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Binomial modulo-sums

Jörg W. Müller

1. Introduction

The fact that the probability of observing an even or an odd number of counts in a given time interval contains useful information about the eventual presence of doubled events (primary and secondary pulses) has recently been exploited to develop a quantitative method for measuring afterpulses [1] in a series of events which is originally of the Poissonian type. This idea has then been extended to multiple afterpulses [2], as a general way was found to calculate the corresponding modulo-sums

$$W(J \mid K) \equiv \text{Prob}\{j = J \pmod K\}, \quad \text{with } 0 \leq J < K, \tag{1}$$

if the probability for observing exactly j events is given by the Poisson law

$$P_\mu(j) = e^{-\mu} \cdot \frac{\mu^j}{j!},$$

where $\mu = \rho t$ is the expected mean number within a time interval t.

As the method used for determining the modulo-sums $W(J \mid K)$ seemed to be quite general, one might apply it to other discrete distributions as well. The initial choice was not quite obvious as there are many candidates (geometric*), binomial, hypergeometric, Pólya, etc.). Finally, in view of its fundamental importance (from a practical as well as theoretical point of view) the binomial distribution has been selected as an example. This choice has also been favoured by the hope to use the result for a check of the Poisson case treated previously and to derive from it some general formulae for modulo-sums of binomial coefficients. The last problem will be treated in a subsequent report.

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*) see the Appendix

2. The binomial case

Let us assume that the probability for observing j events (in a given time) is described by the law of the binomial distribution ($q \equiv 1 - p$)

$$B_{p,n}(j) = \binom{n}{j} p^j q^{n-j} , \qquad \text{with } j = 0, 1, 2, \ldots, n, \tag{2}$$

where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$ is a binomial coefficient giving the number of combinations of n things taken j at a time.

The general idea used in [2] to evaluate for the probability distribution (1) the so-called modulo-sums $W_\mu(J\,|\,K)$, which are defined by

$$W_\mu(J\,|\,K) = \sum_{s=0}^{\infty} P_\mu(J + sK) , \qquad \text{with} \quad 0 \leqslant J < K , \tag{3}$$

was to replace the single sum in (3) by

$$W_\mu(J\,|\,K) = \frac{1}{K} \sum_{j=0}^{\infty} \sum_{k=0}^{K-1} x_k^{j-J} \cdot P_\mu(j) , \tag{4}$$

where the quantities x_k are the K roots of the equation

$$x^K = 1 . \tag{5}$$

That (3) and (4) are identical is assured by the fact that according to Eq. (10) in [2]

$$\sum_{k=0}^{K-1} x_k^{j-J} = \begin{cases} K & \text{for } j = J \pmod K \\ 0 & \text{"} \quad j \neq J \pmod K) . \end{cases}$$

Therefore in the binomial case, i.e. with the probability distribution (2), the expression corresponding to (4) is now

$$W_{p,n}(J\,|\,K) = \frac{1}{K} \sum_{j=0}^{\infty} B_{p,n}(j) \sum_{k=0}^{K-1} x_k^{j-J}$$

$$= \frac{1}{K} \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} \sum_{k=0}^{K-1} x_k^{j-J}$$

$$= \frac{1}{K} \sum_{k=0}^{K-1} x_k^{-J} \sum_{j=0}^{n} \binom{n}{j} (p \cdot x_k)^j q^{n-j}$$

$$= \frac{1}{K} \sum_{k=0}^{K-1} x_k^{-J} (p \cdot x_k + q)^n , \tag{6}$$

since $\binom{n}{j} = 0$ for $j > n$.

Before going further we can make use of the symmetry of the binomial distribution (2). It is well known that

$$B_{p,n}(j) = \binom{n}{j} p^j (1-p)^{n-j}$$

$$= \binom{n}{n-j} (1-p)^{n-j} p^j = B_{1-p,n}(n-j) . \tag{7}$$

This relation simply states that for a probability of "success" p in any single trial, the probability of exactly j "successes" in n independent trials is identical with the probability of exactly $n-j$ "failures", if the probability of "failure" in any single trial is $1-p$.

We therefore obtain with (3) for the modulo-sums the corresponding identity

$$W_{p,n}(J|K) = \sum_{s=0}^{\infty} B_{p,n}(J + sK)$$

$$= \sum_{s=0}^{\infty} B_{1-p,n} (n - J - sK) = W_{1-p,n}(J'|K) , \tag{8}$$

where $J' = n - J \pmod K$.

The main task now consists in transforming (6) into a more easily manageable explicit form where the complex solutions x_k are eliminated. We remember that a similar rearrangement had also to be performed in the Poissonian case [2] .

3. Some rearrangements

As the solution of (5) is

$$x_k = \exp (i \varphi_k) , \tag{9}$$

with $\varphi_k = k \cdot \frac{2\pi}{K}$, $\qquad k = 0, 1, \ldots, K-1$, we get for the first factor in (6)

$$x_k^{-J} = \cos (J \cdot \varphi_k) - i \cdot \sin (J \cdot \varphi_k) \ . \tag{10}$$

For the second factor we first write

$$\begin{aligned} px_k + q &= 1 + p(x_k - 1) = 1 + p(\cos \varphi_k + i \sin \varphi_k - 1) \\ &\equiv a_k + i\, b_k \ , \end{aligned} \tag{11}$$

where $a_k = 1 - p\,(1 - \cos \varphi_k)$

and $\quad b_k = p \cdot \sin \varphi_k$.

With this notation (6) now reads

$$K \cdot W_{p,n}(J|K) = \sum_{k=0}^{K-1} x_k^{-J} (a_k + i\, b_k)^n \ . \tag{6'}$$

To simplify the notation, we shall usually drop the parameters p and n in what follows, thus writing $W(J|K)$ for $W_{p,n}(J|K)$ whenever ambiguities are unlikely.

To evaluate the sum (6') it is practical to separate the real from the complex solutions x_k. The contribution from <u>$k = 0$</u> is always present and particularly simple. We get with the definitions (11)

$$x_o^{-J} (a_o + i\, b_o)^n = 1 \left[1 - p(1 - \cos \varphi_o) + ip \cdot \sin \varphi_o \right]^n = 1 \ , \tag{12}$$

since $x_o = 1$ and $\varphi_o = 0$.

Another real solution of (5) only exists if K is even. Then for <u>$k = K/2$</u> we have $x_k = -1$ and $\varphi_k = \pi$. Therefore

$$a_k + ib_k = 1 - p(1 - \cos \varphi_k) + ip \cdot \sin \varphi_k = 1 - 2p \ ,$$

hence

$$x_k^{-J} (a_k + i\, b_k)^n = (-1)^J (1 - 2p)^n \ . \tag{13}$$

All the other solutions x_k (for $K \geq 3$) appear in the form of complex-conjugate pairs (for k and k', say), where $\varphi_{k'} = -\varphi_k$, hence $a_{k'} = a_k$, but $b_{k'} = -b_k$.

The contribution to the sum (6') from such a pair is therefore

$$x_k^{-J}(a_k + i\, b_k)^n + x_{k'}^{-J}(a_{k'} + i\, b_{k'})^n \tag{14}$$

$$= \left[\cos(J\,\varphi_k) - i\,\sin(J\,\varphi_k)\right](a_k + i\, b_k)^n + \left[\cos(J\,\varphi_k) + i\,\sin(J\,\varphi_k)\right](a_k - i\, b_k)^n$$

$$= \cos\varphi_{k\cdot J}\left\{(a_k + i\, b_k)^n + (a_k - i\, b_k)^n\right\} + i\cdot\sin\varphi_{k\cdot J}\left\{(a_k - i\, b_k)^n - (a_k + i\, b_k)^n\right\}.$$

To form the powers in the curly brackets, it would be possible to apply directly the corresponding binomial expansions. This, however, leads rapidly to rather complicated expressions and it seems more practical to proceed in a slightly different manner. Let us put

$$a_k = r_k \cdot \alpha_k \quad \text{and}$$
$$b_k = r_k \cdot \beta_k\,, \tag{15}$$

where r_k is chosen so that $\alpha_k^2 + \beta_k^2 = 1$. Thus

$$a_k^2 + b_k^2 = r_k^2\,(\alpha_k^2 + \beta_k^2) = r_k^2$$

or, after inserting (11),

$$r_k^2 = 1 - 2p(1-p)\,(1-\cos\varphi_k)\ . \tag{16}$$

But then a_k and b_k can also be readily expressed in polar coordinates by (see Fig. 1)

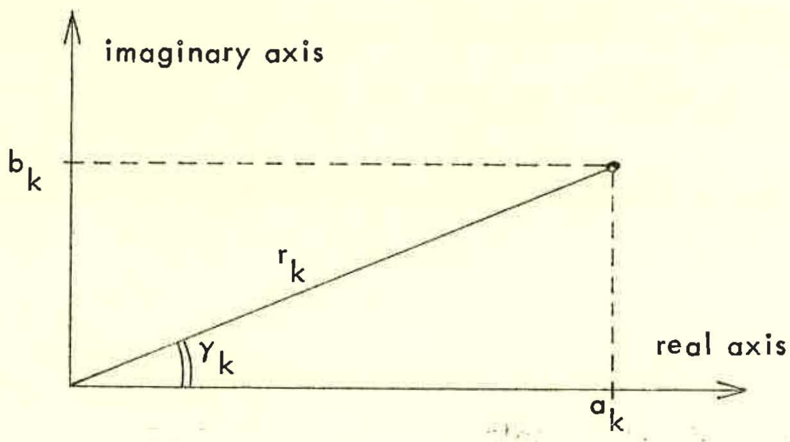

Fig. 1.- Transformation of a_k, b_k to polar coordinates r_k, γ_k

$$a_k = r_k \cdot \cos \gamma_k \quad \text{and}$$

$$b_k = r_k \cdot \sin \gamma_k \quad ,$$

where $$\operatorname{tg} \gamma_k = \frac{\sin \gamma_k}{\cos \gamma_k} = \frac{b_k}{a_k} \quad ,$$

or with (11)

$$\gamma_k = \operatorname{arc\,tg}\left(\frac{b_k}{a_k}\right) = \operatorname{arc\,tg}\left[\frac{p \cdot \sin \varphi_k}{1 - p(1-\cos \varphi_k)}\right] \quad . \tag{17}$$

Because of the periodicity of the trigonometric functions, their inverse functions are indefinitely many-valued. A word of caution may therefore be in place here. By convention, the principal value of arc tg x, written as Arc tg x, lies between $-\pi/2$ and $\pi/2$, and it is this value which is calculated for example by an automatic computer. This practice, however, is not in agreement with the requirements of our present problem. It is easy to see from Fig. 1 that

$$0 < \gamma_k < \pi \quad .$$

For a complex solution x_k, we have $0 < k < K/2$. Hence x_k lies in the upper half of the complex plane and $\sin \varphi_k > 0$, therefore also $b_k > 0$.

If we design the principal value by

$$\operatorname{Arc\,tg}\left[\frac{p \cdot \sin \varphi_k}{1 - p(1-\cos \varphi_k)}\right] \equiv g_k \quad , \tag{18}$$

the correct choice for γ_k is (see Fig. 2)

$$\gamma_k = \begin{cases} g_k & \text{for } g_k > 0 \\ g_k + \pi & \text{"} \quad g_k < 0 \quad . \end{cases} \tag{19}$$

The sign of g_k is determined by $1 - p(1-\cos \varphi_k)$, the denominator of (18).

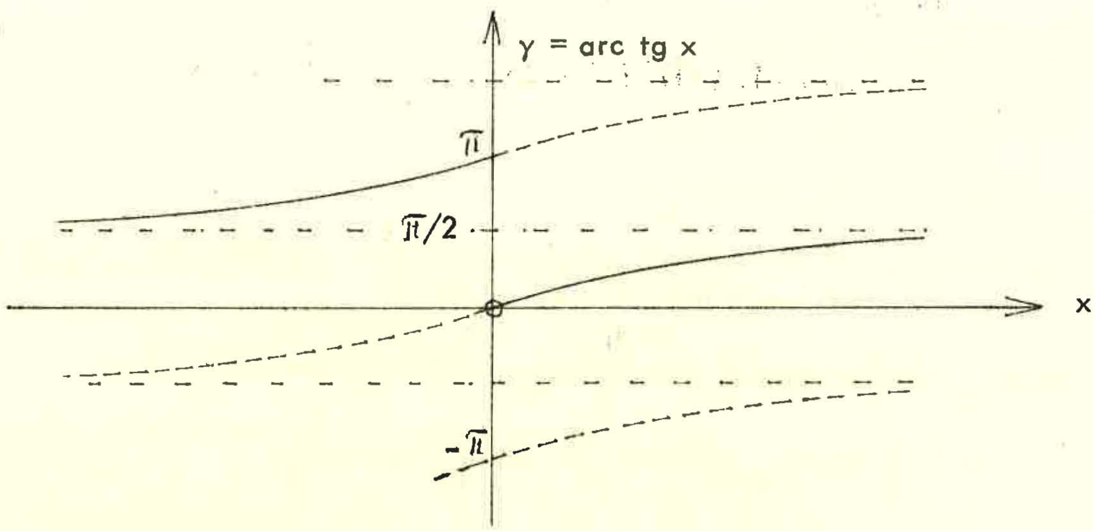

Fig. 2.- Schematic plot of the multiple-valued function arc tg x. The line passing through the origin gives the principal value; our choice is the solid line.

With the new variables r_k and γ_k we get for the powers in (14), by using the expansion

$$(a_k + ib_k)^n = \sum_{j=0}^{n} \binom{n}{j} a_k^{n-j} \cdot (\pm i\, b_k)^j ,$$

the expressions

$$S_+ = 2\, r_k^n \sum_{j=0}^{i_+} (-1)^j \binom{n}{2j} (\cos \gamma_k)^{n-2j} \cdot (\sin \gamma_k)^{2j}$$

and

$$S_- = 2i\, r_k^n \sum_{j=0}^{i_-} (-1)^j \binom{n}{2j+1} (\cos \gamma_k)^{n-2j-1} \cdot (\sin \gamma_k)^{2j+1} ,$$

with

$$S_{\pm} \equiv (a_k + i\, b_k)^n \pm (a_k - i\, b_k)^n$$

and

$$i_+ = \left[\left[\frac{n+1}{2}\right]\right], \qquad i_- = \left[\left[\frac{n}{2}\right]\right] .$$

We note that formally both sums may be extended to infinity ($i_+ = i_- = \infty$) since $\binom{n}{s} = 0$ for $s > n$.

By comparing S_+ and S_- with the multiple-angle formulae for trigonometric functions with multiple arguments to powers of trigonometric functions, they turn out to be simply

$$S_+ = 2\, r_k^n \cdot \cos(n\,\gamma_k) \;,$$
$$S_- = 2i\, r_k^n \cdot \sin(n\,\gamma_k) \;. \qquad (20$$

The contribution of a pair of complex-conjugate solutions $(x_k \,,\, x_{k'})$, previously expressed by (14), may therefore be stated as

$$\cos\varphi_{kJ} \cdot 2\, r_k^n \cdot \cos n\,\gamma_k + i \cdot \sin\varphi_{kJ} \cdot (-2i)\, r_k^n \cdot \sin n\,\gamma_k$$
$$= 2\, r_k^n\, (\cos\varphi_{kJ} \cdot \cos n\,\gamma_k + \sin\varphi_{kJ} \cdot \sin n\,\gamma_k)$$
$$= 2\, r_k^n \cdot \cos(\varphi_{kJ} - n\,\gamma_k) \;. \qquad (21)$$

This shows that the sum (6) can finally be written in the simpler form

$$W(J\,|K) = \frac{1}{K}\left\{1 + (K - 2\varkappa - 1)\,(-1)^J \cdot (1 - 2p)^n + 2 \sum_{k=1}^{\varkappa} r_k^n \cdot \cos(n\,\gamma_k - \varphi_{k\cdot J})\right\} , \quad (22a)$$

where

$$\varphi_i = i \cdot 2\pi/K \;,$$

r_k and γ_k are given by (16) and (1$\overset{9}{\cancel{8}}$), respectively, and

$\varkappa = [\,[K/2]\,]$ is the largest integer below $K/2$,

Therefore

$$K - 2\varkappa - 1 = \begin{cases} 1 & \text{for } K \text{ even} \\ 0 & \text{" } K \text{ odd.} \end{cases}$$

Since γ_k appears in (22a) as the argument of a cosine, it is also possible to resolve the ambiguity of its value in a way which may be slightly more practical than the procedure indicated by (19). As a matter of fact, (22a) can be written likewise in the form

$$W(J\,|K) = \frac{1}{K}\left\{1 + (K - 2\varkappa - 1)\,(-1)^J \cdot (1 - 2p)^n + 2 \sum_{k=1}^{\varkappa} (r_k s_k)^n \cdot \cos(n\,g_k - \varphi_{k\cdot J})\right\} , \quad (22b)$$

where g_k is the principal value (18) and

$$s_k = \operatorname{sgn} g_k \equiv \begin{cases} 1 & \text{for } g_k > 0 \\ -1 & \text{" } g_k < 0 \;. \end{cases}$$

Either of the formulae (22) is the general expression for the modulo-sum of a binomial distribution (with given parameters n and p). They look much like the corresponding formula (Eq. (13) in [2]) for the Poisson process and their exact relation will be studied in section 5.

4. Some explicit formulae

To illustrate the usefulness of the general formula (22), the expressions for the lowest values K will be briefly discussed in what follows.

For the trivial case K = 1, Eq. (5) has only the solution $x_o = 1$ and therefore

$$W(0|1) = 1 \,, \tag{23}$$

confirming the normalization of the binomial probabilities (2).

The case K = 2 deserves much more interest. Since here $x_o = 1$ and $x_1 = -1$, equation (22) gives

$$W(J|2) = \frac{1}{2}\sum_{k=0}^{1} x_k^{-J}\left[p(x_k - 1) + 1\right]^n \,,$$

$$\left.\begin{aligned} \text{thus}\quad W(0|2) &= \frac{1}{2}\left[1 + (1 - 2p)^n\right] \quad \text{and} \\ W(1|2) &= \frac{1}{2}\left[1 - (1 - 2p)^n\right] \,. \end{aligned}\right. \tag{24}$$

In particular, this yields therefore for p = 1/2

$$W(0|2) = W(1|2) = 1/2 \,, \qquad \text{for any } n \,. \tag{25}$$

We note that the result (25) could also be used as the basis of a new statistical test which should be rather similar to the well-known sign-test (see for example [3]): the "null hypothesis" p = 1/2 will be checked by comparing the number of times the outcome of a repeated binomial measurement is even or odd, the permitted deviations of their ratio from unity being deduced from the binomial law with p = 0.5 for a given value of n.

For K = 3 we have $\varkappa = 1$ and $\varphi_1 = 2\pi/3$. Hence

$$r_1^2 = 1 - 2p(1-p)\,(1-\cos\varphi_1) = 1 - 3p(1-p)$$

$$\text{and}\quad \mathrm{tg}\,\gamma_1 = \cancel{\mathrm{tg}}\left[\frac{p\cdot\sin\varphi_1}{1 - p(1-\cos\varphi_1)}\right] = \cancel{\mathrm{tg}}\left(\frac{p\sqrt{3}}{2 - 3p}\right) .$$

We therefore obtain

$$W(J|3) = \frac{1}{3}\left\{1 + 2\left[1-3p(1-p)\right]^{n/2} \cdot \cos(n\,\gamma_1 - J \cdot 2\pi/3)\right\} , \qquad (26)$$

where $\gamma_1 = \text{Arc tg}\,(\frac{p\sqrt{3}}{2-3p}) + U(3p-2)\cdot\pi$.

The case $\underline{K = 4}$ is also readily solved, since $\varkappa = 1$ and $\varphi_1 = \pi/2$ yield

$$r_1^2 = 1 - 2p(1-p) \quad \text{and}$$

$$\text{tg}\,\gamma_1 = \frac{p \cdot \sin\varphi_1}{1 - p(1-\cos\varphi_1)} = \frac{p}{1-p} ,$$

hence

$$W(J|4) = \frac{1}{4}\left\{1 + (-1)^J\,(1-2p)^n + 2\,r_1^n \cdot \cos(n\,\gamma_1 - J\cdot\pi/2)\right\} , \qquad (27)$$

where $\gamma_1 = \text{Arc tg}\,(\frac{p}{1-p})$.

If written more explicitly, this corresponds to

$$W(0|4) = \frac{1}{4}\left\{1 + (1-2p)^n + 2\left[1-2p(1-p)\right]^{n/2} \cdot \cos(n\,\gamma_1)\right\} ,$$

$$W(1|4) = \frac{1}{4}\left\{1 - (1-2p)^n + 2\left[1-2p(1-p)\right]^{n/2} \cdot \sin(n\,\gamma_1)\right\} ,$$

$$W(2|4) = \frac{1}{4}\left\{1 + (1-2p)^n - 2\left[1-2p(1-p)\right]^{n/2} \cdot \cos(n\,\gamma_1)\right\} \quad \text{and} \qquad (27')$$

$$W(3|4) = \frac{1}{4}\left\{1 - (1-2p)^n - 2\left[1-2p(1-p)\right]^{n/2} \cdot \sin(n\,\gamma_1)\right\} .$$

The expressions of $W(J|K)$ for $1 \leqslant K \leqslant 8$ are summarized in Table 1.

The probabilities $W(J|K)$ reach a limiting value in a similar way as they did in the case of a Poisson process. Provided that $n \gg K$ and $p \cdot q \neq 0$, we obtain

$$\lim_{n\to\infty} W(J|K) = 1/K , \text{ for any } J , \qquad (28)$$

whereas

$$\lim_{p\to 0} W(J|K) = \delta_{0,J} , \text{ for any } n \text{ and } K. \qquad (29)$$

Table 1. Formulae for the binomial modulo-sums $W(J|K)$, with $0 \leqslant J < K \leqslant 8$

K	$K \cdot W(J\|K)$	with
1	1	
2	$1 + (-1)^J \cdot (1-2p)^n$	
3	$1 + 2(r\,s)^n \cdot \cos(ng - J \cdot 2\pi/3)$	$\begin{cases} r^2 = 1 - 3p(1-p), \\ g = \text{Arc tg}\,(\frac{p\sqrt{3}}{2-3p}), \\ s = \text{sgn}\,(2-3p). \end{cases}$
4	$1 + (-1)^J \cdot (1-2p)^n + 2\,r^n \cdot \cos(ng - J \cdot \pi/2)$	$\begin{cases} r^2 = 1 - 2p(1-p), \\ g = \text{Arc tg}\,(\frac{p}{1-p}). \end{cases}$
5	$1 + 2\sum_{k=1}^{2} (r_k\,s_k)^n \cdot \cos(ng_k - Jk \cdot 2\pi/5)$	$\begin{cases} r_k^2 = 1 - 2p(1-p)\,[1 - \cos(k \cdot 2\pi/5)], \\ g_k = \text{Arc tg}\left\{\frac{p \cdot \sin(k \cdot 2\pi/5)}{1 - p\,[1-\cos(k \cdot 2\pi/5)]}\right\}, \\ s_1 = 1, \quad s_2 = \text{sgn}\left\{1 - p\,[1+\cos(\pi/5)]\right\}. \end{cases}$

Table 1 (cont'd)

K	$K \cdot W(J \mid K)$	with
6	$1 + (-1)^J \cdot (1-2p)^n + 2 \sum_{k=1}^{2} (r_k s_k)^n \cdot \cos(n g_k - Jk \cdot \pi/3)$	$\begin{cases} r_k^2 = 1 - (2k-1)p(1-p) \,, \\ g_k = \mathrm{Arc\ tg} \left[\dfrac{p\sqrt{3}}{2 - (2k-1)p} \right] \,, \\ s_1 = 1, \quad s_2 = \mathrm{sgn}\,(2-3p) \,. \end{cases}$
7	$1 + 2 \sum_{k=1}^{3} (r_k s_k)^n \cdot \cos(n g_k - Jk \cdot 2\pi/7)$	$\begin{cases} r_k^2 = 1 - 2p(1-p) \left[1 - \cos(k \cdot 2\pi/7) \right] \\ g_k = \mathrm{Arc\ tg} \left\{ \dfrac{p \cdot \sin(k \cdot 2\pi/7)}{1 - p\left[1 - \cos(k \cdot 2\pi/7)\right]} \right\}, \\ s_1 = 1, \quad s_{2,3} = \mathrm{sgn} \left\{ 1 - p \left[1 - \cos(k \cdot 2\pi/7) \right] \right\} . \end{cases}$
8	$1 + (-1)^J \cdot (1-2p)^n + 2 \sum_{k=1}^{3} (r_k s_k)^n \cdot \cos(n g_k - Jk \cdot \pi/4)$	$\begin{cases} r_k^2 = 1 - 2p(1-p) \left[1 + (k-2)/\sqrt{2} \right] , \\ g_2 = \mathrm{Arc\ tg} \left(\dfrac{p}{1-p} \right) , \\ g_{1,3} = \mathrm{Arc\ tg} \left[\dfrac{p}{\sqrt{2} - p(k-2+\sqrt{2})} \right] , \\ s_1 = s_2 = 1 \,, \\ s_3 = \mathrm{sgn} \left[1 - p(1 + 1/\sqrt{2}) \right] . \end{cases}$

However, the limit $1/K$ may also be reached for a finite value n. Thus with an odd K, for example, (22) shows that this limiting value is always obtained if

$$n \cdot g_k - \varphi_{kJ} = (2c_k+1)\frac{\pi}{2}, \quad \text{for all} \quad 1 \leq k \leq \varkappa ,$$

where c_k are integers. This condition can obviously be fulfilled only by certain values of p, if at all.

A computer program (in Fortran IV), based on (22b), has been written, which gives $W(J|K)$ for any combination of $J < K$, p and n; it is available upon request.

5. The Poisson limit

It is a well-known fact that the Poisson law can be obtained as a limiting case from the binomial distribution when $p \to 0$ and $n \to \infty$, while the expectation

$$\mu = p \cdot n \quad \text{remains finite.} \tag{30}$$

It should be possible, therefore, to obtain from the general formula (22) for the binomial the corresponding expression for a Poisson process. As this result has been derived independently before, it would serve as a useful check on the present and on the previous calculations.

The corresponding three limiting processes in (22) are easy to perform. They are

a) $$\lim_{n\to\infty} (1-2p)^n = \lim_{n\to\infty} \left(1 - \frac{2\mu}{n}\right)^n = e^{-2\mu}, \tag{31a}$$

b) $$\lim_{n\to\infty} r_k^{2n} = \lim_{n\to\infty} \left[1 - \frac{2\mu}{n}(1-\cos\varphi_k)\right]^n = e^{-2\mu(1-\cos\varphi_k)}, \tag{31b}$$

and

c) $$\lim_{n\to\infty} n\,\gamma_k = \lim_{n\to\infty} \left\{ n \cdot \operatorname{Arc\,tg} \left[\frac{\frac{\mu}{n}\cdot\sin\varphi_k}{1 - \frac{\mu}{n}(1-\cos\varphi_k)} \right] \right\}$$

$$= \lim_{n\to\infty} \left(n \cdot \frac{\mu\cdot\sin\varphi_k}{n\cdot 1} \right) = \mu\cdot\sin\varphi_k . \tag{31c}$$

If these results are inserted into (22), we obtain

$$W(J|K) = \frac{1}{K}\Big\{ 1 + (K-2\varkappa-1)(-1)^J \cdot e^{-2\mu}$$

$$+ 2\sum_{k=1}^{\varkappa} e^{-\mu(1-\cos\varphi_k)} \cdot \cos(\mu\cdot\sin\varphi_k - \varphi_{kJ}) \Big\}, \tag{32}$$

which is identical with equation (13) in [2] for the Poisson process.

APPENDIX

Geometric modulo-sums

The determination of modulo-sums for the geometric probability law is particularly simple and they can be easily found directly, i.e. without using the general method sketched in section 2.

The geometric probability distribution is given by

$$G(j) = p \cdot q^{j-1}, \qquad \text{with} \quad j = 1, 2, 3, \ldots, \tag{A1}$$

where $0 < p < 1$ and $q \equiv 1 - p$.

$G(j)$ can be interpreted as the probability that the first "success" arrives at trial number j, supposing that p, the probability of "success", is the same for each trial (lack of memory). In determining the modulo-sums which are defined by

$$W(J|K) = \text{Prob}\left\{j = J \ (\text{mod } K)\right\},$$

we have to be careful to exclude the case $j=0$ (as it does not make sense here)*). This can be achieved by now choosing J in the range $1 \leqslant J \leqslant K$, and we obtain

$$\begin{aligned} W(J|K) &= \sum_{s=0}^{\infty} G(j = J + sK) \\ &= \sum_{s} p \cdot q^{J+sK-1} = p \cdot q^{J-1} \sum_{s=0}^{\infty} q^{sK} \\ &= p \cdot q^{J-1} \cdot \frac{1}{1-q^K} = \frac{p\,(1-p)^{J-1}}{1-(1-p)^K}. \end{aligned} \tag{A2}$$

It is readily verified that

$$\sum_{J=1}^{K} W(J|K) = 1, \tag{A3}$$

as one would expect.

The corresponding calculations for the remaining discrete probability distributions seem to be somewhat more involved, but no real attempt has yet been made nor is it planned.

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*) Formally, this annoyance could have been avoided by choosing instead of j a variable $j' = j - 1$, which corresponds to the number of "failures" before the first "success".

References

[1] J.W. Müller: "A new method for distinguishing between pairs and single pulses", Report BIPM-72/14 (December 1972)

[2] id.: "A complex modulo K counter", Report BIPM-73/5 (May 1973)

[3] B.L. van der Waerden: "Mathematical Statistics" (Springer, Berlin, 1969).

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