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Some notes on pair statistics

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1. Introduction

Let us assume that the parent and the daughter pulses stemming from a two-step nuclear decay cannot be readily distinguished, as for example is usually the case for beta particles and highly converted gamma rays. We then say that a "pair" is counted if both the parent and the daughter pulse from a specific decay have been registered within a certain measuring time T. All other pulses are denoted "singles". We assume that the intermediate state has a mean lifetime λ^{-1} and that the detection efficiencies of the counters are ε_p and ε_f for the parent and daughter pulses, respectively. Dead-time effects will not be taken into account. Our interest is focussed on the statistical behaviour of the total process which consists of the random superposition of parent and daughter pulses.

It should be emphasized that this report does not claim to state anything really new. It is rather an attempt to derive fully some basic results from a somewhat naive point of view which, however, will prove useful for some developments to be described later. For an earlier different approach compare [1].

Most results are derived by completely elementary methods as well as by using integral transforms: the second way is usually much shorter and therefore probably easier to overlook.

In order to arrive at the probability distribution for the number k of registered pulses in T, a subdivision into pairs and single pulses is practical, as suggested by their different behaviour in time. We therefore write

$$k = n_1 + 2\, n_2 , \tag{1}$$

where n_1 = number of observed single pulses
and n_2 = " " " pairs,
both for a given time interval T.

n_1 and n_2 are random quantities, but the condition (1) restricts their possible values for a given k, as can be seen from the following simple examples.

k	n_1	n_2
0	0	0
1	1	0
2	0	1
	2	0
3	1	1
	3	0
4	0	2
	2	1
	4	0

k	n_1	n_2
5	1	2
	3	1
	5	0
6	0	3
	2	2
	4	1
	6	0
7	0	4
	2	3
	4	2
	6	1
	8	0
	...	...

Table 1: Some possible numbers for singles and pairs when the total number k is fixed

The general rule is readily shown to be

$$n_1 = \begin{cases} 0,\ 2,\ \ldots,\ 2\left[\!\left[k/2\right]\!\right] & \text{for } k \text{ even} \\ 1,\ 3,\ \ldots,\ k & \text{" } k \text{ odd} \end{cases}$$

and $n_2 = 0,\ 1,\ 2,\ \ldots,\ \left[\!\left[k/2\right]\!\right]$,

where $\left[\!\left[a\right]\!\right]$ denotes the largest integer below a.

For the probability of finding exactly k events in T, this yields the basic equation

$$W(k) = \sum_{j=0}^{K} \underline{P}_1(k-2j) \cdot \underline{P}_2(j) \ , \tag{2}$$

with $K \equiv \left[\!\left[\frac{k+1}{2}\right]\!\right] = \begin{cases} \frac{k}{2} & \text{for } k \text{ even} \\ \frac{k-1}{2} & \text{" } k \text{ odd.} \end{cases}$

The problem of finding W(k) is thus reduced to the determination of the probabilities $\underline{P}_1$ and $\underline{P}_2$ for the number of singles and pairs.

2. Probabilities for singles and pairs

In order to arrive at a useful explicit form of (2), the distributions for the single and paired events have first to be determined.

Our aim is to show that for an original sequence of parent decays forming a Poisson process, the distributions $\underline{P}_1$ and $\underline{P}_2$ are still Poissonian.

Let us first consider the effect of the finite efficiencies. For a sufficiently long measuring interval $(T \gg \lambda^{-1})$ the survival probabilities π are

- for a pair $\qquad \pi_2 = \varepsilon_p \cdot \varepsilon_d \,, \qquad (3)$

whereas

- for a single parent $\qquad \pi_p = \varepsilon_p \cdot (1 - \varepsilon_d) \qquad$ and
- " " " daughter $\qquad \pi_d = \varepsilon_d \cdot (1 - \varepsilon_p) \,,$

thus

$$\pi_1 = \pi_p + \pi_d$$
$$= \varepsilon_p + \varepsilon_d - 2\,\varepsilon_p \cdot \varepsilon_d \,. \qquad (4)$$

However, an original Poisson process with expectation μ, from which events are eliminated at random and independently, results in another Poisson process with new expectation $\pi\mu$, where π is the survival probability. A simple and direct way to see this may run as follows.

In order to arrive at k remaining pulses, we have to start from $j \geq k$ original events which are supposed to be Poisson distributed. Therefore

$$\underline{P}(k) = \sum_{j=k}^{\infty} p_{\mu}(j) \cdot b_{\pi}(j,k) \,, \qquad (5)$$

where $p_{\mu}(j) = e^{-\mu} \cdot \frac{\mu^j}{j!}$ is a Poisson distribution with expectation μ

and $b_{\pi}(j,k) = \binom{j}{k} \cdot \pi^k \cdot (1-\pi)^{j-k}$ is a binomial distribution $(k = 0, 1, \ldots, j)$.

Inserting into (5) gives

$$\underline{P}(k) = \sum_{j=k}^{\infty} e^{-\mu} \cdot \frac{\mu^j}{j!} \binom{j}{k} \pi^k (1-\pi)^{j-k}$$

$$= e^{-\mu} \cdot \pi^k \sum_{r=0}^{\infty} \frac{\mu^{r+k}}{(r+k)!} \binom{r+k}{r} (1-\pi)^r \,, \qquad r = j-k$$

$$= e^{-\mu} \cdot \frac{(\pi\mu)^k}{k!} \sum_{r=0} \frac{[(1-\pi)\mu]^r}{r!}$$

$$= e^{-\mu} \cdot \frac{(\pi\mu)^k}{k!} \cdot e^{(1-\pi)\mu} = e^{-\pi\mu} \cdot \frac{(\pi\mu)^k}{k!}$$

$$= p_{\pi\mu}(k) \,. \qquad (6)$$

Therefore $\underline{P}(k)$ is again a Poisson distribution, but now with the expectation $\pi\mu$, where $0 \leq \pi \leq 1$.

A more elegant way to derive this result would be to consider the present situation as a special case of a so-called branching process. Then, on the grounds of a general relation [2] , the following equation holds for the respective transforms

$$\underline{\tilde{P}}(s) = \tilde{p}_{\mu}\left[\tilde{b}_{\pi}(1,s)\right] . \tag{7}$$

Here $b_{\pi}(1,k)$ is a Bernoulli variable with

$$b_{\pi}(1,0) = 1 - \pi \quad \text{and} \quad b_{\pi}(1,1) = \pi .$$

Applying e.g. Laplace transforms, we obtain

$$\tilde{b}_{\pi}(1,s) = (1-\pi) + \pi \cdot e^{-s}$$

$$\tilde{p}_{\mu}(s) = \exp\left\{\mu(e^{-s}-1)\right\}$$

and therefore with (7)

$$\underline{\tilde{P}}(s) = \exp\left\{\mu\left[(1-\pi + \pi \cdot e^{-s}) - 1\right]\right\}$$

$$= \exp\left\{\pi\mu(e^{-s}-1)\right\} = \tilde{p}_{\pi\mu}(s) ,$$

which is identical with (6).

3. Distribution of the pairs

For the pairs, the result (6) cannot be applied directly because for any measuring interval of finite length T the probability of finding a real pair depends on its location t, with $0 \leq t \leq T$, which may be defined for instance by the arrival time of the parent pulse. Since "earlier" parents have obviously a better chance of finding their daughter pulse in the same time interval (and forming thereby a pair) than "later" parents, the survival probability for pairs is now a function of t. In deriving (6), however, it was essential to assume that π is a constant.

Therefore, it is probably somewhat surprising that, in spite of all that, the number of pairs in T is still Poisson distributed all the same. This fact, as it seems, was first explicitly stated by Foglio Para et al. [1], but some readers might hesitate to accept their proof of this statement since a closer look at the development used (for their type (3)) reveals that this might have anticipated the solution as it corresponds to the differential form of a Poisson process. We therefore prefer to show by means of a very simple argument more clearly the basis of this result, which holds quite generally.

For this purpose we imagine the counting interval T to be subdivided into N parts, of equal length $\Delta t = T/N$. Let μ be the mean number of events in T. Then for any of the resulting partial intervals $(n-1) \cdot \Delta t \leq t \leq n \cdot \Delta t$ for t (with $n = 1, 2, \ldots, N$), the statistics of the parent pulses is clearly still Poissonian with mean $m = \mu/N = \rho \cdot \Delta t$, say, where ρ is the count rate of the parent pulses. We note that m is independent of the position t. On the other hand, the survival probability π for a daughter pulse, the parent of which has been observed at t, is obviously a function of time which decreases monotonically with t.

The new expected number of pairs, with the parent pulses lying in $t \ldots t+\Delta t$, is now

$$\Delta\mu(t) = m \cdot \pi(t) = \rho \cdot \pi(t) \cdot \Delta t \quad (8)$$

Provided that Δt is small enough (i.e. for $N \gg 1$), $\pi(t)$ is a constant for a given subinterval and the distribution of the corresponding pairs (with parents in Δt) is still Poissonian.

4. Sums of Poisson processes

The total number of pairs is made up by all the contributions from the N subintervals. They all form Poisson processes, but with different means. It is a well-known fact, however, that a sum of independent Poisson processes is again a Poisson process, as can easily be shown, either directly (see later) or, perhaps more conveniently, again by the use of transforms. The multiple convolution

$$\underline{P}_{tot}(k) = p_{\Delta\mu_1}(k) * p_{\Delta\mu_2}(k) * \ldots * p_{\Delta\mu_N}(k)$$

of the originals corresponds to the product

$$\tilde{\underline{P}}_{tot}(s) = \tilde{p}_{\Delta\mu_1}(s) \cdot \tilde{p}_{\Delta\mu_2}(s) \cdot \ldots \cdot \tilde{p}_{\Delta\mu_N}(s)$$

$$= \prod_{j=1}^{N} \exp\left\{\Delta\mu_j(s-1)\right\} = \exp\left\{M(s-1)\right\}$$

$$= \tilde{p}_M(s) \, , \quad (9)$$

with $M = \sum_{j=1}^{N} \Delta\mu_j$.

The superposition thus forms another Poisson process, the expectation M of which is equal to the sum of the expectations $\Delta\mu_j$ of the component processes. In our case this yields for the new mean with (8)

$$M = \lim_{M \to \infty} \sum_{i=0}^{N} \Delta\mu(i \cdot \Delta t) = \rho \cdot \int_0^T \pi(t)\, dt = \overline{\pi} \cdot \mu , \tag{10}$$

with $\overline{\pi} = \frac{1}{T} \int_0^T \pi(t)\, dt$.

This shows that even for the case of a time-dependent survival probability it is still permitted to use (6) with π replaced by $\overline{\pi}$.

In order to check this important result empirically, a Monte Carlo simulation has been performed where the number of pairs in a given time interval T has been counted. The result is given in Table 2 and shows indeed a very satisfactory agreement with a Poisson distribution.

n_2	observed: $h(n_2)$	expected: $p_{\mu_2}(n_2) \cdot 10^6$
0	478 359	479 142
1	353 215	352 533
2	129 770	129 690
3	31 736	31 807
4	5 911	5 851
5	879	861
6	119	106
7	10	11
8	1	1

Table 2: Empirical frequencies $h(n_2)$ for n_2 pairs in T
(with $\varepsilon_p = \varepsilon_d = 1$, $T = 1$, $\rho = 2$ and $\lambda = 1$)

The empirical mean number of pairs, based on this sample of 10^6 intervals, is found to be

$$\overline{n}_2 = 10^{-6} \sum n_2 \cdot h(n_2) = 0.737 \pm 0.001 ,$$

whereas on theoretical grounds one would expect (cf. later eq. 20)

$$\mu_2 = \rho - \frac{\rho}{\lambda} (1 - e^{-\lambda}) = 2/e \cong 0.736. \tag{11}$$

It is not difficult to show in a direct way the reproductive property of a Poisson distribution with respect to the mean. We may restrict ourselves to the case of two components.

Let $\underline{P}(k) = p_{\mu_1}(k) * p_{\mu_2}(k)$,

with $p_{\mu}(k) = e^{-\mu} \cdot \frac{\mu^k}{k!}$.

Written in full this is equivalent to

$$\underline{P}(k) = \sum_{j=0}^{k} p_{\mu_1}(k-j) \cdot p_{\mu_2}(j)$$

$$= e^{-(\mu_1+\mu_2)} \sum_{j=0}^{k} \frac{\mu_1^{k-j}}{(k-j)!} \cdot \frac{\mu_2^j}{j!} .$$

However, since

$$(\mu_1 + \mu_2)^k = \sum_{j=0}^{k} \binom{k}{j} \mu_1^{k-j} \cdot \mu_2^j$$

$$= k! \sum \frac{1}{(k-j)! \cdot j!} \mu_1^{k-j} \cdot \mu_2^j ,$$

we obtain immediately

$$\underline{P}(k) = e^{-(\mu_1+\mu_2)} \cdot \frac{(\mu_1+\mu_2)}{k!} = p_{\mu_3}(k)$$

with $\mu_3 = \mu_1 + \mu_2$,

in agreement with (9).

5. Mean and variance for the pairs

We want to show here how the first two moments can be obtained directly from the distribution. Since this is a somewhat lengthy calculation, the next section will indicate a shortcut to arrive at the same results using integral transforms.

Since we know that the number of pairs and (by similar arguments) of single pulses are Poisson distributed, equation (2) can now be written as

$$W(k) = e^{-(\mu_1+\mu_2)} \sum_{j=0}^{K} \frac{\mu_2^j}{j\,!} \cdot \frac{\mu_1^{k-2j}}{(k-2j)!} \quad , \tag{12}$$

where μ_1 and μ_2 are the expectations for the number of singles and pairs in T, respectively,

and $K = \left[\left[\frac{k+1}{2}\right]\right]$.

Another equivalent form would be

$$W(k) = \frac{e^{-(\mu_1+\mu_2)}}{k\,!} \sum_{j=0}^{K} (2j-1)!!\ (2\mu_2)^j \cdot \mu_1^{k-2j} \quad . \tag{12'}$$

We first determine the first moment

$$m_1(k) = \sum_{k=0}^{\infty} k \cdot W(k) = \sum_{k=0}^{\infty} \sum_{j=0}^{K} k \cdot P_2(j) \cdot P_1(k-2j)$$

$$= \sum_k \sum_j k \cdot e^{-\mu_2} \cdot \frac{\mu_2^j}{j\,!} \cdot e^{-\mu_1} \cdot \frac{\mu_1^{k-2j}}{(k-2j)!} \quad .$$

Since $\frac{1}{(k-2j)\,!} = 0$ for $j > K$, the summation over j can be extended to infinity. Upon reversing the order of the sums we get

$$m_1(k) = e^{-(\mu_1+\mu_2)} \sum_{j=0}^{\infty} \frac{\mu_2^j}{j\,!} \sum_{k=0}^{\infty} \frac{k \cdot \mu_1^{k-2j}}{(k-2j)\,!} \quad .$$

With $s = k-2j$ the second sum is

$$\sum_s (s+2j) \frac{\mu_1^s}{s\,!} = \mu_1 \cdot e^{\mu_1} + 2j \cdot e^{\mu_1} \quad ,$$

hence

$$m_1(k) = e^{-\mu_2} \sum_{j=0}^{\infty} (\mu_1+2j) \frac{\mu_2^j}{j\,!}$$

$$= e^{-\mu_2} (\mu_1 \cdot e^{\mu_2} + 2\mu_2 \cdot e^{\mu_2}) = \mu_1 + 2\mu_2 \quad . \tag{13}$$

We have to keep in mind, however, that the means μ_1 and μ_2 depend on the length T of the measuring interval.

For the second moment (12) gives

$$m_2(k) = e^{-(\mu_1+\mu_2)} \sum_{j=0}^{\infty} \frac{\mu_2^j}{j!} \left\{ \sum_{k=0}^{\infty} k^2 \cdot\cdot \frac{\mu_1^{k-2j}}{(k-2j)!} \right\} .$$

Putting again $k-2j = s$, we have

$$k^2 = s\,(s-1) + s\,(1+4j) + 4\,j^2 \qquad ,$$

and therefore

$$\left\{\cdots\right\} = \mu_1^2 \sum \frac{\mu_1^{s-2}}{(s-2)\,!} + (1+4j) \cdot \mu_1 \sum \frac{\mu_1^{s-1}}{(s-1)\,!} + 4\,j^2 \sum \frac{\mu_1^s}{s\,!}$$

$$= \mu_1^2 \cdot e^{\mu_1} + (1+4j) \cdot \mu_1 \cdot e^{\mu_1} + 4j^2 \cdot e^{\mu_1} \quad ,$$

or

$$m_2(k) = e^{-\mu_2} \sum_j \frac{\mu_2^j}{j\,!} \left[\mu_1^2 + (1+4j)\ \mu_1 + 4j^2 \right] .$$

Since $4j^2 + 4j \cdot \mu_1 = 4 \cdot j(j-1) + 4(1+\mu_1) \cdot j$,

we may also write

$$m_2(k) = e^{-\mu_2} \left\{ (\mu_1^2 + \mu_1) \sum_j \frac{\mu_2^j}{j\,!} + 4\mu_2^2 \sum_j \frac{\mu_2^{j-2}}{(j-2)!} + 4\,(1+\mu_1)\,\mu_2 \sum_j \frac{\mu_2^{j-1}}{(k-1)!} \right\}$$

$$= e^{-\mu_2} \left[(\mu_1^2 + \mu_1)\, e^{\mu_2} + 4\mu_2^2 \cdot e^{\mu_2} + 4\,\mu_2 (1+\mu_1)\, e^{\mu_2} \right]$$

$$= \mu_1^2 + \mu_1 + 4\mu_2^2 + 4\,\mu_2 + 4\,\mu_1\mu_2$$

$$= (\mu_1 + 2\mu_2)^2 + \mu_1 + 4\mu_2 \quad . \tag{14}$$

This then yields for the variance of k

$$\sigma^2(k) \equiv m_2(k) - m_1^2(k) = \mu_1 + 4\mu_2 \quad . \tag{15}$$

Actually, the results (13) and (15) for the mean and the variance just reflect the simple fact that the total process can be thought of as the superposition of two independent Poisson processes, namely for the singles and for the pairs.

Since for the superimposed process, according to (14),

$$\sigma^2(k) = m_1(k) + 2\,\mu_2 \quad ,$$

equality of mean and variance is only possible if pairs are absent. In that case we have obviously again a simple Poisson process. The experimental value of the variance can therefore be used for estimating μ_2 .

6. Transform for the pair distribution

Moments of a variable are always simple to determine once the transform of the corresponding probability distribution is known. This is what we now try to obtain for (2) which can be written symbolically in the form of the convolution

$$W(k) = \underline{P}_1(k) * \underline{P}_2(2k) \quad . \tag{2'}$$

The transform for the singles is straightforward. Using Laplace transforms we obtain

$$\tilde{\underline{P}}_1(s) \equiv \mathcal{L}\left\{\underline{P}_1(k),\ s\right\} = E\left\{e^{sk}\right\} = \sum_{k=0}^{\infty} \underline{P}_1(k) \cdot e^{-sk} \quad .$$

For the second factor which describes the pairs we put

$$\underline{P}_2(2k) = \underline{Q}_2(k) \quad ,$$

where $\underline{Q}_2(k)$ is now the probability distribution for observing exactly k pairs (i.e. $2k$ pulses) in T. Its transform is

$$\mathcal{L}\left\{\underline{P}_2(2k),\ s\right\} = \mathcal{L}\left\{\underline{Q}_2(k),\ s\right\} = \sum_{k=0}^{\infty} \underline{Q}_2(k) \cdot e^{-s\cdot 2k}$$

$$= \mathcal{L}\left\{\underline{Q}_2(k),\ 2s\right\} = \tilde{\underline{Q}}_2(2s) \quad . \tag{16}$$

As singles and pairs are described by Poisson distributions with respective means μ_1 and μ_2 , we obtain for the transformed total distribution

$$\tilde{W}(s) = \mathcal{L}\left\{\underline{P}_1(k),\ s\right\} \cdot \mathcal{L}\left\{\underline{P}_2(2k),\ s\right\} = \tilde{\underline{P}}_1(s) \cdot \tilde{\underline{Q}}_2(2s)$$

$$= e^{\mu_1(e^{-s}-1)} \cdot e^{\mu_2(e^{-2s}-1)} = e^{-(\mu_1+\mu_2)} \cdot \exp\left\{\mu_1 \cdot e^{-s} + \mu_2 \cdot e^{-2s}\right\} \quad . \tag{17}$$

The moments of k (of order r) are obtained by differentiation according to

$$m_r(k) = (-1)^r \cdot \left.\frac{d^r\, \tilde{W}(s)}{ds^r}\right|_{s=0} \quad . \tag{18}$$

A simple calculation leads to

$$\frac{d\tilde{W}}{ds} = - \exp(-\mu_1-\mu_2) \cdot \exp(\mu_1 \cdot e^{-s} + \mu_2 \cdot e^{-2s}) \cdot (\mu_1 \cdot e^{-s} + 2\mu_2 \cdot e^{-2s}) ,$$

$$\frac{d^2\tilde{W}}{ds^2} = \exp(-\mu_1-\mu_2) \cdot \exp(\mu_1 \cdot e^{-s} + \mu_2 \cdot e^{-2s}) \left\{(\mu_1 \cdot e^{-s} + 2\mu_2 \cdot e^{-2s})^2 + \mu_1 \cdot e^{-s} + \mu_2 \cdot e^{-2s}\right\} .$$

Hence (17) gives now

$$m_1(k) = \mu_1 + 2\mu_2 \qquad \text{and}$$

$$m_2(k) = (\mu_1 + 2\mu_2)^2 + \mu_1 + 4\mu_2 ,$$

in agreement with (13) and (14).

7. The mean rates

The general structure of the superimposed process is now reasonably clear, but we still have to determine the expected mean rates μ_1 and μ_2 for the singles and the pairs, respectively.

For this purpose let us consider the survival probability of a pair with parent pulse at an arbitrary location t. If an exponential distribution with mean distance λ^{-1} is assumed for the time lag between parent and daughter pulse, the probability $q(t)$ for the daughter to fall in the same measuring interval of duration T as her parent is

$$q(t) = \lambda \int_t^T e^{-\lambda(x-t)} \, dx = 1 - e^{-\lambda(T-t)} .$$

The corresponding average probability is therefore

$$\bar{q} = \frac{1}{T} \int_0^T q(t) \, dt = 1 - \frac{1}{\lambda T} (1 - e^{-\lambda T}) . \tag{19}$$

For $\lambda T \gg 1$ the effect of the finite interval length disappears since then $\bar{q} = 1$, independently of the exact time distribution of the daughters. Taking into account the finite counter efficiencies, the average survival probability of a pair becomes

$$\pi_2 = \varepsilon_p \, \varepsilon_d \cdot \bar{q} ,$$

which generalizes (3). With an original rate ϱ for the parent pulses, we therefore arrive at an expectation of

$$\mu_2 = \varrho \cdot \pi_2 \cdot T = \varrho T \cdot \varepsilon_p \varepsilon_d \left[1 - \frac{1}{\lambda T}(1 - e^{-\lambda T})\right] \tag{20}$$

for the number of pairs in an interval T.

As the expectation for the total number of pulses registered in T is given by

$$\mu = \varrho\,(\varepsilon_p + \varepsilon_d)\,T\,,$$

we finally obtain for the mean number of uncorrelated single pulses in the interval T

$$\mu_1 = \mu - 2\mu_2 = \varrho T \left\{\varepsilon_p + \varepsilon_d - 2\,\varepsilon_p \varepsilon_d \left[1 - \frac{1}{\lambda T}(1 - e^{-\lambda T})\right]\right\}. \tag{21}$$

Putting

$$\mu_1 = \varrho \cdot \pi_1 \cdot T\,,$$

with $$\pi_1 = \varepsilon_p + \varepsilon_d - 2\,\varepsilon_p \varepsilon_d \left[1 - \frac{1}{\lambda T}(1 - e^{-\lambda T})\right], \tag{22}$$

we see that equation (22) reduces to (4) for $\lambda T \gg 1$.

If a mean number b of background pulses is registered in T, where cross-over transitions and any other non-correlated events are included, this should clearly be added to the number μ_1 of single pulses (and to the total number μ).

We note that according to (21) and (20) the mean number of singles or pairs is no longer proportional to the length T of the measuring interval. Single pulses and pairs thus form what the statisticians call a non-homogeneous Poisson process or a Poisson process with non-stationary increments [3].

Let us recall that all the above reasonings assume the complete absence of dead times in the counters. It is not clear at the present time how dead-time effects could be taken into account properly in the case of a two-step decay.

References

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