Explicit interval densities for equilibrium counting processes

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1. General relations

For a renewal process [1], the interval $t \equiv t_{i+1} - t_i$ between the arrival times of two consecutive pulses is described by a probability density f(t). Since the process is independent of its "history" (renewal property), the case of multiple intervals of order k, where $t \equiv t_{i+k} - t_i$, is equally simple to treat and the corresponding density is known to be given by the k-fold self-convolution

$$f_k(t) = \{f(t)\}^{*k} , \qquad k = 1, 2, \ldots, \tag{1}$$

with $f_1(t) \equiv f(t)$.

The only difficulty arises in connection with the arrival time of the first pulse. As a matter of fact, the density for this event is in general only equal to f(t) if the beginning of the measuring interval coincides with a pulse. Although such a synchronisation could easily be achieved for most practical cases, this is seldom really done. Instead, the usual experimental situation is such that there is no definite relation between the arrival times of the pulses and the time origin, which is then said to be chosen "at random". The corresponding series of events is called an equilibrium renewal process. In this case, the density g(t) for the first pulse will be modified accordingly and it can be shown to be given by

$$g(t) = \mu \int_t^{\infty} f(x)\, dx , \tag{2}$$

with $\mu^{-1} \equiv \int_0^{\infty} t \cdot f(t)\, dt$.

Since μ^{-1} is the average time interval between pulses, μ corresponds to the mean count rate of the process for a measuring time which is much longer than μ^{-1}.

The arrival time of event number $k > 1$ can thus be decomposed into $k-1$ intervalswith density f and one (the first) with density g. Therefore, for an equilibrium process, the probability density for the arrival time of event number k is given by

$$g_k(t) = g(t) * f_{k-1}(t) , \qquad k = 1, 2, \ldots, \tag{3}$$

where $g_1(t)$ is identified with $g(t)$. For some applications, it is also practical to define $g_o(t) = \delta(t)$.

The evaluation of $g_k(t)$ is greatly simplified by the use of Laplace transforms. By putting

$$\mathcal{L}\left\{f(t)\right\} \equiv \tilde{f}(s) ,$$

the equations (1) to (3) now read [2]

$$\tilde{f}_k(s) = \left[\tilde{f}(s)\right]^k , \tag{4}$$

whence $f_o(t) = \delta(t)$;

$$\tilde{g}(s) = \frac{\mu}{s}\left[1 - \tilde{f}(s)\right] \tag{5}$$

and therefore

$$\tilde{g}_k(s) = \tilde{g}(s) \cdot \tilde{f}_{k-1}(s) = \frac{\mu}{s}\left[\tilde{f}_{k-1}(s) - \tilde{f}_k(s)\right] . \tag{6}$$

The cumulative distributions of $f_k(t)$ and $g_k(t)$ are defined as

$$F_k(t) \equiv \int_o^t f_k(x)\, dx \qquad \text{and} \qquad G_k(t) \equiv \int_o^t g_k(x)\, dx , \tag{7}$$

with $F_o(t) = G_o(t) = U(t)$.

Therefore, the original of (6) can also be written in the form

$$g_k(t) = \mu\left[F_{k-1}(t) - F_k(t)\right] . \tag{8}$$

We conclude from (6) that

$$\frac{d}{dt}\, g_k(t) = \mu\left[f_{k-1}(t) - f_k(t)\right] ; \tag{9}$$

thus, in particular for $k = 1$;

$$\frac{d}{dt} g(t) \equiv g'(t) = \mu [\delta(t) - f(t)] .$$

Hence

$$g'(t) = -\mu \cdot f(t) , \qquad \text{for } t > 0. \tag{10}$$

It is the aim of this report to give explicit expressions for the equilibrium densities $g_k(t)$ in the case of an original Poisson process which has been modified by the insertion of a dead time which is either of the non-extended or of the extended type.

As a matter of fact, an expression for $G_k(t)$ has been used previously in deriving the probabilities $W_k(t)$ for observing exactly k counts within an interval t, and related quantities, for an original Poisson process distorted by a non-extended dead time [3]. In recent problems, however, explicit forms of $g_k(t)$ were needed as for instance in the determination of the arrival time of pulse j, when the total number $N \geqslant j$ of observed events in t is given. For the case of an extended dead time, we know of no previous attempt to determine $g_k(t)$.

2. Non-extended dead time

For a Poisson process (with original count rate ρ) which has been modified by a non-extended (n) dead time τ, the density for the k-fold interval is known to be [4]

$${}_nf_k(t) = U(t - k\tau) \cdot \rho \cdot \frac{[\rho(t - k\tau)]^{k-1}}{(k-1)!} \cdot e^{-\rho(t-k\tau)} . \tag{11}$$

This gives for the cumulative distribution

$${}_nF_k(t) = \int_0^t {}_nf_k(x)\, dx = \int_0^{\rho(t-k\tau)} U(z) \cdot \frac{z^{k-1}}{(k-1)!} \cdot e^{-z}\, dz$$

$$= \frac{U(T_k)}{(k-1)!} \int_0^{T_k} z^{k-1} \cdot e^{-z}\, dz , \tag{12}$$

where $T_k \equiv \rho(t - k\tau)$.

The transformation of the incomplete gamma function (by integrating by parts) into a cumulative Poisson distribution according to the relation

$$\int_0^x e^{-z} \cdot z^{k-1} \, dz = (k-1)! \sum_{i=k}^{\infty} \frac{x^i}{i!} \cdot e^{-x} \tag{13}$$

yields in our case

$$_nF_k(t) = U(T_k) \left[1 - \sum_{i=0}^{k-1} P_k(i) \right] , \tag{14}$$

where $P_k(i)$ is an abbreviation for the "shifted" Poisson probability

$$P_k(i) \equiv \frac{T_k^i}{i!} \cdot e^{-T_k} \; . \tag{15}$$

According to (2) and (11)

$$\mu_n^{-1} = \rho \int_{\tau}^{\infty} t \cdot e^{-\rho(t-\tau)} \, dt = \frac{1}{\rho} + \tau \; .$$

Eq. (8) now yields for the k-fold interval density with a non-extended dead time $(k = 1, 2, \ldots)$

$$_ng_k(t) = \mu_n \left[U(T_{k-1}) - U(T_k) + U(T_k) \sum_{i=0}^{k-1} P_k(i) - U(T_{k-1}) \sum_{i=0}^{k-2} P_{k-1}(i) \right] , \tag{16}$$

or in a slightly different, but equivalent form

$$\frac{_ng_k(t)}{\mu_n} = \begin{cases} 0 & \text{for} \quad t < (k-1)\tau \\ 1 - \sum_{i=0}^{k-2} P_{k-1}(i) & \text{"} \quad (k-1)\tau < t \leq k\tau \\ \sum_{i=0}^{k-1} P_k(i) - \sum_{i=0}^{k-2} P_{k-1}(i) & \text{"} \quad t \geq k \; , \end{cases} \tag{16'}$$

where $\mu_n = \dfrac{\rho}{1 + \rho\tau}$.

For the lowest values of k, this gives in particular

- for <u>k = 1</u>:

$$_ng_1(t) = \begin{cases} \dfrac{\rho}{1+\rho\tau} & \text{for} \quad 0 < t \leq \tau \\ \dfrac{\rho}{1+\rho\tau} \cdot e^{-\rho(t-\tau)} & \text{"} \quad t \geq \tau \; , \end{cases} \tag{17}$$

- for $\underline{k = 2}$:

$$
{}_n g_2(t) = \begin{cases} 0 & \text{for } t \leq \tau \\ \dfrac{\rho}{1+\rho\tau}\left[1 - e^{-\rho(t-\tau)}\right] & \text{" } \tau \leq t \leq 2\tau \\ \dfrac{\rho}{1+\rho\tau}\cdot e^{-\rho(t-2\tau)}\left[1 + \rho(t-2\tau) - e^{-\rho\tau}\right] & \text{" } t \geq 2\tau , \end{cases} \tag{18}
$$

- for $\underline{k = 3}$:

$$
{}_n g_3(t) = \begin{cases} 0 & \text{for } t \leq 2\tau \\ \dfrac{\rho}{1+\rho\tau}\left[1 - e^{-\rho(t-2\tau)}\left\{1 + \rho(t-2\tau)\right\}\right] & \text{" } 2\tau \leq t \leq 3\tau \\ \dfrac{\rho}{1+\rho\tau}\cdot e^{-\rho(t-3\tau)}\Big[1 + \rho(t-3\tau) & \\ \qquad + \dfrac{\rho^2}{2}(t-3\tau)^2 - e^{-\rho\tau}\left\{1 + \rho(t-2\tau)\right\}\Big] & \text{" } t \geq 3\tau . \end{cases} \tag{19}
$$

In Fig. 1 the interval densities ${}_n f(t)$ and ${}_n g(t)$ for the first pulse are shown. These experimental distributions agree very well with the theoretical shapes as given by (11), for k = 1, and (17).

3. Extended dead time

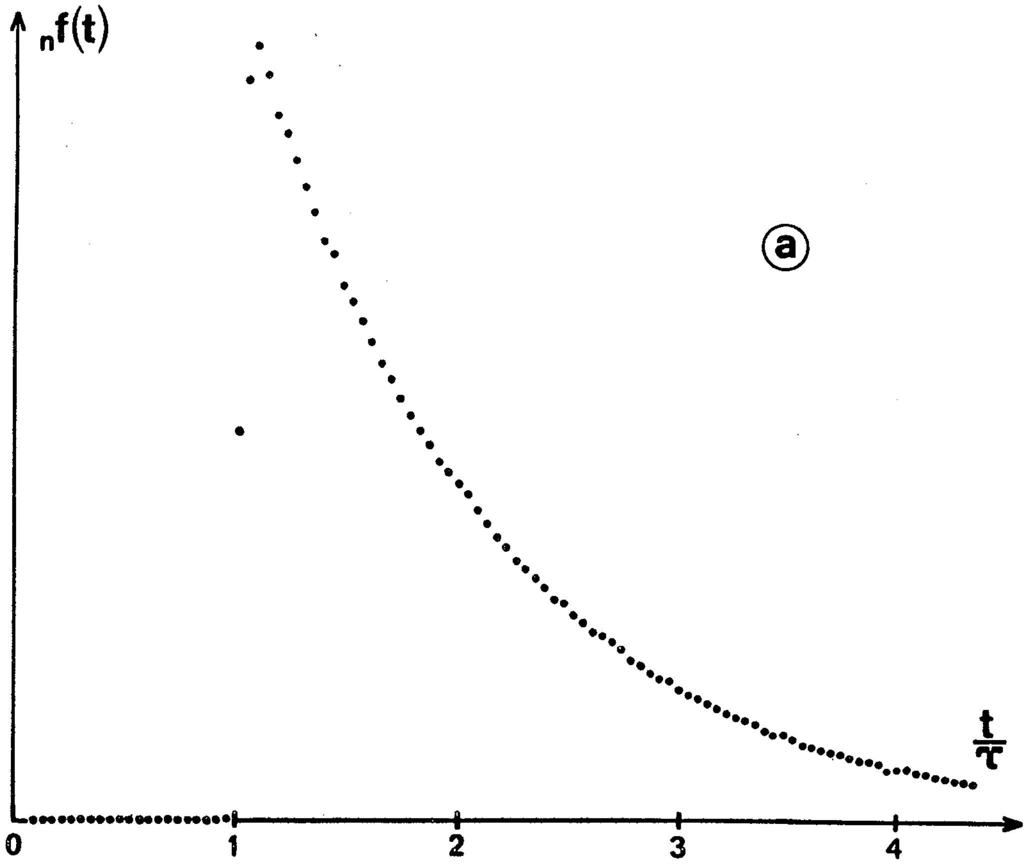
The interval density for a Poisson process (original rate ρ), distorted by a dead time τ of the extended type (e), has been found previously to be (compare [5], eq. 26)

$$
{}_e f_k(t) = \rho\,(-1)^{k-1} \sum_{j=k}^{J} {}_k A_j(t) , \qquad k = 1, 2, \ldots, \tag{20}
$$

where ${}_k A_j(t) \equiv \binom{j-1}{k-1} \cdot \dfrac{(-T_j)^{j-1}}{(j-1)!} \cdot e^{-j\rho\tau} = \dfrac{1\,(-T_j)^{j-1}}{(k-1)!\,(j-k)!} \cdot e^{-j\rho\tau}$

and $\qquad J \equiv [[t/\tau]]$.

This may also be written more explicitely as



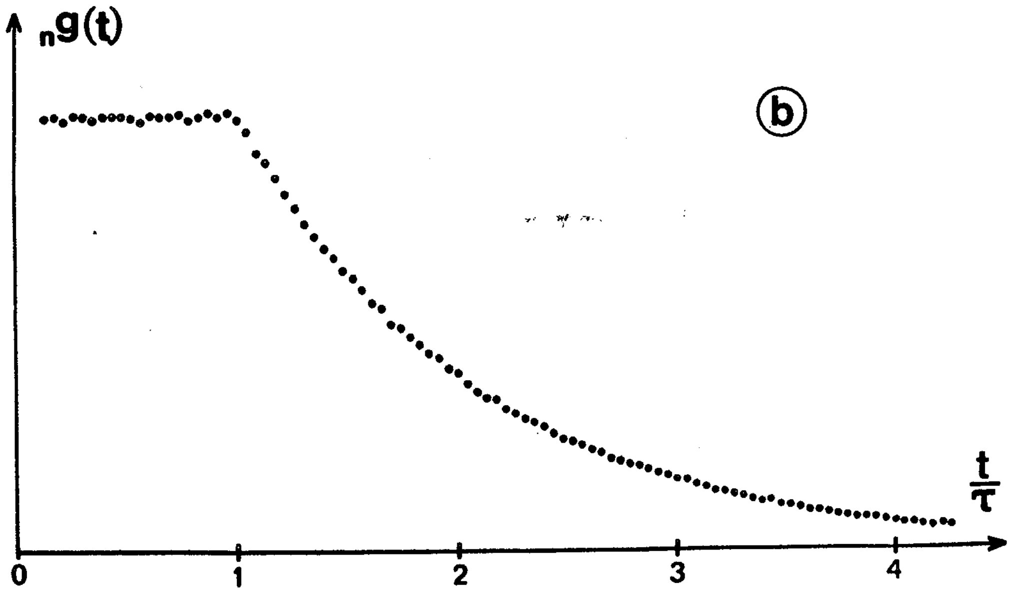


Fig. 1 - Experimental measurement of the interval density of pulses from a radioactive source. The measuring conditions are $\rho \cong 2\ 000\ s^{-1}$, $\tau \cong 400\ \mu s$ and

a) time origin given by a pulse

b) random start.

The dead time is of the non-extended type.

$$\frac{{}_e f_k(t)}{\rho(-1)^{k-1}} = \begin{cases} 0 & \text{for } t < k\tau \\ {}_k A_k(t) & \text{"} \quad k\tau < t \leqslant (k+1)\tau \\ {}_k A_k(t) + {}_k A_{k+1}(t) & \text{"} \quad (k+1)\tau \leqslant t \leqslant (k+2)\tau \\ {}_k A_k(t) + {}_k A_{k+1}(t) + {}_k A_{k+2}(t) & \text{"} \quad (k+2)\tau \leqslant t \leqslant (k+3)\tau \\ \quad \cdots & \quad \cdots \end{cases} \tag{20'}$$

In what follows, two different ways will be sketched to obtain the required densities $g_k(t)$. Whereas in the first approach we first determine $F_k(t)$ and then use eq. (8), the second derivation is based on integral transforms.

a) Direct derivation of ${}_e g_k(t)$

A term-by-term integration of (20) requires the evaluation of

$$\begin{aligned} \int_0^t {}_k A_i(x)\, dx &= \binom{i-1}{k-1} \frac{e^{-i\rho\tau}}{(i-1)!} (-\rho)^{i-1} \int_{i\tau}^{t} (x - i\tau)^{i-1}\, dx \\ &= \binom{i-1}{k-1} \frac{e^{-i\rho\tau}}{(i-1)!} (-\rho)^{i-1} \frac{1}{i} (t - i\tau)^i \\ &= \frac{-1}{\rho} \binom{i-1}{k-1} \frac{e^{-i\rho\tau}}{i!} (-T_i)^i , \end{aligned} \tag{21}$$

which gives for the cumulative distribution of ${}_e f_k(t)$

$${}_e F_k(t) = (-1)^k \sum_{i=k}^{J} {}_k B_i(t) , \qquad k \geqslant 1 , \tag{22}$$

where ${}_k B_i(t) = \binom{i-1}{k-1} \dfrac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau}$.

For an extended dead time, the asymptotic mean count rate is known to be

$$\mu_e = \rho \cdot e^{-\rho\tau}, \tag{23}$$

if the underlying original Poisson process had a rate ρ .

The corresponding equilibrium densities $g_k(t)$ are again derived by using the general relation (8). Since according to (7) and (22)

$${}_e F_o(t) = U(t)$$

and

$$_eF_1(t) = -\sum_{i=1}^{J} \frac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau},$$

we obtain from (8) for $t > 0$

$$_eg_1(t) = \mu_e \left[1 + \sum_{i=1}^{J} \frac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau} \right] = \rho \cdot e^{-\rho\tau} \sum_{i=0}^{J} \frac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau}.$$

For $k \geqslant 2$, we can write

$$
\begin{aligned}
_eg_k(t) &= \mu_e \left[F_{k-1}(t) - F_k(t) \right] \\
&= \mu_e \left[(-1)^{k-1} \sum_{i=k-1}^{J} \binom{i-1}{k-1} \frac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau} - (-1)^k \sum_{i=k}^{J} \binom{i-1}{k-1} \frac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau} \right] \\
&= \mu_e \left[(-1)^{k-1} \sum_{i=k}^{J} \frac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau} \left\{ \binom{i-1}{k-2} + \binom{i-1}{k-1} \right\} \right. \\
&\qquad \left. + (-1)^{k-1} \binom{k-2}{k-2} \frac{(-T_{k-1})^{k-1}}{(k-1)!} \cdot e^{-(k-1)\rho\tau} \right] \\
&= \mu_e \left[(-1)^{k-1} \sum_{i=k}^{J} \frac{(-T_i)^i}{i!} \binom{i}{k-1} \cdot e^{-i\rho\tau} + \frac{T_{k-1}^{k-1}}{(k-1)!} \cdot e^{-(k-1)\rho\tau} \right] \\
&= \rho \cdot e^{-\rho\tau} \cdot (-1)^{k-1} \sum_{i=k-1}^{J} \binom{i}{k-1} \frac{(-T_i)^i}{i!} \cdot e^{-i\rho\tau} \\
&= \rho \cdot e^{-\rho\tau} \cdot \frac{(-1)^{k-1}}{(k-1)!} \sum_{i=k-1}^{J} \frac{(-T_i \cdot e^{-\rho\tau})^i}{(i-k+1)!},
\end{aligned}
\tag{24}
$$

since

$$\binom{k-2}{k-2} = 1 \qquad \text{for } k \geqslant 2,$$

while

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

and

$$\binom{i}{k-1} \frac{1}{i!} = \frac{1}{(k-1)!\,(i-k+1)!}.$$

A comparison with the formula for $_eg_1(t)$ derived above shows that (24) is also valid for $k = 1$.

The explicit forms of $_eg_k$ for the lowest values of k are therefore

- for $\underline{k = 1}$:

$$_eg_1(t) = \rho \cdot e^{-\rho\tau} \sum_{j=0}^{J} \frac{[-\rho(t-j\tau)]^j}{j!} \cdot e^{-j\rho\tau}, \tag{25}$$

- for $\underline{k = 2}$:

$$_eg_2(t) = -\rho \cdot e^{-\rho\tau} \sum_{j=1}^{J} \frac{[-\rho(t-j\tau)]^j}{(j-1)!} \cdot e^{-j\rho\tau}, \tag{26}$$

- for $\underline{k = 3}$:

$$_eg_3(t) = \frac{\rho \cdot e^{-\rho\tau}}{2} \sum_{j=2}^{J} \frac{[-\rho(t-j\tau)]^j}{(j-2)!} \cdot e^{-j\rho\tau}, \tag{27}$$

where always $J \equiv [[t/\tau]]$.

Fig. 2 gives experimental realizations for the interval densities $_ef(t)$ and $_eg(t)$. Again, there is very good agreement with the shapes predicted by (20), for k = 1, and (25).

b) Application of integral transforms

Before making more detailed calculations, let us first have a closer look at the Laplace transform of $_eg(t)$. From previous results [5] we know that

$$_e\tilde{f}(s) = \frac{\rho}{\rho + s \cdot e^{(s+\rho)\tau}} = \frac{\rho \cdot e^{-(s+\rho)\tau}}{s + \rho \cdot e^{-(s+\rho)\tau}} = \frac{-X(s)}{1 - X(s)}, \tag{28}$$

with

$$X(s) \equiv -\frac{\rho}{s} \cdot e^{-(s+\rho)\tau} = -\frac{\mu_e}{s} \cdot e^{-s\tau}. \tag{29}$$

The general relation (5) therefore leads to

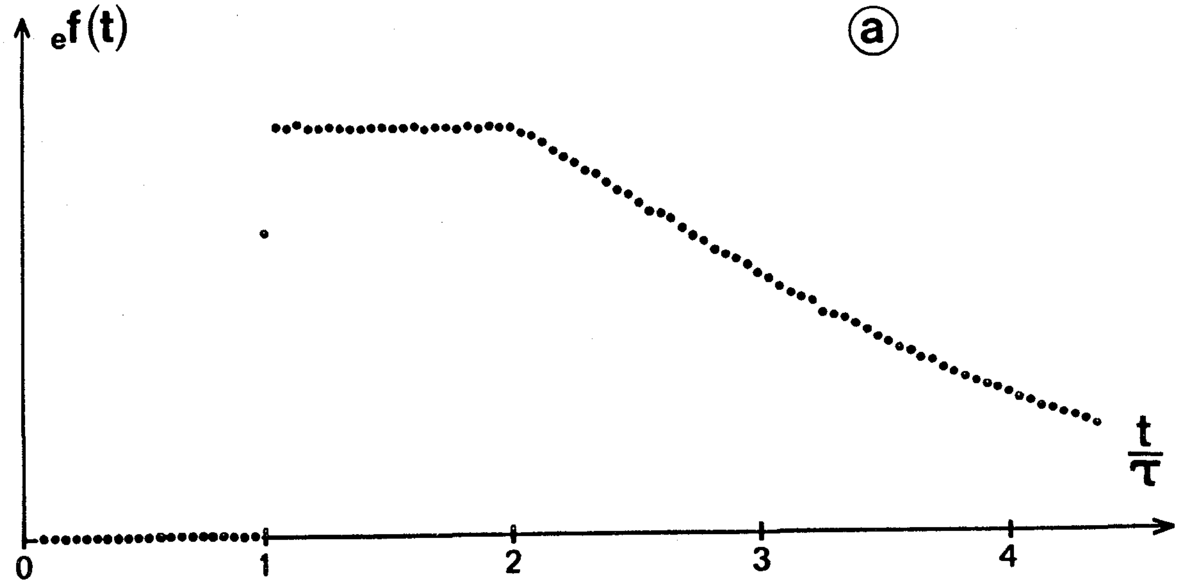
$$_e\tilde{g}(s) = \frac{\mu_e}{s}\left[1 - {_e\tilde{f}(s)}\right] = \frac{\mu_e}{s}\left[1 + \frac{X(s)}{1 - X(s)}\right] = \frac{\mu_e}{s} \cdot \frac{1}{1 - X(s)}. \tag{30}$$

By inserting (23), we get with (29) the surprisingly simple result

$$_e\tilde{g}(s) = \frac{\rho \cdot e^{-\rho\tau}}{s} \cdot \frac{1}{1 - X(s)} = \frac{-X(s)}{1 - X(s)} \cdot e^{s\tau} = e^{s\tau} \cdot {_e\tilde{f}(s)}, \tag{31}$$

which corresponds in the original space to the relation

$$_eg(t) = {_ef(t + \tau)}, \qquad \text{for } t > 0 . \tag{32}$$



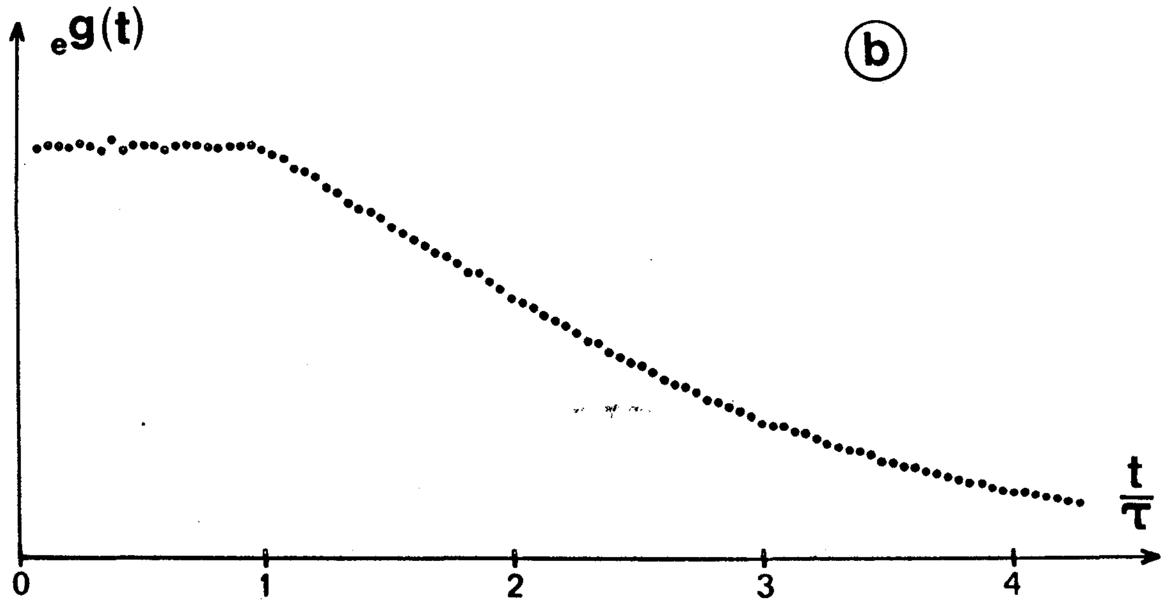


Fig. 2 - Interval densities observed under the same measuring conditions as in Fig. 1, but for an extended dead time.

For an extended dead time, therefore, the densities f(t) and g(t) are identical, apart from a shift by τ in time. It is easy to see from (3) that this can be generalized for $k \geqslant 1$ to

$$ {}_e g_k(t) = {}_e f_k(t+\tau) , \qquad \text{for } t > 0 . \tag{33} $$

Fig. 2 confirms graphically the relation (32).

Therefore, instead of looking for the original of (30) or even for ${}_e g_k(t)$ - which could be done in the way already described in [5] -, a shortcut is now possible. Application of (33) leads with (20) immediately to

$$ {}_e g_k(t) = \varrho \, (-1)^{k-1} \sum_{j=k}^{J+1} {}_k C_j(t) , \qquad k = 1, 2, \ldots , \tag{34} $$

where

$$ {}_k C_j(t) \equiv \binom{j-1}{k-1} \frac{(-T_{j-1})^{j-1}}{(j-1)!} \cdot e^{-j\varrho\tau} . $$

Hence

$$ \begin{aligned} {}_e g_k(t) &= \varrho \, (-1)^{k-1} \sum_{j=k}^{J+1} \binom{j-1}{k-1} \frac{(-T_{j-1})^{j-1}}{(j-1)!} \cdot e^{-j\varrho\tau} \\ &= \mu_e \, (-1)^{k-1} \sum_{j=k-1}^{J} \binom{j}{k-1} \frac{(-T_j)^j}{j!} \cdot e^{-j\varrho\tau} \\ &= \mu_e \cdot \frac{(-1)^{k-1}}{(k-1)!} \sum_{j=k-1}^{J} \frac{(-S_j)^{-j}}{(j-k+1)!} , \end{aligned} \tag{35} $$

with

$$ S_j \equiv e^{-\varrho\tau} \cdot T_j = \mu_e \cdot (t - j\tau) . $$

This result clearly agrees with the previous formula (24) obtained in the more conventional direct way.

In this context the question arises whether a relation like (32) will also exist for other experimental conditions (type of process and/or dead time). If we assume that

$$ g(t) = f(t+\tau') , \tag{36} $$

then (5) allows us to write

$$ \tilde{g}(s) = \frac{\mu}{s} \left[1 - \tilde{f}(s) \right] = \tilde{f}(s) \cdot e^{s\tau'} , $$

from which we deduce

$$\tilde{f}(s) = \frac{\mu/s}{\mu/s + e^{s\tau'}} = \frac{\rho'}{\rho' + s \cdot e^{(s+\rho)\tau'}} , \tag{37}$$

where $\rho' = \mu \cdot e^{\rho\tau'}$.

However, this is just the transformed interval distribution (25) for an original Poisson process (with rate ρ') after an extended dead time τ'. This case is therefore the only instance where the functional equation (36) is fulfilled for a non-vanishing dead time. In particular, no such relation between f and g exists for a non-extended dead time, a conclusion which is confirmed by comparing the explicit forms given in (11) and (16).

4. Evaluation of some moments for f and g

Let us denote the ordinary moments of t (of order r) by

$m_r(t)$ if t has the density f(t)

and

$M_r(t)$ " t " " " g(t) .

If we take into account that the integral transform of, say, f(t) can be written in the form

$$\tilde{f}(s) = 1 - s \cdot m_1(t) + \frac{s^2}{2!} \cdot m_2(t) - \frac{s^3}{3!} \cdot m_3(t) \pm \ldots, \tag{38}$$

then a direct application of (5) leads to (since here $\mu = 1/m_1$)

$$\tilde{g}(s) = \frac{1}{s \cdot m_1} \left[1 - (1 - s \cdot m_1 + \frac{s^2}{2} \cdot m_2 - \frac{s^3}{6} \cdot m_3 \pm \ldots) \right]$$

$$= 1 - \frac{s}{2} \cdot \frac{m_2}{m_1} + \frac{s^2}{6} \cdot \frac{m_3}{m_1} \mp \ldots . \tag{39}$$

By comparison with

$$\tilde{g}(s) = 1 - s \cdot M_1(t) + \frac{s^2}{2!} \cdot M_2(t) \mp \ldots ,$$

we conclude from (39) that g(t) is correctly normalized to unity and that

$$M_1 = \frac{m_2}{2\, m_1} \qquad \text{and} \qquad M_2 = \frac{m_3}{3\, m_1} . \tag{40}$$

Therefore, the variance of g(t) is given by

$$\sigma_g^2 \equiv M_2 - M_1^2 = \frac{1}{12\, m_1^2} (4\, m_1\, m_3 - 3\, m_2^2) \,. \tag{41}$$

We now look separately at the moments m_r for the case of a non-extended and an extended dead time as these are needed for the evaluation of the moments M_r of g(t) according to the relations (40) and (41).

For a non-extended dead time, with ${}_n f_1(t)$ given by (11), the moments are

$$m_r(t) = \int_0^\infty t^r \cdot {}_n f_1(t)\, dt = \rho \int_\tau^\infty t^r \cdot e^{-\rho(t-\tau)}\, dt \,.$$

By applying (13), this can be shown to be

$$m_r(t) = \frac{r!}{\rho^r} \sum_{i=0}^{r} \frac{(\rho\tau)^i}{i!} \,. \tag{42}$$

For an extended dead time, the moments have been determined previously (see [5], eq. 32). For both cases, the first few are summarized in Table 1.

	${}_n f(t)$	${}_e f(t)$
$m_1(t)$	$\frac{1}{\rho}(1 + x)$	$\frac{1}{\rho} \cdot y$
$m_2(t)$	$\frac{2}{\rho^2}(1 + x + x^2/2)$	$\frac{2}{\rho^2} \cdot y\,(y - x)$
$m_3(t)$	$\frac{6}{\rho^3}(1 + x + x^2/2 + x^3/6)$	$\frac{3}{\rho^3} \cdot y\,(2\,y^2 - 4\,xy + x^2)$
$\sigma_f^2 \equiv m_2 - m_1^2$	$\frac{1}{\rho^2}$	$\frac{1}{\rho^2} \cdot y\,(y - 2x)$

Table 1: The first three moments for the ordinary interval density f(t) in the case of a non-extended (n) or an extended (e) dead time, where $x \equiv \rho\tau$ and $y \equiv e^{\rho\tau}$.

By virtue of (40) and (41), the moments of the corresponding functions g(t) are easily obtained; they are listed in Table 2.

	${}_n g(t)$	${}_e g(t)$
$M_1(t)$	$\frac{1}{\varrho} \cdot \frac{1 + x + x^2/2}{1 + x}$	$\frac{1}{\varrho}(y - x)$
$M_2(t)$	$\frac{2}{\varrho^2} \cdot \frac{1 + x + x^2/2 + x^3/6}{1 + x}$	$\frac{2}{\varrho^2}(y^2 - 2\,xy + x^2/2)$
$\sigma_g^2(t)$	$\frac{1}{\varrho^2}\left[1 + \frac{x^3}{3} \cdot \frac{1 + x/4}{(1+x)^2}\right]$	$\frac{1}{\varrho^2} \cdot y\,(y - 2\,x)$

Table 2: The first two moments for the equilibrium interval density g(t), with abbreviations as in Table 1.

5. The normalization of $g_k(t)$

Although (3) implies that if f(t) is normalized to unity then so is $g_k(t)$, it is useful to check the normalization explicitly.

Let $N \equiv \int_0^\infty g_k(t)\,dt = G_k(\infty)$.

According to the Tauber theorem [6], we are also allowed to write

$$N = \lim_{t \to \infty} G_k(t) = \lim_{s \to 0} s \cdot \widetilde{G}_k(s) = \lim_{s \to 0} \widetilde{g}_k(s) \ , \tag{43}$$

which permits to control if N = 1, as we should expect.

a) Non-extended dead time

Since

$$_n\widetilde{f}_k(s) = \left[\frac{\varrho \cdot e^{-s\tau}}{s + \varrho}\right]^k , \tag{44}$$

we get with (6) for $_n\widetilde{g}_k(s)$

$$_n\widetilde{g}_k(s) = \frac{\mu_n}{s}\left[\frac{\varrho \cdot e^{-s\tau}}{s + \varrho}\right]^{k-1}\left[1 - \frac{\varrho \cdot e^{-s\tau}}{s + \varrho}\right]$$

$$= \frac{\mu_n}{s}\left[\frac{\varrho \cdot e^{-s\tau}}{s + \varrho}\right]^{k-1}\left[\frac{\varrho\,(1 - e^{-s\tau}) + s}{s + \varrho}\right] , \tag{45}$$

and therefore by means of (43)

$$ {}_nN = \mu_n \cdot \lim_{s \to 0} \left\{ \frac{1}{s} \left[\frac{\rho\,(1 - e^{-s\tau}) + s}{s + \rho} \right] \right\}. $$

Since $1 - e^{-s\tau} \cong s\tau - \frac{1}{2}(s\tau)^2 \pm \ldots,$ for $s \to 0$, we obtain really

$$ {}_nN = \mu_n \cdot \lim_{s \to 0} \left[\frac{1}{s} \cdot \frac{\rho \cdot s\tau + s}{s + \rho} \right] = \mu_n \cdot \frac{1 + \rho\tau}{\rho} = 1 \,. \tag{46} $$

b) Extended dead time

From the previous results (28) and (30) we know that

$$ {}_e\tilde{f}(s) = \frac{-X(s)}{1 - X(s)} $$

and

$$ {}_e\tilde{g}(s) = \frac{\mu_e}{s} \cdot \frac{1}{1 - X(s)} \,, $$

where $X(s) = -\frac{\rho}{s} \cdot e^{-(s+\rho)\tau}$;

hence, from (6),

$$ {}_e\tilde{g}_k(s) = \frac{\mu_e}{s} \left[\frac{-X(s)}{1 - X(s)} \right]^{k-1} \left[1 + \frac{X(s)}{1 - X(s)} \right] $$

$$ = -\frac{\mu_e}{s} \frac{\left[-X(s)\right]^{k-1}}{\left[1 - X(s)\right]^k} = -\mu_e \cdot \frac{1}{s \cdot X(s)} \left[\frac{-X(s)}{1 - X(s)} \right]^k . \tag{47} $$

However, since

$$ \lim_{s \to 0} \left[-s \cdot X(s) \right] = \lim \left[\rho \cdot e^{-(s+\rho)\tau} \right] = \rho \cdot e^{-\rho\tau} \cdot \lim_{s \to 0} (e^{-s\tau}) = \mu_e $$

and

$$ \lim_{s \to 0} \left[\frac{-X(s)}{1 - X(s)} \right] = \lim \left[\frac{\rho \cdot e^{-(s+\rho)\tau}}{s + \rho \cdot e^{(s+\rho)\tau}} \right] $$

$$ = \rho \cdot e^{-\rho\tau} \cdot \lim_{s \to 0} \left[\frac{e^{-s\tau}}{s + \rho \cdot e^{-\rho\tau} \cdot e^{-s\tau}} \right] = \mu_e \cdot \frac{1}{\mu_e} = 1 \,, $$

we get indeed for the normalization by means of (43)

$$ {}_eN = \lim_{s \to 0} {}_e\tilde{g}_k(s) = 1 \,. \tag{48} $$

6. Total equilibrium density

The total density D of an equilibrium process, defined by

$$D(t) \equiv \sum_{k=1}^{\infty} g_k(t) \,, \tag{49}$$

has a constant value which depends only on the (asymptotic) count rate. This is easily shown as follows:

$$\tilde{D}(s) = \sum_{k=1}^{\infty} \tilde{g}_k(s) = \tilde{g}(s) \sum_{k=0}^{\infty} \left[\tilde{f}(s)\right]^k = \frac{\tilde{g}(s)}{1 - \tilde{f}(s)} = \frac{\mu}{s} \,,$$

and therefore

$$D(t) = \mu \cdot U(t) \,, \tag{50}$$

with μ defined in (2).

As a further check, let us control whether the explicit forms of $g_k(t)$ agree with this general result.

For a non-extended dead time, we obtain from (45)

$$_n\tilde{D}(s) = \sum_{k=1}^{\infty} {_n\tilde{g}_k(s)} = \frac{\mu_n}{s}\left[1 - \frac{\rho \cdot e^{-s\tau}}{s+\rho}\right] \sum_{k=0}^{\infty} \left[\frac{\rho \cdot e^{-s\tau}}{s+\rho}\right]^k$$

$$= \frac{\mu_n}{s}\left[1 - \frac{\rho \cdot e^{-s\tau}}{s+\rho}\right]\left[\frac{1}{1 - \frac{\rho \cdot e^{-s\tau}}{s+\rho}}\right] = \frac{\mu_n}{s} \,. \tag{51}$$

Likewise, the case of an extended dead time gives, by applying (47),

$$_e\tilde{D}(s) = \sum_{k=1}^{\infty} {_e\tilde{g}_k(s)} = \frac{\mu_e}{s}\left[1 + \frac{X(s)}{1 - X(s)}\right] \sum_{k=0}^{\infty} \left[\frac{-X(s)}{1 - X(s)}\right]^k$$

$$= \frac{\mu_e}{s}\left[1 + \frac{X(s)}{1 - X(s)}\right]\left[\frac{1}{1 + \frac{X(s)}{1 - X(s)}}\right] = \frac{\mu_e}{s} \,. \tag{52}$$

In both cases, (50) has thus been verified.

7. Expectation and variance of $g_k(t)$

As a consequence of the independent addition of the individual intervals, which is formally described by (3), we can write with respect to the arrival time t_k of event $k \geq 1$ in an equilibrium process

- for the expectation:

$$E(t_k) = M_1(t) + (k - 1) \cdot m_1(t) , \tag{53}$$

- for the variance:

$$V(t_k) = \sigma_g^2(t) + (k - 1) \cdot \sigma_f^2(t) . \tag{54}$$

By applying the values of the respective moments as given in Tables 1 and 2, we find

a) for a non-extended dead time:

$$_nE(t_k) = \frac{1}{\rho} \cdot \frac{1 + x + x^2/2}{1 + x} + \frac{k - 1}{\rho} (1 + x)$$

$$= \frac{k}{\rho} \cdot \frac{1}{1 + x} \left[1 + x (2 + x) \cdot (1 - \frac{1}{2 k}) \right] , \tag{55}$$

$$_nV(t_k) = \frac{1}{\rho^2} \left[1 + \frac{x^3}{3} \cdot \frac{1 + x/4}{(1 + x)^2} \right] + (k - 1) \frac{1}{\rho^2}$$

$$= \frac{k}{\rho^2} \left[1 + \frac{x^3}{3 k} \cdot \frac{1 + x/4}{(1 + x)^2} \right] ; \tag{56}$$

b) for an extended dead time:

$$_eE(t_k) = \frac{1}{\rho} (y - x) + (k - 1) \frac{y}{\rho} = \frac{k}{\rho} (y - x/k) , \tag{57}$$

$$_eV(t_k) = \frac{1}{\rho^2} y (y - 2 x) + (k - 1) \frac{y}{\rho^2} (y - 2 x)$$

$$= \frac{k}{\rho^2} \cdot y (y - 2 x) . \tag{58}$$

It can be readily verified that for both types of dead time the following relations hold

- for $k = 1$:

$$E(t_1) = M_1(t) , \qquad V(t_1) = \sigma_g^2(t) ; \tag{59}$$

- for $k \gg 1$:

$$E(t_k) \longrightarrow k \cdot m_1(t) , \qquad V(t_k) \longrightarrow k \cdot \sigma_f^2(t) , \tag{60}$$

where the respective explicit expressions for the quantities on the right side of (59) and (60) can be found in Tables 1 and 2.

We finally mention that

$$_eV(t_k) = k \cdot {_e\sigma_f^2}(t) \tag{60'}$$

is rigorously true for any $k \geqslant 1$.

There is no doubt that the above results for the moments could also have been obtained either directly from the densities $g_k(t)$ and/or by differentiation of the transforms $\tilde{g}(s)$. However, as this risks to be a somewhat lengthy arithmetic procedure, it will be left here as an exercise to the reader who is interested in some practice.

In looking back upon the results given in this report, it is interesting to notice that in many cases the formulae for an extended dead time are rather simpler than those which correspond to the usually preferred non-extended type. This is not only contrary to a probably widespread opinion, but it may also be a useful hint for future developments.

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