On the limiting behaviour of the interval density for an extended dead time

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1. Introduction

An essential part of a recent semi-empirical method for determining the correction to the number of counts due to a non-ideal dead time [1] is based on the assumption that the interval density f(t) goes over into an exponential function for sufficiently large time intervals t. In order to render such a supposition more plausible, it seems desirable to show that this result holds for the two extreme cases of a dead time, namely the extended or the non-extended type. We always assume that the dead time has been inserted into a process of originally Poissonian nature with count rate ρ .

The required property is only trivial for a non-extended dead time, since here

$$f(t) = \rho \exp\left[-\rho(t-\tau)\right] = \rho \exp(\rho\tau) \cdot e^{-\rho t} \qquad (1)$$

is exactly an exponential (with the original count rate ρ) for any $t > \tau$. For extended dead times, however, things are much less evident and in a recent attempt to tackle this problem [2], none of our three different approaches led to really convincing and unambiguous results. But in the meantime - we hope at least - the arguments have been somewhat improved and considerably simplified so that it might be worthwhile to submit them to those who will be interested in such matters. Since our solutions are still far from being as clear as definite answers should be, further improvements or entirely new ways are certainly possible and would be welcome.

Two different approaches will be discussed in what follows. The first is based on the original interval density f(t), while the second starts with its integral transform $\tilde{f}(s)$. Different features are used for arriving at the results which supplement each other.

2. Proof for the original density

The general expression for the interval density with an extended dead time τ , when the rate of the original Poisson process was ρ , is given according to equation (21) of [3] by

$$f(t) = \sum_{k=1}^{K} A_k(t,\tau) \quad , \qquad (2)$$

where

$$A_k(t,\tau) = U(t - k\,\vartheta)\cdot\frac{(-1)^{k-1}}{(k-1)!}\cdot\rho^k\cdot(t-k\tau)^{k-1}\cdot e^{-k\rho\tau},$$

$K \equiv \left[\!\left[t/\tau\right]\!\right]$ is the largest integer below t/τ

and U is the unit step function.

Derivation with respect to t yields ($\rho\tau \equiv x$)

$$\frac{dA_k}{dt} = U(t-k\tau)\cdot\frac{(-1)^{k+1}}{(k-2)!}\cdot e^{-kx}\cdot\rho^2\cdot(\rho t - kx)^{k-2} \quad . \tag{3}$$

In order to be able to compare (3) with

$$A_{k-1}(t,\tau) = -U(t-[k-1]\,\tau)\cdot\frac{e^x}{\rho}\cdot\frac{(-1)^{k+1}}{(k-2)!}\cdot e^{-kx}\cdot\rho^2\cdot(\rho t-[k-1]\,x)^{k-2},$$

we introduce a fictitious dead time

$$\tau' \equiv \frac{k-1}{k}\cdot\tau \equiv \tau/\gamma \tag{4}$$

and put accordingly $x' = \rho\tau'$, thus

$$x = \frac{k}{k-1}\cdot x' = \gamma\cdot x' \quad . \tag{4'}$$

This then leads to

$$A_{k-1}(t,\tau) = -U(t-k\cdot\tau/\gamma)\cdot\frac{e^{\gamma x'}}{\rho}\cdot\frac{(-1)^{k+1}}{(k-2)!}\cdot e^{-k\gamma x'}\cdot\rho^2\,(t-kx')^{k-2}$$

$$= -U(t-k\tau')\cdot\frac{e^{\gamma x'-k\gamma x'+\cancel{kx'}}}{\rho}\cdot\frac{(-1)^{k+1}}{(k-2)!}\cdot\rho^2\cdot(\rho t-kx')^{k-2}\cdot e^{-kx'} .$$

But since the exponent

$$\gamma x' - kx'(\gamma-1) = \frac{k}{k-1}x' - kx'\cdot\left(\frac{k}{k-1}-1\right) = \frac{k}{k-1}x' - kx'\cdot\frac{1}{k-1} = 0$$

vanishes, we simply get

$$A_{k-1}(t,\tau) = -\frac{1}{\rho}\cdot\frac{d}{dt}A_k(t,\tau')\cdot\frac{U(t-k\tau')}{U(t-k\tau)} \quad . \tag{5}$$

However, asymptotically $k \gg 1$ for $t \longrightarrow \infty$, thus $\gamma = 1$ and $\tau' = \tau$, and therefore also in the same limit

$$A_{k-1}(t,\tau) = -\frac{1}{\rho}\cdot\frac{d}{dt}A_k(t,\tau) . \tag{6}$$

By summing over k we get

$$\sum A_{k-1}(t,\tau) = f(t) \quad \text{and} \tag{7}$$

$$\frac{d}{dt} A_k(t,\tau) = \frac{d}{dt} f(t) .$$

We thus obtain asymptotically the differential equation

$$\frac{df(t)}{dt} = -\rho \cdot f(t) , \tag{8}$$

the solution of which is

$$\lim_{t \to \infty} f(t) = A_o \cdot e^{-\rho t} \tag{9}$$

This gives the expected limiting exponential behaviour where A_o is an undetermined constant.

3. Proof by help of the transform

We still want to show that in the limit

$$\lim_{t \to \infty} f(t) = A_o \cdot e^{-\rho t} ,$$

where f(t) is given by (1). For that purpose we form the new function

$$g(t) \equiv e^{\rho' \cdot t} \cdot f(t) \tag{10}$$

and shall show that a finite, non-vanishing limiting value

$$\lim_{t \to \infty} g(t) = A_o$$

can only result provided that $\rho' = \rho$.

Application of the well-known shift rule for Laplace transforms [4] gives

$$\tilde{g}(s) \equiv \mathcal{L}\left\{e^{\rho' t} \cdot f(t)\right\} = \tilde{f}(s - \rho') . \tag{11}$$

By inserting the known form of the transform of the interval density (1), which was given in equation (11) of [3] as

$$\tilde{f}(s) = \frac{\rho}{\rho + s \cdot e^{(s+\rho)\tau}} , \tag{12}$$

we obtain according to (11)

$$\tilde{g}(s) = \frac{\rho}{\rho + (s - \rho') \cdot e^{(s-\rho'+\rho)}} .$$

By taking advantage of the Tauber theorem [4] in the form :

$$\lim_{t \to \infty} g(t) = \lim_{s \to \infty} s \cdot \tilde{g}(s) \quad , \tag{13}$$

we get for the limit

$$\lim_{t \to \infty} g(t) = \lim_{s \to 0} \frac{s\rho}{\rho + (s - \rho') \cdot e^{(\rho - \rho')\tau} \cdot e^{s\tau}}$$

$$= \lim_{s \to 0} \frac{\rho s}{\left[\rho - \rho' \cdot e^{(\rho - \rho')\tau} \cdot e^{s\tau}\right] + s \cdot e^{(\rho - \rho')\tau} e^{s\tau}} \, .$$

But for $s \to 0$ this expression can only lead to a finite value A_o if a factor s can be split off from the square bracket, i.e. for

$$\rho - \rho' \cdot e^{(\rho - \rho')\tau} \cdot (1 + s\tau + \frac{1}{2} s^2 \tau^2 + \ldots) = s \cdot B \, . \tag{14}$$

This, in turn, requires that for any dead time τ

$$\rho = \rho' \cdot e^{(\rho - \rho')\tau} \quad ,$$

thus $\rho' = \rho$. (15)

Then, it follows that

$$A_o = \lim_{t \to \infty} g(t) = \lim_{s \to 0} \frac{\rho \cdot s}{\rho - \rho e^{s\tau} + s e^{s\tau}} = \lim_{s \to 0} \frac{\rho \cdot s}{\rho(-s\tau) + s e^{s\tau}} = \frac{\rho}{1 - \rho\tau} \, . \tag{16}$$

Accordingly, we conclude that an extrapolation of the asymptotic exponential distribution (or a linear extrapolation in the logarithmic plot) leads to an intersection with the ordinate at the origin $t = 0$ at the value

$$A_o = \frac{\rho}{1 - \rho\tau} \, .$$

4. Experimental checks

By measuring the distribution of the time intervals following an extended dead time, we can check the limiting behaviour of the density $f(t)$.

As an example such a measurement is shown in Fig. 1. For a comparison with the theoretical curve, ratios of ordinates have to be taken since our experimental data are not normalized. According to equation (2), the density $f(t)$ has in the range $\tau < t < 2\tau$ the constant value

$$A_1 = \rho \cdot e^{-\rho\tau} .$$

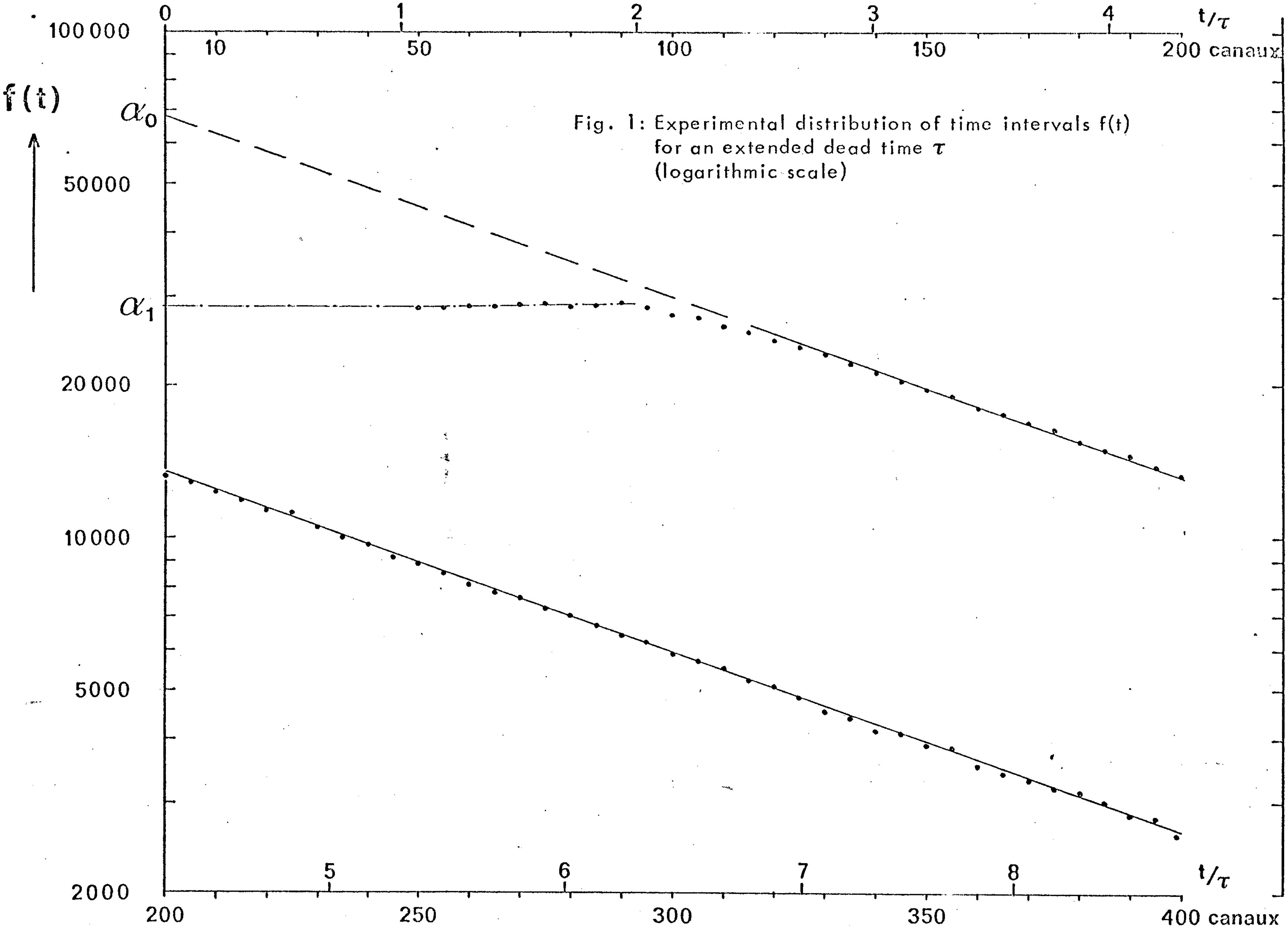

Fig. 1: Experimental distribution of time intervals f(t) for an extended dead time τ (logarithmic scale)

The empirical interval distribution in Fig. 1 was obtained in 16 hours on a kicksorter. The relevant parameters for this run were

$$\left.\begin{aligned} \rho &= (412 \pm 1)\ \mathrm{s}^{-1} \\ \tau &= (930 \pm 3)\ \mu\mathrm{s} \end{aligned}\right\} \rho\tau \cong 38\%$$

time per channel: $(20.00 + 0.05)\ \mu\mathrm{s}$.

This gives for the theoretical values

$$A_o = \frac{\rho}{1 - \rho\tau} = (667.9 \pm 1.7)\ \mathrm{s}^{-1}$$

and

$$A_1 = \rho \cdot e^{-\rho\tau} = (280.8 \pm 0.7)\ \mathrm{s}^{-1} ,$$

which have to be compared with the experimental quantities (counts per channel)

$$\alpha_o = 68\ 000 \pm 500 ,$$

$$\alpha_1 = 28\ 000 \pm 50 .$$

This can be easily performed by means of the ratios

$$\alpha_o / A_o = \frac{68\ 000}{667.9} = 101.8 \pm 0.8 \quad \text{and}$$

$$\alpha_1 / A_1 = \frac{28\ 800}{280.8} = 102.6 \pm 0.4 .$$

The closeness of these two ratios demonstrates that the value for the constant A_o , as expected according to formula (16), is well confirmed by the experimental data.

Furthermore, the empirical value of ρ' , i.e. the limiting slope of the experimental distribution, drawn on a logarithmic scale, has been determined in order to check if the density really tends towards an exponential with the true count rate ρ , as has to be expected on the basis of (9).

The data of a similar experiment as illustrated in Fig. 1, but with better statistics (within 64 h), with the parameters

$$\left.\begin{aligned} \rho &= (411 \pm 1)\ \mathrm{s}^{-1} \\ \tau &= (400 \pm 2)\ \mu\mathrm{s} \end{aligned}\right\} \rho\tau \cong 16\%$$

were analyzed by two different methods, namely

a) least-squares fit of an exponential over 25 channels each time,

b) differential determination of the derivative after smoothing of the original data, a procedure similar to the one used earlier in a different context [5].

The numerical results are

- method a): $\rho' = (409.8 \pm 0.8)\ s^{-1}$,

- method b): $\rho' = (411.5 \pm 0.5)\ s^{-1}$.

The agreement with ρ is satisfactory. We can therefore conclude that a dead time is not capable of changing the exponential interval distribution at distances which are large compared with the dead time, as one would have expected naively.

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References

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