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Sums of factorials

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Abstract

Since factorials often give rise to simpler mathematical expressions than powers of the same order, we study the behaviour of sums of factorials. They can indeed be expressed in a remarkably compact form. A comparison with Bernoulli sums leads to two relations, one of them implying a surprising link between Bernoulli numbers and sums over Stirling numbers of the first kind.

1. Introduction

As is well known, sums of the form

$$ {}_rS_n \equiv \sum_{j=1}^{n} j^r , \quad \text{for } r = 1, 2, \dots , \tag{1}$$

lead to the expression first found by J. Bernoulli, in which the Bernoulli numbers made their appearance [1]. Equation (1) will be called a Bernoulli sum in what follows. However, factorials often lead to less complicated formulae than powers; for a striking example, one may compare ordinary with factorial moments for some discrete statistical distributions. We therefore wonder if sums of the form

$$ ({}_rS_n) \equiv \sum_{j=r}^{n} (j)_r , \tag{2}$$

where $(j)_r = j(j-1)\,(j-2)\,\dots\,(j-r+1)$ is a so-called "falling factorial" [2], would lead to similar, and possibly simpler, expressions than for (1). The notation used in (2) for factorial sums is practical, but requires special attention.

2. A direct way of summing

In this first approach, we make use of a formula which allows us to express (2) by a single falling factorial. A formula suitable for this purpose has been found [3] and is written in the form

$$ \sum_{m=1}^{n} m\,(m+1)\,\dots\,(m+k) = \frac{n\,(n+1)\,\dots\,(n+k+1)}{k+2} . \tag{3}$$

Putting m+k = j we find

$$\sum_{j=k+1}^{k+n} j\,(j-1)\ \dots\ (j-k) = \frac{(n+k-1)_{k+2}}{k+2}$$

or, with k+1 = r and k+n = N ,

$$\sum_{j=r}^{N} (j)_r = \frac{(N+1)_{r+1}}{r+1}\,. \tag{4}$$

Hence, the factorial sum (2) can be written in the simple form

$$({}_rS_n) = \frac{(n+1)_{r+1}}{r+1}\,. \tag{5}$$

This result can be linked to the Bernoulli sum (1).

Since, according to [2], for r = 1, 2, ...,

$$j^r = \sum_{k=1}^{r} S(r,k)\,(j)_k\,, \tag{6}$$

where S(r,k) is a Stirling number of the second kind, we also have

$${}_rS_n = \sum_{k=1}^{r} S(r,k)\,({}_kS_n) = \sum_{k=1}^{r} S(r,k)\,\frac{(n+1)_{k+1}}{k+1}\,. \tag{7}$$

This is apparently a novel and, it seems, conveniently compact formula for evaluating a Bernoulli sum of type (1). For illustration, we take the example

$${}_4S_7 = \sum_{k=1}^{4} S(4,k)\,\frac{(8)_{k+1}}{k+1} = \frac{1}{2}\,(8)_2 + \frac{7}{3}\,(8)_3 + \frac{6}{4}\,(8)_4 + \frac{1}{5}\,(8)_5 = 4\,676\,,$$

which is in agreement with a direct evaluation based on (1).

3. An alternative method

We begin with a relation which allows us to transform the factorial $(j)_r$ in (2) into a sum of powers, namely [2]

$$(j)_r = \sum_{k=1}^{r} s(r,k)\,j^k\,, \tag{8}$$

where s(r,k) is a Stirling number of the first kind. We note that (6) and (8) form a symmetric pair of relations.

In addition, we make use of the known explicit expression for a Bernoulli sum (1), which can be written, in the form of increasing powers of n, as

$${}_rS_n = \sum_{j=1}^{r+1} {}_r\alpha_j\, n^j\,. \tag{9}$$

The coefficients ${}_r\alpha_j$ of lowest order are assembled in Table 1.

Table 1. Coefficients ${}_r\alpha_j$ appearing in (9) for the Bernoulli sums ${}_rS_n$.

	${}_r\alpha_1$	${}_r\alpha_2$	${}_r\alpha_3$	${}_r\alpha_4$	${}_r\alpha_5$	${}_r\alpha_6$	${}_r\alpha_7$	${}_r\alpha_8$	${}_r\alpha_9$
r = 1	1/2	1/2							
2	1/6	1/2	1/3						
3	0	1/4	1/2	1/4					
4	-1/30	0	1/3	1/2	1/5				
5	0	-1/12	0	5/12	1/2	1/6			
6	1/42	0	-1/6	0	1/2	1/2	1/7		
7	0	1/12	0	-7/24	0	7/12	1/2	1/8	
8	-1/30	0	2/9	0	-7/15	0	2/3	1/2	1/9

This gives

- for r = 2

$$(j)_2 = -j + j^2,$$

thus

$$({}_2S_n) = \sum_{j=1}^{n} (j)_2 = -\sum_j j + \sum_j j^2$$
$$= -\left(\frac{1}{2}\,n + \frac{1}{2}\,n^2\right) + \frac{1}{6}\,n + \frac{1}{2}\,n^2 + \frac{1}{3}\,n^3$$
$$= -\frac{1}{3}\,n + \frac{1}{3}\,n^3,$$

- for r = 3

$$(j)_3 = 2j - 3j^2 + j^3,$$

$$({}_3S_n) = \sum_{j=1}^{n} (j)_3 = 2\sum j - 3\sum j^2 + \sum j^3$$
$$= \ldots = \frac{1}{2}\,n - \frac{1}{4}\,n^2 - \frac{1}{2}\,n^3 + \frac{1}{4}\,n^4,$$

- for r = 4

$$({}_4S_n) = -6\sum j + 11\sum j^2 - 6\sum j^3 + \sum j^4$$
$$= \ldots = -\frac{6}{5}\,n + n^2 + n^3 - n^4 + \frac{1}{5}\,n^5,$$

etc.

By assembling the numerical results thus obtained in the form of the expansion

$$({}_rS_n) = \sum_{j=1}^{r+1} {}_r\beta_j\, n^j\,. \qquad (10)$$

we find for the coefficients ${}_r\beta_j$ the values given in Table 2.

Table 2. The coefficients ${}_r\beta_j$ appearing in 10, for $1 \le r \le 8$.

	${}_r\beta_1$	${}_r\beta_2$	${}_r\beta_3$	${}_r\beta_4$	${}_r\beta_5$	${}_r\beta_6$	${}_r\beta_7$	${}_r\beta_8$	${}_r\beta_9$
r = 1	1/2	1/2							
2	-1/3	0	1/3						
3	1/2	-1/4	-1/2	1/4					
4	-6/5	1	1	-1	1/5				
5	4	-13/3	-5/2	25/6	-3/2	1/6			
6	-120/7	22	7	-20	10	-2	1/7		
7	90	-261/2	-35/2	889/8	-70	77/4	-5/2	1/8	
8	-560	892	-64/9	-707	4 809/9	-182	98/3	-3	1/9

By putting $(r+1)\,{}_r\beta_j = {}_r\gamma_{j-1}$, we arrive at new coefficients which are all integers. With them, the factorial sum (2) now takes the form

$$({}_rS_n) = \sum_{j=1}^{n} (j)_r = \frac{n}{r+1} \sum_{j=0}^{r} {}_r\gamma_j\, n^j\,, \qquad (11)$$

with the coefficients ${}_r\gamma_j$ listed in Table 3.

Table 3. The coefficients ${}_r\gamma_j$ appearing in (11), for $1 \le r \le 8$.

	${}_r\gamma_0$	${}_r\gamma_1$	${}_r\gamma_2$	${}_r\gamma_3$	${}_r\gamma_4$	${}_r\gamma_5$	${}_r\gamma_6$	${}_r\gamma_7$	${}_r\gamma_8$
r = 1	1	1							
2	-1	0	1						
3	2	-1	-2	1					
4	-6	5	5	-5	1				
5	24	-26	-15	25	-9	1			
6	-120	154	49	-140	70	-14	1		
7	720	-1 044	-140	889	-560	154	-20	1	
8	-5 040	8 028	-64	-6 363	4 809	-1 638	294	-27	1

At first sight, the coefficients ${}_r\gamma_j$ show hardly any regularity (except for the limiting values ${}_r\gamma_0$ and ${}_r\gamma_r$). However, one can find that

$$\sum_{j=0}^{r} {}_r\gamma_j = 0 , \qquad \text{for} \quad r \geq 2,$$

or also (12)

$$\sum_{j=0}^{r} (-1)^j \, {}_r\gamma_j = 0 , \qquad \text{for} \quad r \geq 1 .$$

In addition, a closer inspection reveals a surprisingly simple rule, namely that all the coefficients follow the general relation

$$ {}_r\gamma_j = s(r,j+1) + s(r,j) , \tag{13}$$

where s(r,k) are again Stirling numbers of the first kind, for which we put, as usual, $s(r,j) = 0$ for $j > r$ or $j \leq 0$.

As a simple check of (11) and (13) we consider the case r = 1, where

$$\begin{aligned} \sum_{j=1}^{n} (j)_1 &= \frac{n}{2} \sum_{j=0}^{1} \left[s(1,j) + s(1,j+1) \right] n^j \\ &= \frac{n}{2} \left[(0+1)\, n^0 + (1+0)\, n^1 \right] \\ &= \frac{n}{2} (1+n) = \sum_{j=1}^{n} j , \end{aligned}$$

as expected.

4. Relation with the Bernoulli sums

By means of (6), the Bernoulli sum (1) can also be written as

$$ {}_rS_n = \sum_{j=1}^{n} j^r = \sum_{j=1}^{n} \sum_{k=1}^{r} S(r,k)\, (j)_k = \sum_{k=1}^{1} S(r,k)\, ({}_kS_n) ,$$

which, taking advantage of the decomposition given in (11), becomes

$$ {}_rS_n = \sum_{k=1}^{r} S(r,k) \frac{1}{k+1} \sum_{j=0}^{k} {}_k\gamma_j \, n^{j+1} ,$$

with $1 \leq j+1 \leq r+1$.

Using the known expression (13) for ${}_k\gamma_j$ we find

$$ {}_rS_n = \sum_{k=1}^{r} \frac{S(r,k)}{k+1} \sum_{j=0}^{k} [s(k,j+1) + s(k,j)] \, n^{j+1} . \tag{14}$$

It is convenient for later use to introduce the shorthand notation

$$ {}_rS_n = \sum_{f=0}^{r} \Lambda_f \, n^{r+1-f} , \qquad \text{with } f = r\text{-}j . \tag{15} $$

The powers of n are within the limits 1 and r+1, as they should be for a Bernoulli sum, and the coefficient of $n^{j+1} = n^{r+1-f}$ has the general form (t = r-k)

$$ \Lambda_f = \sum_{t=0}^{f} \frac{S(r,r-t)}{r+1-t} \, [s(r-t,r+1-f) + s(r-t,r-f)] . \tag{16} $$

Let us look at the coefficients Λ_f for some low values of f. The simplest case is f = 0, which corresponds to the power n^{r+1}, according to (15). One finds, since j = r,

$$ \Lambda_0 = \frac{S(r,r)}{r+1} \, [s(r, r+1) + s(r,r)] = \frac{1}{r+1} , $$

since S(r,r) = s(r,r) = 1 and s(r,r+1) = 0.

Similarly, we have for f = 1

$$ \Lambda_1 = \frac{S(r,r)}{r+1} \, [s(r,r) + s(r,r-1)] + \frac{S(r,r-1)}{r} \, [s(r-1,r) + s(r-1,r-1)] . $$

By means of the relations given in (A7) we can write

$$ \Lambda_1 = \frac{1}{r+1} \left[1 - \binom{r}{2} \right] + \frac{1}{r} \binom{r}{2} [0 + 1] = \frac{1}{2} . $$

The results for Λ_0 and Λ_1 agree with what we expect on the basis of the Bernoulli development (compare the coefficients in Table 1).

There are several equivalent ways to write the general expression for a Bernoulli sum. A convenient one (see (23) in [4]) is to put, using decreasing powers of n,

$$ {}_rS_n = \sum_{f=0}^{r} C_f \, n^{r+1-f} , \tag{17} $$

with $$ C_0 = \frac{1}{r+1} , \quad C_1 = \frac{1}{2} \tag{18} $$

and $$ C_f = \frac{1}{f} \binom{r}{f-1} B_f , \quad \text{for } f \geq 2 , \tag{19} $$

where B_f are the so-called Bernoulli numbers, with $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, ...

A comparison with (15) leads to the identity

$$ \Lambda_f = \frac{1}{f} \binom{r}{f-1} B_f , \quad \text{for } f \geq 2 . \tag{20} $$

Since the Bernoulli numbers B_f vanish for $f \geq 3$ and odd, we have also the relation

$$\Lambda_f = 0, \qquad \text{for} \quad f = 3, 5, 7, \ldots . \tag{21}$$

No simple and general way to verify (20) or (21) is known. However, by showing explicitly their correctness for some low values of f, our belief in their general validity will be strengthened.

In the case of <u>f = 2</u> we have, according to (16),

$$\begin{aligned}\Lambda_2 = \; & \frac{S(r,r)}{r+1} \, [s(r,r-1) + s(r,r-2)] \\ & + \frac{S(r,r-1)}{r} \, [s(r-1,r-1) + s(r-1,r-2)] \\ & + \frac{S(r,r-2)}{r-1} \, [s(r-2,r-1) + s(r-2,r-2)] \, .\end{aligned}$$

Making use of the results in (A7) for the Stirling numbers, we obtain

$$\Lambda_2 = \frac{1}{r+1} \left[-\binom{r}{2} + 3\binom{r}{4} + 2\binom{r}{3} \right] + \frac{1}{r}\binom{r}{2} \left[1 - \binom{r-1}{2} \right] + \frac{1}{r-1} \left[3\binom{r}{4} + \binom{r}{3} \right] .$$

After some lengthy rearrangements this can be reduced to $\Lambda_2 = r/12$. This is in agreement with (20).

For the case <u>f = 3</u> we expect from (21) that $\Lambda_3 = 0$. A direct evaluation according to (16) gives

$$\begin{aligned}\Lambda_3 = \; & \frac{S(r,r)}{r+1} \, [s(r,r-2) + s(r,r-3)] \\ & + \frac{S(r,r-1)}{r} \, [s(r-1,r-2) + s(r-1,r-3)] \\ & + \frac{S(r,r-2)}{r-1} \, [s(r-2,r-2) + s(r-2,r-3)] \\ & + \frac{S(r,r-3)}{r-2} \, [s(r-3,r-2) + s(r-3,r-3)] \, .\end{aligned}$$

By means of the relations given in the Appendix we can also write

$$\begin{aligned}\Lambda_3 = \; & \frac{1}{r+1} \left[3\binom{r}{4} + 2\binom{r}{3} - 15\binom{r}{6} - 20\binom{r}{5} - 6\binom{r}{4} \right] \\ & + \binom{r}{2} \frac{1}{r} \left[-\binom{r-1}{2} + 3\binom{r-1}{4} + 2\binom{r-1}{3} \right] \\ & + \frac{1}{r-1} \left[3\binom{r}{4} + \binom{r}{3} \right] \left[1 - \binom{r-2}{2} \right] \\ & + \frac{1}{r-2} \left[15\binom{r}{6} + 10\binom{r}{5} + \binom{r}{4} \right] 1 \, .\end{aligned}$$

After some elementary but lengthy algebra, Λ_3 can be shown to vanish, as expected.

Similar, although even more strenuous efforts are required to verify that $\Lambda_4 = -\binom{r}{3}/120$, again in agreement with (20).

Thus, although formula (16) for Λ_r is based on the unproven relation (12), there can be no real doubt that both (20) and (21) are valid in general.

APPENDIX

Stirling numbers as sums of binomial coefficients

Since relatively few relations are known which involve Stirling numbers (many of them are somewhere in [5], but difficult to find), it is useful to have relations which allow us to transform Stirling numbers into a form which lends itself to later algebraic manipulation. Otherwise, general reasoning must be stopped and replaced by numerical treatment, which is usually more cumbersome and less efficient.

A possible way to circumvent such a limitation is given by transforming Stirling numbers into binomial coefficients, for which more relations are available for subsequent handling.

The relevant relations have been known for long, but they are rarely mentioned in modern texts. My main source is a book by Jordan [6], the tabulations of which have been extended. The notation follows Riordan [2].

a) *Stirling numbers of the first kind*

Stirling numbers of the first kind, written as s(n,k), allow the following decomposition into binomial coefficients:

$$s(n,n-d) = \sum_{j=0}^{d-1} J^{(1)}_{d,j} \binom{n}{2d-j}, \tag{A1}$$

with $d = 1, 2, \ldots, n-1$.

The Jordan coefficients $J^{(1)}_{d,j}$ are listed in Table A1.

This tabulation can be continued by applying the recursion formula

$$J^{(1)}_{d,j} = -(2d-j-1)\left[J^{(1)}_{d-1,j-1} + J^{(1)}_{d-1,j}\right]. \tag{A2}$$

For checking purposes, it may be useful to know that

$$\sum_{j=0}^{d-1} (-1)^j\, J^{(1)}_{d,j} = (-1)^d . \tag{A3}$$

b) *Stirling numbers of the second kind*

For Stirling numbers of the second kind, written as S(n,k), we have the analogous relation

$$S(n,n-d) = \sum_{j=0}^{d-1} J^{(2)}_{d,j} \binom{n}{2d-j} , \tag{A4}$$

with the coefficients $J^{(2)}_{d,j}$ listed in Table A2.

Here the recurrence is

$$J^{(2)}_{d,j} = (d-j)\ J^{(2)}_{d-1,j-1} + (2d-j-1)\ J^{(2)}_{d-1,j} , \tag{A5}$$

while the alternate sum is

$$\sum_{j=0}^{d-1} (-1)^j J^{(2)}_{d,j} = d!\ . \tag{A6}$$

Let us end by giving some simple examples which illustrate the use of the tables.

$$\begin{aligned}
s(n,n-1) &= -\binom{n}{2}, & S(n,n-1) &= \binom{n}{2}, \\
s(n,n-2) &= 3\binom{n}{4} + 2\binom{n}{3}, & S(n,n-2) &= 3\binom{n}{4} + \binom{n}{3}, \\
s(n,n-3) &= -15\binom{n}{6} - 20\binom{n}{5} - 6\binom{n}{4}, & S(n,n-3) &= 15\binom{n}{6} + 10\binom{n}{5} + \binom{n}{4}, \\
s(n,n-4) &= 105\binom{n}{8} + 210\binom{n}{7} + 130\binom{n}{6} + 24\binom{n}{5}, & S(n,n-4) &= 105\binom{n}{8} + 105\binom{n}{7} + 25\binom{n}{6} + \binom{n}{5}.
\end{aligned} \tag{A7}$$

Table A1 - Jordan coefficients $J^{(1)}_{d,j}$, for $d \leqslant 8$.

	j=0	1	2	3	4	5	6	7
d = 1	-1							
2	3	2						
3	-15	-20	-6					
4	105	210	130	24				
5	-945	-2 520	-2 380	-924	-120			
6	10 395	34 650	44 100	26 432	7 308	720		
7	-135 135	-540 540	-866 250	-705 320	-303 660	-64 224	-5 040	
8	2 027 025	9 459 450	18 288 270	18 858 840	11 098 780	3 678 840	623 376	40 320

Table A1 - Jordan coefficients $J^{(2)}_{d,j}$, for $d \leqslant 8$.

	j=0	1	2	3	4	5	6	7
d = 1	1							
2	3	1						
3	15	10	1					
4	105	105	25	1				
5	945	1 260	490	56	1			
6	10 395	17 325	9 450	1 918	119	1		
7	135 135	270 270	190 575	56 980	6 825	246	1	
8	2 027 025	4 729 725	4 099 095	1 636 635	302 995	22 935	501	1

References

[1] J. Bernoulli: "Ars conjectandi" (Basel, 1713), p. 97

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