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A complex modulo K counter

Jörg W. Müller

1. Introduction

This report describes an attempt to generalize a previous suggestion [1] for distinguishing between single and paired events to the case of multiple pulses. Although this problem is at present probably only of minor practical importance, it has been occasionally touched upon, especially in the context of afterpulses, but the traditional methods do not lend themselves easily to a convenient treatment (see for instance [2]). The present approach is based on a rather straightforward extension of the correlation function to more than two states. As an experimental realization should not pose any special problems, it seemed worthwhile to sketch briefly the basic ideas. Useful applications, for instance for quantitative checking purposes, might be found later.

2. Choice of the states

If the development in time of a stationary stochastic process is described by a function $x(t)$, the corresponding autocorrelation function is defined by the expectation

$$R(\delta) \equiv E\left\{x(t) \cdot x^*(t - \delta)\right\} . \qquad (1)$$

We recall that in general R exhibits conjugate symmetry, i.e.

$$R(-\delta) = R^*(\delta) . \qquad (2)$$

This means that the real part of R is an even function of δ, and the imaginary part is odd. In particular, the correlation function is thus symmetric for $x(t)$ real.

Up to now, $x(t)$ has been a real function which could only take the two values +1 and -1, say [3]. Our first problem is to find an appropriate generalization for more than two states. Their corresponding values $x(t)$ will now be denoted by the discrete states x_k, with $k = 1, 2, \ldots, K$. For physical reasons we require that R not depend on the initial state (at $t-\delta$), but only on the development of the process during the time interval δ, as the origin of time

is chosen at random and should be irrelevant in a stationary process. Thus

$$x_{k_1} \cdot x^*_{k_2} = \text{a function of } (k_1 - k_2) \ , \tag{3a}$$

and we shall demand in particular that $R(0) = 1$,

i.e. $x_k \cdot x^*_k = 1$ for any state k . (3b)

These conditions can be fulfilled by choosing for the states the K solutions of the equation

$$x^K = 1 \ . \tag{4}$$

These roots of unity, as is well known, can be represented in the complex plane (cf. Fig. 1) and are

$$x_k = \sqrt[K]{1} = e^{i\varphi_k} = \cos \varphi_k + i \sin \varphi_k \ , \tag{5}$$

with $\varphi_k = k \frac{2\pi}{K}$, $k = 1, 2, \ldots, K$.

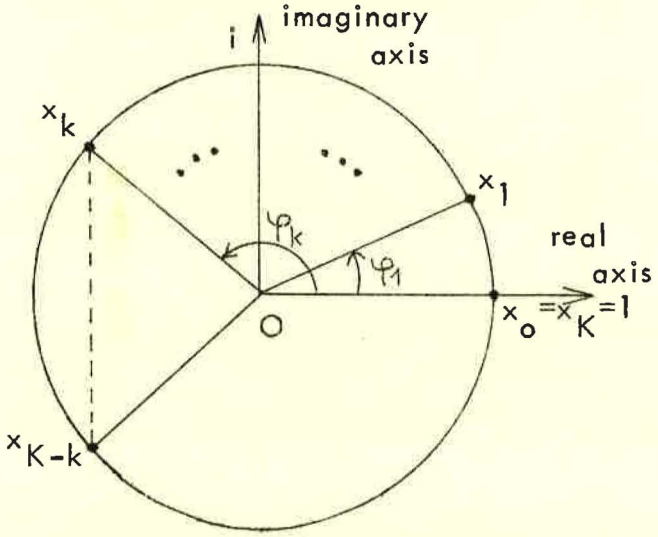

Figure 1 - Roots of (4) in the complex plane (schematic)

One verifies easily that (3b) is fulfilled because

$$x_k \cdot x^*_k = \cos^2 \varphi_k - i^2 \cdot \sin^2 \varphi_k = 1 \ . \tag{3b'}$$

For the sake of convenience, we shall occasionally also use x_o instead of x_K , both of which are obviously equal to 1. Since $\varphi_k = k \cdot \varphi_1$, according to (5), we can write

$$x_k = x_1^k \ . \tag{6}$$

We thus see that the solutions x_k of (4) form a cyclic group of order K, with x_1 as the generating element. We note in passing that if K is a prime number, any other element x_i could have been used as well as a "primitive root". In general, however, the powers of x_i will not generate the full group,

but only a subgroup, the order of which is K/d, if d denotes the largest common divisor of K and j . For other general properties of cyclic groups we refer to any book on algebra (for instance [4]).

As $x_k^* = x_{K-k}$, equation (3b') can now be easily generalized to

$$x_{k_1} \cdot x_{k_2}^* = x_{k_1} \cdot x_{K-k_2} = x_1^{k_1-k_2} = x_k \quad , \tag{3a'}$$

with $k = k_1 - k_2 \pmod K$,

fulfilling thereby the requirement of (3a) that R only depend on the difference of the final and initial states. The effect of the modulo notation, which generalizes the concept of divisibility, is simply that all those numbers k now belong to the same residue class k', for which k-k' can be divided by K.

3. Modulo-probabilities for a Poisson process

The mere fact that a generalized function x(t) with K states x_k can be constructed, and that a corresponding (in general complex) correlation function $R(\delta)$ thus exists with properties which agree with what we expect, is by no means a sufficient guarantee for its usefulness. In addition, we have to demand that such states can be actually "realized" and measured, and that this information can be extracted and interpreted, i.e. compared with theoretical expectation which is based on some realistic model for the process under study. Let us first tackle the last problem and defer the experimental aspects for later.

As was done previously for the simpler case (with K=2), which will be used as a guide-line whenever possible, the basic stochastic process is supposed to be Poissonian with count rate ϱ . One of the main problems we were faced with was the evaluation of the sums for all the even or all the odd numbers of events for an expectation $\mu = \varrho \cdot \delta$. If the probability for observing exactly j events in the interval δ is

$$P(j) = e^{-\mu} \cdot \frac{\mu^j}{j!} \quad , \tag{6}$$

this requires an evaluation of the sums

$$\text{Prob (j even)} = \sum_{s=0}^{\infty} P(2s) = e^{-\mu} \sum_{s} \frac{\mu^{2s}}{(2s)!} = e^{-\mu} \cdot \cosh \mu$$

and (7)

$$\text{Prob (j odd)} = \sum_{s=0}^{\infty} P(2s+1) = e^{-\mu} \sum_{s} \frac{\mu^{2s+1}}{(2s+1)!} = e^{-\mu} \cdot \sinh \mu ,$$

respectively.

Although these closed forms look quite attractive, they are not very useful for finding the generalization we are looking for. What we now need are infinite sums of Poisson probabilities which are e.g. of the form

$$P(0) + P(3) + P(6) + P(9) + \ldots$$

or $$P(2) + P(6) + P(10) + P(14) + \ldots .,$$

The general type is thus

$$\sum_{k=0}^{\infty} P(j=J+k \cdot K) = \text{Prob}\left\{ j = J \ (\text{mod } K) \right\} \equiv W(J \mid K) , \tag{8}$$

with $J = 0, 1, 2, \ldots, K-1$.

K may be called the period (or module), whereas J can be taken as a phase, as it depends on the initial conditions. In this notation the sums evaluated in (7) are denoted by

$$\text{Prob (j even)} = W(0 \mid 2) \quad \text{and}$$

$$\text{Prob (j odd)} = W(1 \mid 2) .$$

A more useful form of (7) would be

$$\text{Prob (j even)} = e^{-\mu} \cdot \frac{1}{2} \left\{ \sum_{j=0}^{\infty} \frac{\mu^j}{j!} + \sum_{j=0}^{\infty} \frac{(-\mu)^j}{j!} \right\} . \tag{7'}$$

Whereas all terms with j even in the curly bracket appear twice, those with j odd cancel, which proves the formula.

A decisive hint can now be obtained by observing that the signs in the two sums of (7') may be interpreted in terms of the two roots x_k of equation (4), namely as $x_0^j = 1$ and $x_1^j = (-1)^j$, respectively.

It might be worthwhile, therefore, to consider similar sums with roots from (4) for a general value of K, thus expressions of the form

$$S(J\,|\,K) \equiv \sum_{i=0}^{\infty} x_o^{i-J} \cdot \frac{\mu^i}{i!} + \sum_{i} x_1^{i-J} \cdot \frac{\mu^i}{i!} + \ldots + \sum_{i} x_{K-1}^{i-J} \cdot \frac{\mu^i}{i!}$$

$$= \sum_{i=0}^{\infty} \frac{\mu^i}{i!} \sum_{k=0}^{K-1} x_k^{i-J} \quad . \tag{9}$$

Whereas (7') can now be written as

$$\text{Prob}\,(j \text{ even}) = \frac{e^{-\mu}}{2} \cdot S(0\,|\,2) \quad ,$$

we might also expect that the two sums mentioned above could be expressed by means of $S(0\,|\,3)$ and $S(2\,|\,4)$, respectively, but this has first to be shown.

In order to do this, let us now have a closer look at the second sum in (9). In the simple case K=2 this is just

$$\sum_{k=0}^{1} x_k^{i-J} = x_o^{i-J} + x_1^{i-J} = \begin{cases} 2 & \text{for } i = J \\ 0 & \text{"} \quad i \neq J \end{cases} .$$

For a general value of K, these sums correspond to the columns in Table 1, where use has been made of the simplifications resulting from the fact that (n = integer)

$$x_k^i = x_{i \cdot k} \quad , \quad x_{n \cdot K+k} = x_k \quad \text{and} \quad x_o = x_K = 1 \quad .$$

Table 1 - The coefficients x_k^{i-J}

	$i-J=0$	1	2	3	...	K-1	K	K+1	...
k = 0	1	x_o	x_o	x_o	...	x_o	1	x_o	...
1	1	x_1	x_2	x_3		x_{K-1}	1	x_1	
2	1	x_2	x_4	x_6		x_{K-2}	1	x_2	
...		...				...	...	...	
K-1	1	x_{K-1}	x_{K-2}	x_{K-3}	...	x_1	1	x_{K-1}	...
sums:	K	0	0	0	...	0	K	0	...

As can be seen from this table, the elements in one column (and also in one row) in general do not contain all the K roots x_k, but only those of a subgroup. In this case, they all appear d times, as the order of the subgroup is K/d. However, this fact will present no inconvenience for the further reasoning.

The sums of the roots as they appear in (9) are readily evaluated by observing that they form a geometric series, thus

$$\sum_{k=0}^{K-1} x_k^{j-J} = \sum_{k=0}^{K-1} x_{j-J}^{k}$$

$$= \begin{cases} K \cdot x_o = K & \text{for } j=J \pmod K \\ \dfrac{1 - x_{j-J}^{K}}{1 - x_{j-J}} = 0 & " \quad j \neq J \pmod K \end{cases} , \tag{10}$$

as indicated in Table 1.

By using the property that (10) vanishes for all values j which belong to a different residue class than J, the partial Poisson sum (8) can also be written as

$$W(J\,|\,K) = \sum_{s=0}^{\infty} \frac{\mu^{J+sK}}{(J+sK)!} \cdot e^{-\mu} = \sum_{j=0}^{\infty} \frac{\mu^j}{j!} \cdot e^{-\mu} \sum_{k=0}^{K-1} x_k^{j-J}$$

$$= e^{-\mu} \cdot \frac{1}{K} \cdot S(J\,|\,K) \ . \tag{11}$$

This answers the question raised after (9).

On the other hand, S(J|K) from (9) can be expressed as

$$S(J\,|\,K) = \sum_{k=0}^{K-1} x_k^{-J} \sum_{j=0}^{\infty} \frac{(\mu x_k)^j}{j!} = \sum_{k=0}^{K-1} x_k^{-J} \cdot e^{\mu x_k} \ . \tag{9'}$$

Inserting this into (11) yields as our final formula (with $0 \leq J \leq K-1$)

$$W(J\,|\,K) = e^{-\mu} \cdot \frac{1}{K} \sum_{k=0}^{K-1} x_k^{-J} \cdot e^{\mu x_k} = \frac{1}{K} \sum_{k=0}^{K-1} x_k^{-J} \cdot e^{\mu (x_k - 1)} , \tag{12}$$

with W(J|K) defined by (8).

As a computer in general cannot handle directly complex quantities, it may be advantageous to express (12) in a somewhat different form. According to (5) the roots x_k can be written as

$$x_k = e^{i\varphi_k} \quad , \quad \text{with} \quad \varphi_k = k\,\frac{2\pi}{K} \; ;$$

we then get readily

$$x_k^{-J} = \cos(J\varphi_k) - i\cdot\sin(J\varphi_k) \qquad \text{and}$$

$$e^{\mu(x_k-1)} = \exp\left[\mu(\cos\varphi_k - 1 + i\cdot\sin\varphi_k)\right]$$

$$= e^{-\mu(1-\cos\varphi_k)}\left[\cos(\mu\cdot\sin\varphi_k) + i\cdot\sin(\mu\cdot\sin\varphi_k)\right] \; .$$

A term in the sum (12) corresponds therefore to

$$x_k^{-J}\cdot e^{\mu(x_k-1)} = e^{-\mu(1-\cos\varphi_k)}\left[\cos(J\varphi_k) - i\cdot\sin(J\varphi_k)\right]\cdot\left[\cos(\mu\cdot\sin\varphi_k) + i\cdot\sin(\mu\cdot\sin\varphi_k)\right] \; .$$

For the corresponding complex conjugate solution $x_{k'}$, which is assumed to be not real and therefore different from x_k (thus k' neither 0 nor K/2). we can obviously just replace $\varphi_{k'}$ by $-\varphi_k$ in the above expression. The sum for a pair of complex conjugate solutions is in (12)

$$x_k^{-J}\cdot e^{\mu(x_k-1)} + x_{k'}^{-J}\cdot e^{\mu(x_{k'}-1)}$$

$$= e^{-\mu(1-\cos\varphi_k)}\cdot 2\left[\cos(J\varphi_k)\cdot\cos(\sin\varphi_k) + \sin(J\varphi_k)\cdot\sin(\sin\varphi_k)\right] \; .$$

Whereas the contribution from k=0 (with x_o=1) always exists and simply amounts to

$$x_o^{-J}\cdot e^{\mu(x_o-1)} = 1 \quad ,$$

the term corresponding to k=K/2 (i.e. x_k=-1) only exists for K even and then is

$$x_k^{-J}\cdot e^{\mu(x_k-1)} = (-1)^J\cdot e^{-2\mu} \; .$$

After some simple rearrangements, the general formula (12) can now be written in the following form which lends itself more easily to a computer evaluation

$$W(J\,|\,K) = \frac{1}{K}\Big\{ 1 + (K-2\varkappa-1)\cdot(-1)^J \cdot e^{-2\mu} + 2\sum_{k=1}^{\varkappa} e^{-\mu(1-\cos\varphi_k)} \Big[\cos(J\varphi_k)\cdot\cos(\mu\cdot\sin\varphi_k) + \sin(J\varphi_k)\cdot\sin(\mu\sin\varphi_k)\Big]\Big\}, \qquad (13)$$

where $\varkappa \equiv [\,[K/2]\,]$ is the largest integer below $K/2$ and $\varphi_k = k\cdot 2\pi/K$.

The factor

$$K-2\varkappa-1 = \frac{1}{2}\Big[1+(-1)^K\Big] = \begin{cases} 1 & \text{for } K \text{ even} \\ 0 & \text{" } K \text{ odd} \end{cases}$$

restricts the contribution from $x_k = -1$ to the case where K is even.

Some explicit expressions for $W(J\,|\,K)$ are given in Table 2. A simple computer program which calculates $W(J\,|\,K)$ for any count rate μ and arbitrary choice of K and $J < K$ on the basis of equation (13) is available upon request.

For $J=0$ the formula (13) simplifies to

$$W(0\,|\,K) = \frac{1}{K}\left[1 + (K-2\varkappa-1)\cdot e^{-2\mu} + 2\sum_{k=1}^{\varkappa} e^{-\mu(1-\cos\varphi_k)}\cdot\cos(\mu\cdot\sin\varphi_k)\right] . \qquad (13')$$

One can also verify that indeed

$$\sum_{J=0}^{K-1} W(J\,|\,K) = 1 \qquad \text{for any } K. \qquad (14)$$

K	J	$K \cdot W(J \mid K)$
1	0	1
2	0	$1 + e^{-2\mu}$
	1	$1 - e^{-2\mu}$
3	0	$1 + 2 \cdot \exp(-\frac{3}{2}\mu) \cdot \cos(\frac{\mu\sqrt{3}}{2})$
	1	$1 - \exp(-\frac{3}{2}\mu) \left[\cos(\frac{\mu\sqrt{3}}{2}) - \sqrt{3} \cdot \sin(\frac{\mu\sqrt{3}}{2})\right]$
	2	$1 - \exp(-\frac{3}{2}\mu) \left[\cos(\frac{\mu\sqrt{3}}{2}) + \sqrt{3} \cdot \sin(\frac{\mu\sqrt{3}}{2})\right]$
4	0	$1 + e^{-2\mu} + 2 \cdot e^{-\mu} \cdot \cos\mu$
	1	$1 - e^{-2\mu} + 2 \cdot e^{-\mu} \cdot \sin\mu$
	2	$1 + e^{-2\mu} - 2 \cdot e^{-\mu} \cdot \cos\mu$
	3	$1 - e^{-2\mu} - 2 \cdot e^{-\mu} \cdot \sin\mu$

Table 2 - Formulae for the Poisson modulo-sums $W(J \mid K) = \sum_{k=0}^{\infty} P(J + kK)$, with $P(j) = e^{-\mu} \cdot \mu^j / j!$, for the four lowest values of K.

As a numerical check let us compare some values obtained by (13) with those we get by summing the corresponding individual Poisson probabilities as they are taken from a table with 6 decimals [5] .

For $\mu = 2$ and $K = 2$ to 6, for instance, we get the following:

K	J	W(J|K)
2	0	0.509 158
	1	0.490 842
3	0	0.328 004
	1	0.364 370
	2	0.307 626
4	0	0.226 419
	1	0.306 951
	2	0.282 739
	3	0.183 891
5	0	0.171 463
	1	0.282 707
	2	0.274 109
	3	0.181 306
	4	0.090 414
6	0	0.147 366
	1	0.274 108
	2	0.271 530
	3	0.180 638
	4	0.090 262
	5	0.036 096

These values are in complete agreement with the sums formed by the individual values taken from [5] .

The following limiting values are also worth noting:

$$\lim_{\mu \to 0} W(J|K) = \delta_{0,J} \,, \qquad \lim_{\mu \to \infty} W(J|K) = 1/K \,. \tag{15}$$

As an example, the distributions $W(J|K)$ for $K = 4$ are plotted in Fig. 2 as a function of the mean value μ .

4. Realization of a modulo counter

Generally speaking, the practical construction of such a counter is guided by the application made previously for the case $K = 2$, whenever this is possible [1] . In particular, both the original and the delayed series of pulses command a modulo K counter, each of which in turn acts on a series of gates (denoted by circles). The state of these gates is periodically explored by a train of pulses from an oscillator (with frequency ν) which can reach a given "output" only if the corresponding two gates they have to pass are both "open". For $K = 3$, one might therefore first imagine an arrangement similar to the one sketched in Fig. 3. For higher values of K, however, the set-up would soon become prohibitive as the total number of gates needed here equals $K(K+1)$.

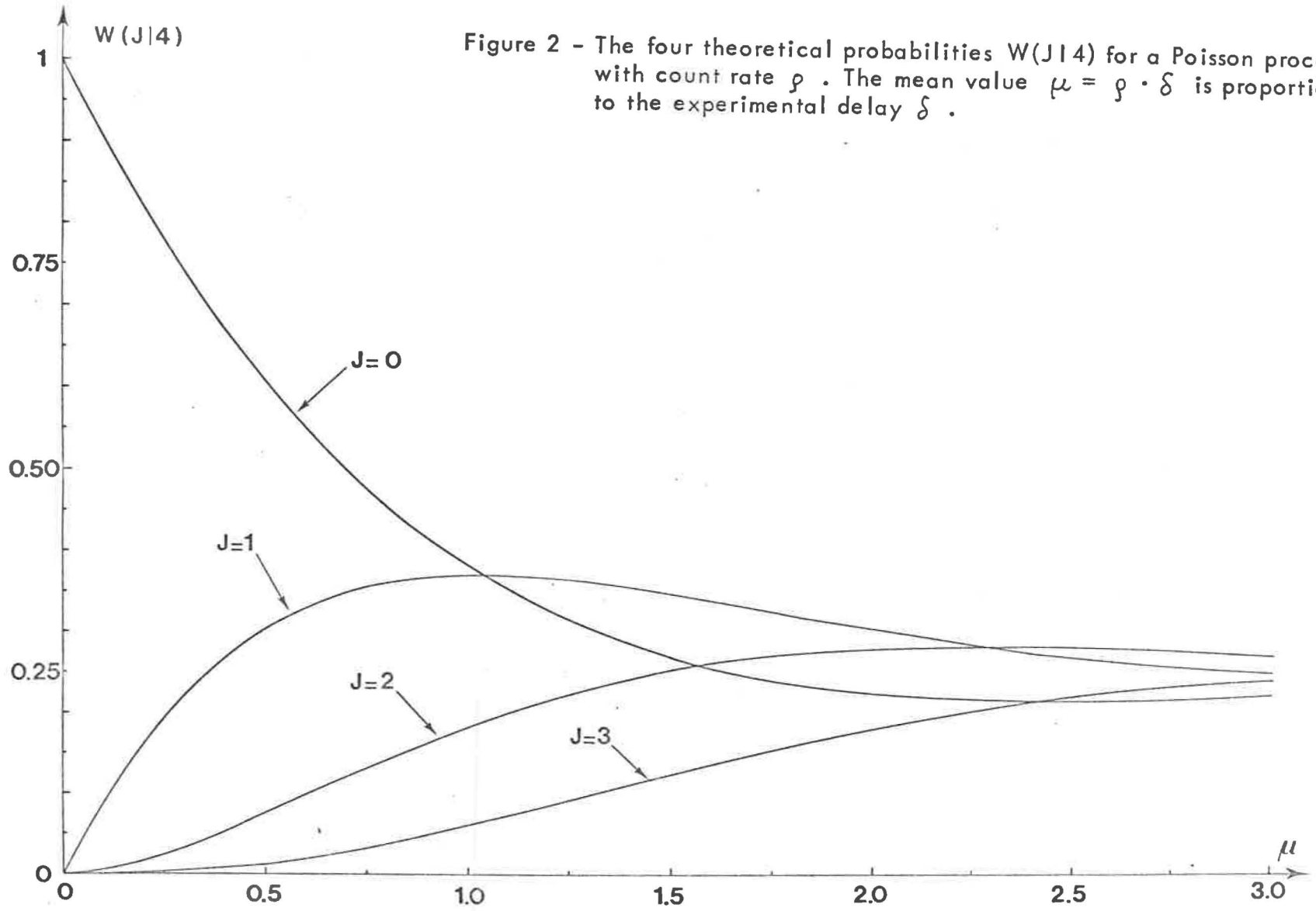

Figure 2 - The four theoretical probabilities W(J|4) for a Poisson process with count rate ϱ . The mean value $\mu = \varrho \cdot \delta$ is proportional to the experimental delay δ .

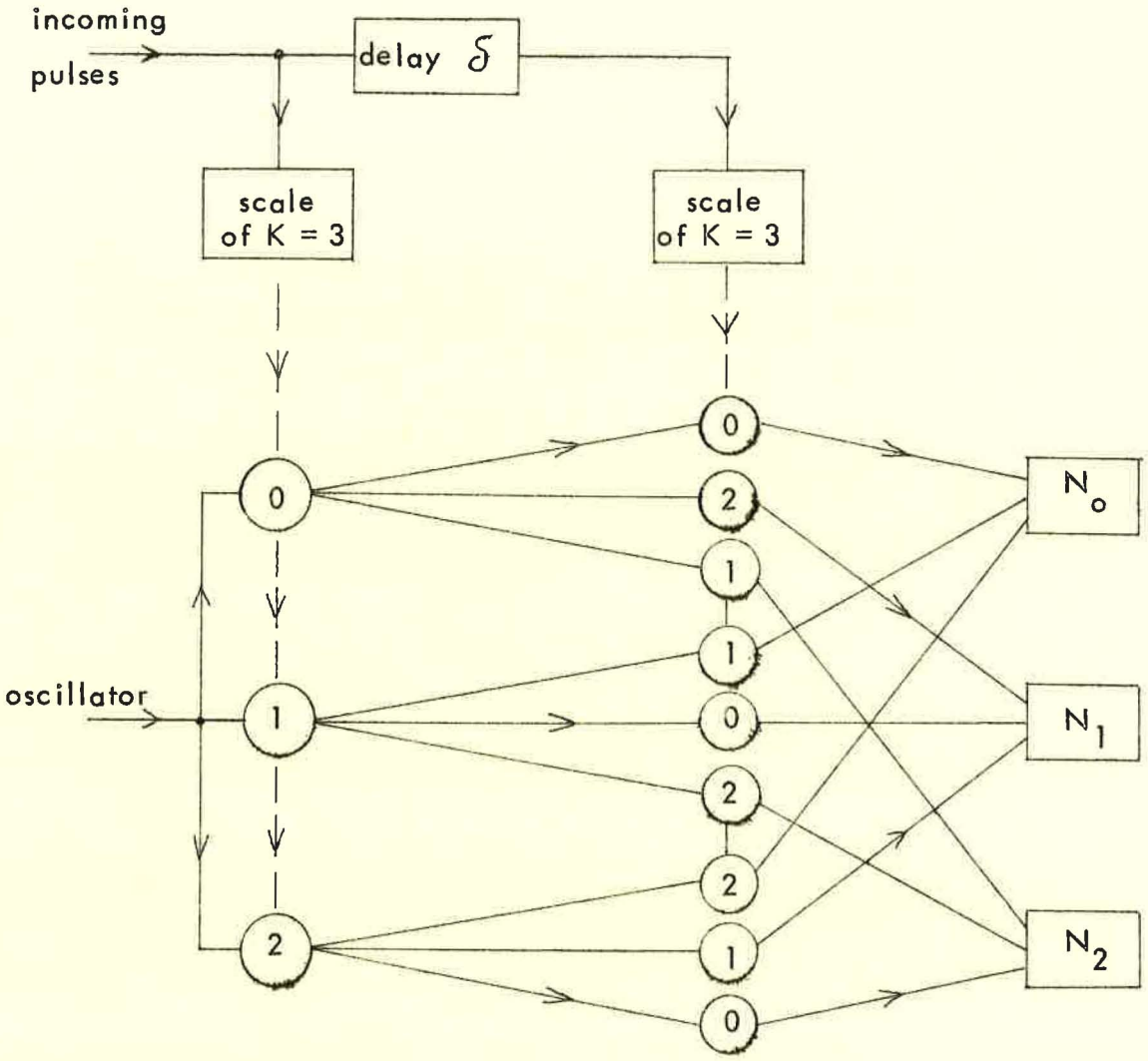

Figure 3 - Schematic diagram for a "complete" correlation measurement with K=3. The gates are open only if the corresponding scale of K is in the position indicated by the number in the circles. The three counters at the right indicate the number of times (N_J) that the oscillator pulses found the two scales of K differing by J, which means that J (mod K = 3) pulses had arrived in the time interval δ .

In such a "complete" arrangement, one specific path (i.e. the corresponding two gates) would always be open for any given oscillator pulse which is therefore registered by one of the K counters at the output. After a measuring

time t, the ratio

$$\pi(J\,|\,K) = \frac{N_j}{\nu \cdot t} , \qquad J = 0, 1, \ldots, K-1 , \tag{16}$$

would then approach the probability for observing J(mod K) events in the given time δ .

It is also possible, of course, to deduce from the experimental frequencies π the correlation function as it has been defined in (1) by simply forming

$$R_K^{(exp)}(\delta) = \sum_{J=0}^{K-1} \pi(J\,|\,K) \cdot x_J , \tag{17}$$

or, since R will be complex in general,

$$\mathrm{Re}\left\{ R_K^{(exp)}(\delta) \right\} = \sum_J \pi(J\,|\,K) \cdot \cos\left(J \cdot \frac{2\pi}{K}\right) \tag{17'}$$

and

$$\mathrm{Im}\left\{ R_K^{(exp)}(\delta) \right\} = \sum_J \pi(J\,|\,K) \cdot \sin\left(J \cdot \frac{2\pi}{K}\right) .$$

The empirical function (17) may then be compared with the corresponding theoretical shape

$$R_K^{(th)}(\delta) = \sum_{J=0}^{K-1} W(J\,|\,K) \cdot x_J . \tag{18}$$

For a Poisson process and the lowest values of K, this can be shown to be

$$\begin{aligned} R_2^{(th)}(\delta) &= \exp\left[-2\mu\right] , \\ R_3^{(th)}(\delta) &= \exp\left[-\mu\left(\frac{3}{2} - i \cdot \frac{\sqrt{3}}{2}\right)\right] \\ R_4^{(th)}(\delta) &= \exp\left[-\mu(1 - i)\right] , \end{aligned} \tag{19}$$

with $\mu = \varrho \cdot \delta$, for $\delta \geq 0$.

However, it seems more natural here to consider directly the measured real quantities $\pi(J\,|\,K)$ as a function of the delay δ since they can be readily compared with the calculated values W(J|K). In particular, any significant

difference would be indicative of a deviation of the process under study from a purely Poissonian one. Besides, such a comparison can be performed for any specific value of J. Instead of the previous "complete" measurement, where all K values $\pi(J|K)$ needed for forming the empirical correlation function, according to (17), are obtained, one such quantity is now sufficient. In this case, the experimental set-up, as sketched in Fig. 3, can be considerably simplified. Furthermore, as two "realizations" such as, for example,

$$x_{j+J} \cdot x_j^* \qquad \text{and} \qquad x_{k+J} \cdot x_k^*$$

both correspond to the same state x_J, as can be seen from (3a'), it is obvious that for reasons of symmetry all the K possible positions of the initial counter (here j or k) will give the same contribution to $\pi(J|K)$. This allows us to use an experimental arrangement which is not only much simpler, but also more flexible, as can be seen from Fig. 4.

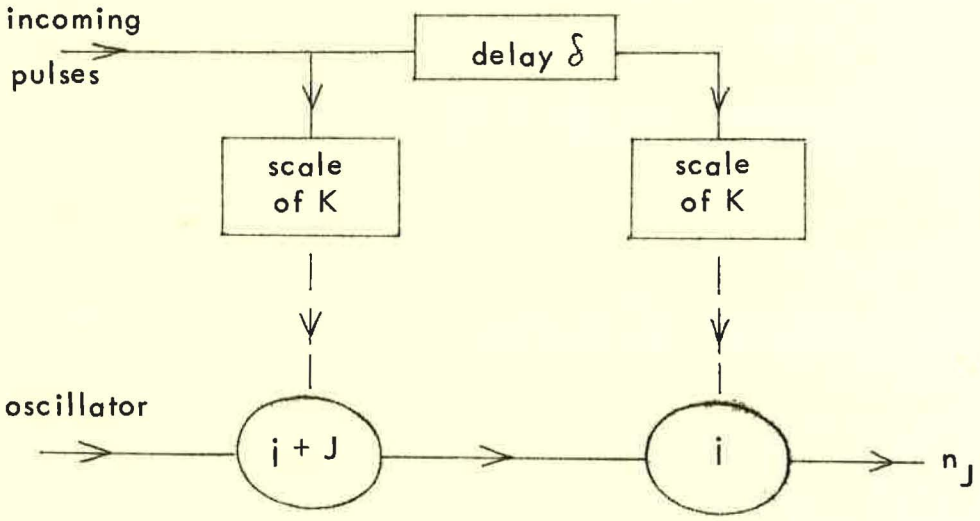

Figure 4 - Simplified schematic arrangement for measuring the probability $\pi(J|K)$.

As just mentioned, the choice of j is irrelevant; it is only the difference J for the two gates which has a physical meaning. As for J, one might prefer the value for which the probability W(J|K) is highest. On the other hand, J = 0 could also be a good choice as the corresponding value W(0|K) which is used for comparison has in general the simplest form (compare 13'). In any case, the empirical probability $\pi(J|K)$ is given by

$$\pi(J|K) = \frac{K \cdot n_J}{\nu \cdot t} , \tag{20}$$

where ν is the oscillator frequency and t the measuring time.

Instead of checking at a given moment the original and the delayed process, as suggested in Fig. 3, it is also possible to get the same information by controlling the original pulse series at two different times, delaying thus the oscillator pulse before it passes the second gate. This method, which has several practical advantages, especially if shift registers are used for producing the delay, is sketched in Fig. 5. and will be used for the experimental checks now under way.

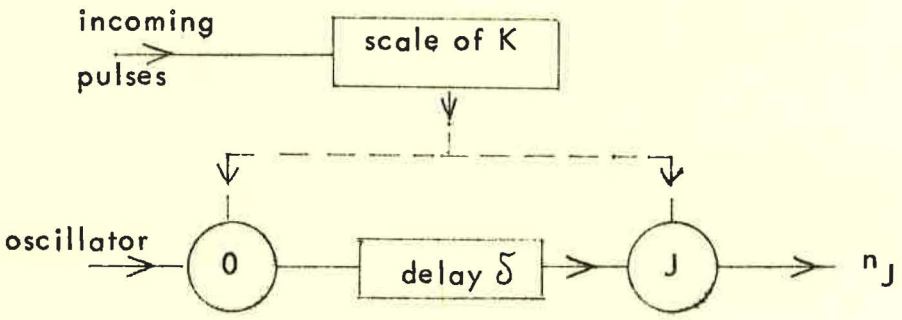

Figure 5 - Final proposal for the experimental arrangement to measure $\pi(J \mid K)$.

The measured values $\pi(J \mid K)$, again given by (20), are now determined for a number of delays δ_i and for a fixed value of J (e.g. 0). On the other hand, a direct measurement of the incoming pulses yields its mean count rate ϱ_{tot}. As the "beginning" of a time interval δ is determined by an oscillator test pulse and thus randomly chosen with respect to the process to be analyzed, the pulses arriving within δ form an equilibrium process, where the average number μ of events is known to be strictly proportional to the measuring time (e.g. [6], eq. 23). The chosen delays therefore correspond to the means $\mu_i = \varrho_{tot} \cdot \delta_i$ and the theoretical probabilities $W(J \mid K)$ can be calculated for each δ_i.

But how should these data now be interpreted to obtain useful information about afterpulses? In the case where the quantities $\pi(J \mid K)$ and $W(J \mid K)$ agree for all the measured points (within the experimental uncertainties, of course), we can conclude that the process studied does not differ from a pure Poisson process. This means here in particular that there is no indication for the presence of afterpulses (of the order K-1).

However, if the differences between the corresponding pairs of values $\pi(J \mid K)$ and $W(J \mid K)$ are significant, the conclusion is in general less straightforward and demands a more careful analysis. Nevertheless, the interpretation is again

simple for a Poisson process (with original rate ϱ), where a certain fraction θ of the events are followed by (exactly) K-1 afterpulses*. Then

$$\varrho_{tot} = \varrho\left[1 + (K-1)\,\theta\right] \quad . \tag{21}$$

Let us assume the effective measuring time δ to be sufficiently long compared to the average distance between the main pulse and its afterpulses so that in general they all will arrive in the same time interval δ (edge effect neglected, see appendix B of [1]). As all the groups of K events (i.e. main pulse and afterpulses) are not "seen" by a modulo K counter, the measured probabilities $\pi(J \mid K)$ only agree with the calculated values $W(J \mid K)$, if these are based on a reduced count rate

$$\varrho_{fit} = \varrho(1 - \theta) = \varrho_{tot} - K \cdot \theta \cdot \varrho \quad , \tag{22}$$

which corresponds to the best fit to the measured data. A significant difference between ϱ_{tot} and ϱ_{fit} thus allows one to determine the count rate due to the (K-1)-fold afterpulses (or of θ).

In practice, more complicated situations may exist; for example, the assumption of an original Poisson process might not be justified, or afterpulses of various order K could be produced. One might expect such a modulo K arrangement to be normally selective enough to pick out a component, the contributions from $K' \neq K$ being largely averaged out. However, this could only be asserted on the basis of a more careful and detailed analysis. Such a study would also give a firm ground for estimating the accuracy of this method.

References

[1] J.W. Müller: "A new method for distinguishing between pairs and single pulses", Report BIPM-72/14 (December 1972)

[2] V.E. Lewis, D. Smith, A. Williams: "Correlation counting applied to the determination of absolute disintegration rates for nuclides with delayed states", Metrologia 9, 14 (1973), eq. 28 and 29

[3] J.W. Müller: "Etude de comptages à l'aide d'un corrélateur", Rapport BIPM-70/6 (July 1970)

[4] B.L. van der Waerden: "Algebra I" (Springer, Berlin, 1971[8])

[5] E.C. Molina: "Poisson's exponential binomial limit" (Van Nostrand, Princeton, 1942)

[6] J.W. Müller: "Dead-time problems" (Herceg Novi paper, to be published in the Proceedings) (August 1972).

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* For simplicity, the detection efficiencies are already included in the count rates or supposed to be unity.