewhat confusing) literature on this subject, an attempt to make a complete survey would hardly be profitable. On the other hand, we made some effort to trace the first reference for a given explicit result $\hat{k}(t)$, although we may have missed some of them. For this task, the recent bibliography [8] has proved useful. One of the main difficulties we met in comparing the results of different authors stems from the fact that in many cases the exact experimental conditions (i.e. type of process and dead time) are not clearly stated. Complements and rectifications by readers would be very welcome.

- For a non-extended dead time:

Type of process	$\hat{k}(t)$	Earliest references		Comment
ordinary process	$\lambda(\rho t - x + \lambda x^2/2)$	[9], eq. 20	[10], eq. 29	1
equilibrium "	$\lambda \rho t$	-	-	2
free counter "	$\lambda(\rho t + \lambda x^2/2)$	[11], eq. 29	[12], eq. 11	-

- For an extended dead time:

Type of process	$\hat{k}(t)$	Earliest references		Comment
ordinary process	$(\rho t - x)/y$	[10], eq. 42	[13], p. 56	3
equilibrium "	$\rho t/y$	-	-	2
free counter "	$(\rho t - x + y - 1)/y$	[11]*, eq. 45	[14], eq. 22	3

Table 3 - Summary of the asymptotic formulae for the expectation $\hat{k}(t)$ of the number of events in a measuring interval t, if an original Poisson process with rate ρ has been distorted by a dead time τ. The abbreviations used are $x = \rho\tau$, $\lambda = (1+x)^{-1}$ and $y = e^x$.

Comments:

1. Campbell's result $\frac{\rho t}{1 + x} - \frac{2x + x^2}{2(1 + x)^2}$ is readily shown to be identical to (17).

2. This is a "classical" formula, although it has rarely been recognized to be valid only for an equilibrium process.

3. This result has actually been first obtained as the exact solution for $t > \tau$.

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* After allowance for an obvious misprint which has, however, been taken over in [15].

APPENDIX

On the effective length of an extended dead time

Let us consider a Poisson process of count rate ρ , where every incoming pulse is followed by a dead time of duration τ . After each outgoing or registered pulse there is a period of paralysis which may last from τ to any arbitrarily large value. We are now going to determine the density for the effective duration of this paralysis for a nominal value τ of the extended dead time as well as the corresponding first few moments.

The Laplace transform of the desired density D(t) has been indicated a long time ago by Feller [11]*; it is

$$\tilde{D}(s) \equiv \mathcal{L}\{D(t)\} = \frac{(\rho + s)\cdot e^{-(\rho+s)\tau}}{s + \rho\cdot e^{-(\rho+s)\tau}} . \qquad \text{(A1)}$$

It may be interesting to find the corresponding original. For this purpose, we write (A1) in the form

$$\tilde{D}(s) = \frac{\rho\cdot e^{-(\rho+s)\tau}}{s+\rho\cdot e^{-(\rho+s)\tau}} + \frac{s\cdot e^{-(\rho+s)\tau}}{s+\rho\cdot e^{-(\rho+s)\tau}}$$

$$= {}_e\tilde{f}(s) + \frac{s}{\rho}\cdot {}_e\tilde{f}(s) , \qquad \text{(A2)}$$

or likewise for the original

$$D(t) = {}_ef(t) + \frac{1}{\rho}\frac{d}{dt}\,{}_ef(t) . \qquad \text{(A2')}$$

The reason for this decomposition lies in the fact that ${}_e\tilde{f}(s)$ is a well-known transform we met before in a similar context (see [7], eq. 11'). The corresponding original density has been found to be

$${}_ef(t) = \rho\sum_{j=1}^{\infty} U(T_j)\cdot\frac{(-T_j)^{j-1}}{(j-1)!}\cdot e^{-jx} , \qquad \text{(A3)}$$

where $T_j = \rho t - jx$, $x = \rho\tau$ and U is the unit step function.

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* His equation (42), after correction of an obvious misprint.

If we remember that

$$\frac{d}{dt}\,U(t - t_o) = \delta(t - t_o)\ ,$$

the contribution of the discontinuity of ${}_e f(t)$ at $t = \tau$ to the derivative is seen to be $\rho\, e^{-x} \cdot \delta\,(t - \tau)$. Hence we can write

$$\frac{1}{\rho}\,\frac{d}{dt}\,{}_e f(t) = \delta\,(t - \tau)\cdot e^{-x} + \sum_{i=1}^{\infty} U(T_i)\cdot \frac{(i-1)\,(-T_i)^{i-2}}{(i-1)\,!}\cdot(-\rho)\cdot e^{-ix}$$

$$= \delta\,(t - \tau)\cdot e^{-x} - \rho\sum_{i=2}^{\infty} U(T_i)\cdot\frac{(-T_i)^{i-2}}{(i-2)\,!}\cdot e^{-ix}\ . \qquad \text{(A4)}$$

By summing (A3) and (A4) we obtain for the interval density of the paralysis

$$D(t) = \delta\,(t-\tau)\cdot e^{-x} + \rho\cdot e^{-x}\cdot U(T_1) + \rho\sum_{i=2}^{\infty} U(T_i)\left\{\frac{(-T_i)^{i-1}}{(i-1)\,!}\cdot e^{-ix} - \frac{(-T_i)^{i-2}}{(i-2)\,!}\cdot e^{-ix}\right\}$$

$$= \delta\,(t-\tau)\cdot e^{-x} + \rho\cdot e^{-x}\cdot U(T_1) + \rho\sum_{i=2}^{\infty} U(T_i)\cdot e^{-ix}\cdot(-T_i - i + 1)\cdot\frac{(-T_i)^{i-2}}{(i-1)\,!}$$

$$= \delta\,(t-\tau)\cdot e^{-x} - \rho\sum_{i=1}^{\infty} U(T_i)\cdot e^{-ix}\cdot(T_i + i - 1)\cdot\frac{(-T_i)^{i-2}}{(i-1)\,!}\ . \qquad \text{(A5)}$$

For a very low count rate ($\rho \longrightarrow 0$), (A5) reduces to

$$D(t) \longrightarrow \delta\,(t - \tau)\ ,$$

as it obviously should do since in this limit overlappings of dead times disappear and the extended dead times become non extended with a fixed length τ .

More explicitly, the density (A5) can be written for the various intervals as

$$D(t) = \begin{cases} 0 & \text{for} \quad t < \tau \\ \gamma_o & \text{"} \quad t = \tau \\ \gamma_1 & \text{"} \quad \tau < t \leqslant 2\tau \\ \gamma_1 + \gamma_2 & \text{"} \quad 2\tau < t \leqslant 3\tau \\ \gamma_1 + \gamma_2 + \gamma_3 & \text{"} \quad 3\tau < t \leqslant 4\tau \\ \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 & \text{"} \quad 4\tau < t \leqslant 5\tau \\ \quad \cdots & \quad \cdots , \end{cases} \tag{A5'}$$

where

$$\gamma_o = \delta(t-\tau) \cdot e^{-x} ,$$

$$\gamma_1 = \rho e^{-x} ,$$

$$\gamma_2 = -\rho e^{-2x} (\rho t - 2x + 1) ,$$

$$\gamma_3 = \frac{1}{2} \rho e^{-3x} (\rho t - 3x + 2)(\rho t - 3x) ,$$

$$\gamma_4 = -\frac{1}{6} \rho e^{-4x} (\rho t - 4x + 3)(\rho t - 4x)^2 ,$$

$$\cdots$$

As usual, the quantities of direct physical interest are the moments of lowest order. However, (A5) does not lend itself easily to their evaluation. Instead it is preferable to go back to the simple decomposition (A2') and derive the moments by algebraic manipulations with the transform. Using the result for the transform ${}_e\tilde{f}(s)$ as given after eq. 31 in [7], we find (since k=1 and $R=y/\rho$)

$$\tilde{D}(s) = {}_e\tilde{f}(s) \cdot (1 + \frac{s}{\rho}) \tag{A6}$$

$$= 1 - \frac{s}{\rho}(y-1) + \frac{s^2}{\rho^2} y (y - x - 1) - \frac{s^3}{\rho^3} y \left[y(y-2x-1) + x(1+x/2)\right] \pm \ldots$$

For the corresponding ordinary moments of order r, which can be derived by applying the formula

$$m_r(t) = (-1)^r \frac{d^r}{ds^r} \tilde{D}(s) \Big|_{s=0} , \tag{A7}$$

we obtain

$$\begin{aligned} m_o(t) &= 1 , \\ m_1(t) &= (y-1)/\rho , \\ m_2(t) &= 2y(y-x-1)/\rho^2 \quad \text{and} \\ m_3(t) &= 6y\left[y^2 - y(1+2x) + x(1+x/2)\right]/\rho^3 . \end{aligned} \tag{A8}$$

For the central moments, this leads after some rearrangements to *

$$\mu_2(t) \equiv m_2(t) - m_1^2(t)$$
$$= (y^2 - 2\,xy - 1)/\rho^2, \qquad (A9)$$

$$\mu_3(t) \equiv m_3(t) - 3\,m_1(t)\cdot m_2(t) + 2\,m_1^3(t)$$
$$= (2\,y^3 - 6\,xy^2 + 3\,x^2y - 2)/\rho^3\ . \qquad (A10)$$

For checking purposes, we are also interested in the limiting values for $\rho \longrightarrow 0$. A short calculation gives, including the lowest non-vanishing order of x,

$$m_1(t) \cong \tau(1 + x/2) \longrightarrow \tau\,,$$
$$\mu_2(t) \cong \tfrac{1}{3}\,x^3/\rho^2 \longrightarrow 0\,,$$
$$\mu_3(t) \cong \tfrac{1}{4}\,x^4/\rho^3 \longrightarrow 0\,,$$

as was to be expected for the limit where $D(t) = \delta\,(t - \tau)$.

It may be worthwhile mentioning that there exists also a simple direct way to obtain the effective length $\tau_{eff} = m_1(t)$ of the paralysis following a registered pulse.

For an original count rate ρ, the output rate R is known to be (say for an equilibrium process, or just for t sufficiently large)

- for a non-extended dead time τ :

$$_nR = \frac{\rho}{1 + \rho\tau}\,, \qquad (A11)$$

- for an extended dead time τ :

$$_eR = \rho \cdot e^{-\rho\tau}. \qquad (A12)$$

It follows from (A11) that the original count rate is given by

$$\rho = \frac{_nR}{1 - {_nR}\tau}\ . \qquad (A13)$$

Thereby the term $_nR\,\tau$ indicates the fraction of time occupied by the dead times following the registered events. Hence, there must also exist a formula equivalent to (A13) for the case of an extended dead time,

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* (A9) agrees with a result given by Feller [16] .

namely

$$\rho = \frac{e^R}{1 - e^R \tau_{eff}} \; . \qquad \text{(A14)}$$

Inserting (A12) into (A14) gives readily

$$\rho = \frac{\rho \cdot e^{-\rho\tau}}{1 - \tau_{eff} \cdot e^{-\rho\tau}} \; ,$$

from which we obtain

$$\tau_{eff} = \frac{1}{\rho} (e^{\rho\tau} - 1) \; , \qquad \text{(A15)}$$

in agreement with (27) and (A8).

As for the moments of order two and three determined before, and which cannot be found in such a simple way, they may prove useful in a general description of counting processes where the dead times are supposed to be non-extended, but of a random nature. Using the respective moments up to the third order then gives an approximation which should be adequate for most practical purposes.

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