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Is there a shortcut for treating dead times in series ?

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Abstract

In an attempt to circumvent the complicated general formulae for a series arrangement of two generalized dead times, simple models are analyzed which rely on an intuitive approach suggested in the literature. It turns out that, in a series development for the output count rate, the second-order terms are already unreliable. Alternatively, simpler expansions, based on the exact formulae, can be obtained. These are rigorous if applied under somewhat restrictive, but well-defined experimental conditions.

1. Introduction

The rigorous analysis of a series arrangement of two dead times leads, as is well known, to rather complicated formulae. This is already the case for the simple situation where the input process is Poissonian and the dead times are of the traditional type [1]. Obviously, the expressions become still more intricate if we allow the two dead times to be of the generalized type.

It is legitimate, therefore, to explore the possible utility of some simplifying assumptions - even if one cannot expect them to have a rigorous basis - hoping that they will lead to useful approximations. Such an alternative approach, if it exists, would allow us to replace the actual formal analysis by a treatment where the various steps can be easily interpreted.

The original problem of two generalized dead times in series is sketched symbolically in Fig. 1. We now try to replace it by a similar scheme capable of leading to an approximate expression for the output count rate R. For reasons of mathematical convenience, we restrict our search to solutions which are in the form of series expansions (powers of $x = \rho\tau$). The coefficients of lowest order appearing in such a development can then be compared with those of an analogous expansion which is based on exact relations.

Fig. 1 - Series arrangement of two generalized dead times, with the adopted notation for count rates, dead times and type parameters.

2. The basic model adopted

It will be assumed throughout that the original input process, with count rate ρ, is Poissonian, and the effect of the first dead time is rigorously taken into account. All the approximations concern the second dead time. We note that this procedure is just opposite to the one adopted for the rigorous treatment, where the additional influence of the smaller first dead time is considered as a perturbation.

In the simplest model that one might be tempted to apply, the effects produced by the two dead times are assumed to be independent. So, the output of the first element, serving as input for the second, is considered still to be Poissonian (contrary to the facts). It can readily be seen that in this form the model is too crude, for it would lead for the output count rate to an expression of the form $\rho[1 - (1+\alpha)x + ...]$ which is already wrong to first order in x. If we choose $\alpha = 1$, the second element should have no effect at all since the incoming pulses are already separated by a time interval of at least τ, i.e. the first dead time.

Obviously, there is a simple way to avoid the main shortcoming of this model: it leads us to an improved version which - although rather artificial - preserves the advantage of simplicity. While we still assume the input process for the second dead time to be Poissonian, we now require in addition that the actual value τ of the second dead time be replaced by an "effective length" of $(1-\alpha)\tau$.

If my interpretation is correct, a similar model has been used previously (for the special case $\theta_1 = 1$ and $\theta_2 = 0$) by Fleming and Lindstrom ([2], [3]) in the context of a decaying source. If the simple replacement is assumed to be applicable to any sequence of types, we are led to the equivalent scheme sketched in Fig. 2.

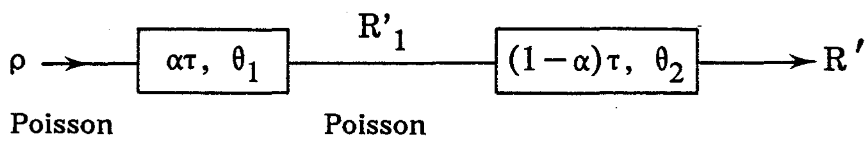

Fig. 2 - Scheme adopted for the simplified model of a series arrangement.

For each of the two parts in the series arrangement we assume that input and output count rates are connected by a relation of the form (with i = 1 or 2)

$$R_i' = \rho_i T_i' \, , \tag{1}$$

where the transmission factor for a (single) generalized dead time is given [4] by the expansion

$$T_i' = 1 - \rho_i\tau_i + (1 - \frac{1}{2}\theta_i)(\rho_i\tau_i)^2 - (1 - \theta_i + \frac{1}{6}\theta_i^2)(\rho_i\tau_i)^3 \pm \dots \qquad (2)$$

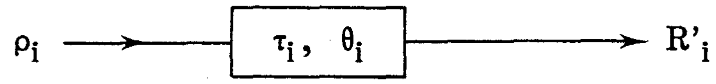

Fig. 3 - Notation used for a single dead-time element in the model of Fig. 2.

We now introduce a further (quite arbitrary) change which concerns the value of the input count rate ρ_2 to be used for the evaluation of the second transmission factor T'_2. It seems natural that this should be taken as $\rho_2 = \rho\, T'_1$, but since we have already manipulated both the nature of the process and the length of the dead time, there can be little objection to making an attempt with the more general relation

$$\rho_2 = \beta\,\rho, \qquad (3a)$$

where the (artificial) transmission factor β is supposed to have the series expansion

$$\beta = 1 + b_1 x + b_2 x^2 + \dots . \qquad (3b)$$

Let us now determine, for our adopted model (Fig. 2), the output count rate R' which can be formally expressed by

$$R' = \rho\, T_1'\, T_2' . \qquad (4)$$

Obviously, this requires the explicit evaluation of the transmission factors T'_1 and T'_2.

For the first part of the series arrangement we have $\rho_1 = \rho$ and $\tau_1 = \alpha\tau$, therefore from (2)

$$\begin{aligned} T_1' &\cong 1 - \rho_1\tau_1 + (1 - \frac{1}{2}\theta_1)(\rho_1\theta_1)^2 - (1 - \theta_1 + \frac{1}{6}\rho_1^2)(\rho_1\tau_1)^3 \\ &\cong 1 - \alpha x\ (1 - \frac{1}{2}\theta_1)(\alpha x)^2 - (1 - \theta_1 + \frac{1}{6}\theta_1^2)(\alpha x)^3 . \end{aligned} \qquad (5)$$

For the second element we have to use $\rho_2 = \beta\rho$ and $\tau_2 = (1-\alpha)\tau$, hence

$$T_2' \cong 1 - \beta\rho(1-\alpha)\tau + (1 - \frac{1}{2}\theta_2)\left[\beta\rho(1-\alpha)\tau\right]^2 - (1 - \theta_2 + \frac{1}{6}\theta_2^2)\left[\beta\rho(1-\alpha)\tau\right]^3 ,$$

or after some rearrangement

$$T_2' \cong 1 - (1-\alpha)x + (1-\alpha)\left[-b_1 + (1-\alpha)\left(1 - \frac{1}{2}\theta_2\right)\right]x^2$$

$$- (1-\alpha)\left[b_2 - 2(1-\alpha)\left(1 - \frac{1}{2}\theta_2\right)b_1 + (1-\alpha)^2\left(1 - \theta_2 + \frac{1}{6}\theta_2^2\right)\right]x^3 . \tag{6}$$

We are now in a position to evaluate by means of (4) the expected output count rate for our model. After a tedious multiplication of the series expansions given in (5) and (6) we finally arrive at

$$\frac{R'}{\rho} = T_1' T_2' = 1 - x + {}_0A_2 x^2 - {}_0A_3 x^3 \pm \ldots , \tag{7a}$$

with

$${}_0A_2 = 1 - \frac{1}{2}\theta_2 - b_1 - \alpha(1 - \theta_2 - b_1) + \alpha^2\left(1 - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2\right) \quad \text{and} \tag{7b}$$

$$\begin{aligned} {}_0A_3 = {} & 1 - \theta_2\left(1 - \frac{1}{6}\theta_2\right) - b_1(2 - \theta_2) + b_2 \\ & - \alpha\left[2 - \frac{1}{2}\theta_2(5 - \theta_2) - b_1(3 - 2\theta_2) + b_2\right] \\ & + \alpha^2\left[2 - \frac{1}{2}\theta_1 - \theta_2\left(2 - \frac{1}{2}\theta_2\right) - b_1(1 - \theta_2)\right] \\ & - \alpha^3\left[\frac{1}{2}(\theta_1 - \theta_2) - \frac{1}{6}(\theta_1^2 - \theta_2^2)\right] . \end{aligned} \tag{7c}$$

If the model considered above, for a specific choice of β, actually turns out to be of practical use, then (7) may be considered as one of the main results of the present study. For the time being, however, it is no more than the outcome of a calculation based on a model of unknown value. The utility of (7) depends on its ability to approximate the real output count rate R, which has to be evaluated independently; this will be done in section 5.

A rough check is already possible for the limiting cases $\alpha = 0$ or 1, for which we hope to come back to the situation of a single dead time. However, we have to remember that $\beta \longrightarrow 1$ for $\alpha \longrightarrow 0$, since $\rho_2 = \rho$ in the absence of a first dead time. Hence we now readily find in (7)

- for $\alpha = 0$ (and $b_1 = b_2 = 0$):

$$ {}_0A_2 = 1 - \frac{1}{2}\theta_2 , \qquad {}_0A_3 = 1 - \theta_2 + \frac{1}{6}\theta_2^2 , $$

- for $\alpha = 1$:

$$ {}_0A_2 = 1 - \frac{1}{2}\theta_1 , \qquad {}_0A_3 = 1 - \theta_1 + \frac{1}{6}\theta_1^2 , $$

as expected. Yet we should keep in mind that this agreement has in fact been imposed on the model by means of the "effective" second dead time.

In order to make the model more specific, we now consider for the parameter β two special cases: we choose $\beta = 1$ or $\beta = T'_1$, with the latter quantity given in (5).

3. The first model ($\beta = 1$)

In this first model we make the (unphysical) assumption that the count rate appearing in the second transmission factor T'_2 is ρ, i.e. the original one. This corresponds in (3) to putting $\beta = 1$, thus assuming that $b_1 = b_2 = 0$. If we write (7a) in the form

$$ \frac{R'}{\rho} = 1 - x + A'_2 x^2 - A'_3 x^3 \pm \dots , \tag{8a} $$

the new coefficients are readily obtained as

$$ A'_2 = 1 - \frac{1}{2}\theta_2 - \alpha(1 - \theta_2) + \alpha^2(1 - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_2) , \tag{8b} $$

$$ \begin{aligned} A'_3 = {} & 1 - \theta_2 (1 - \frac{1}{6}\theta_2) - \alpha\left[2 - \frac{1}{2}\theta_2 (5 - \theta_2)\right] \\ & + \alpha^2 \left[2 - \frac{1}{2}\theta_1 - \theta_2 (2 - \frac{1}{2}\theta_2)\right] \\ & - \alpha^3 \left[\frac{1}{2}(\theta_1 - \theta_2) - \frac{1}{6}(\theta_1^2 - \theta_2^2)\right] . \end{aligned} \tag{8c} $$

Of particular interest are the special cases which correspond to the four series arrangements of traditional dead times (with types abbreviated as N or E). The results are assembled in Table 1.

Table 1 - Coefficients appearing in the output count rate (8a) of the first model, for two traditional dead times in series.

Type sequence	A'_2	A'_3
N,N	$1 - \alpha + \alpha^2$	$1 - 2\alpha + 2\alpha^2$
E,N	$1 - \alpha + \frac{1}{2}\alpha^2$	$1 - 2\alpha + \frac{3}{2}\alpha^2 - \frac{1}{3}\alpha^3$
N,E	$\frac{1}{2}(1 + \alpha^2)$	$\frac{1}{6}(1 + 3\alpha^2 + 2\alpha^3)$
E,E	$\frac{1}{2}$	$\frac{1}{6}$

4. The second model ($\beta = T'_1$)

In this second model we identify the count rate ρ_2, which is used in the evaluation of the transmission factor T'_2, with the output count rate R'_1 of the first stage, as it seems natural to do. This amounts to equating β with T'_1, given in (5). Comparison with (3b) yields

$$b_1 = -\alpha , \qquad b_2 = \alpha^2 (1 - \tfrac{1}{2}\theta_1) . \tag{9}$$

On substituting this in (7) we obtain, after some algebra, expressions for the coefficients appearing in

$$\frac{R'}{\rho} = 1 - x + A''_2 x^2 - A''_3 x^3 \pm \dots \tag{10a}$$

which are

$$A''_2 = 1 - \tfrac{1}{2}\left[\alpha^2 \theta_1 + (1-\alpha)^2 \theta_2\right] , \tag{10b}$$

$$A''_3 = 1 - \theta_2 (1 - \tfrac{1}{6}\theta_2) + \alpha\, \theta_2 (\tfrac{3}{2} - \tfrac{1}{2}\theta_2)$$

$$- \alpha^2 (\theta_1 - \tfrac{1}{2}\theta_2^2) - \alpha^3 \left[\tfrac{1}{2}\theta_2 - \tfrac{1}{6}(\theta_1^2 - \theta_2^2)\right] . \tag{10c}$$

The results corresponding to the series arrangement of traditional dead times are again assembled in tabular form.

Table 2 - Coefficients appearing in the output count rate (10a) of the second model, for two traditional dead times in series.

Type sequence	A''_2	A''_3
N,N	1	1
E,N	$1 - \frac{1}{2}\alpha^2$	$1 - \alpha^2 + \frac{1}{6}\alpha^3$
N,E	$1 - \frac{1}{2}(1-\alpha)^2$	$\frac{1}{6}(1 + 6\alpha + 3\alpha^2 - 4\alpha^3)$
E,E	$\frac{1}{2} + \alpha(1-\alpha)$	$\frac{1}{6}(1 + 6\alpha - 3\alpha^2 - 3\alpha^3)$

Although a real discussion has to be delayed until the approximations can be compared with the expected rigorous results (to be evaluated in section 5), two of the above approximations immediately appear suspicious, namely the case E,E in Table 1 and the case N,N in Table 2, where both developments correspond to a situation where the first dead time is assumed to be zero.

5. Derivation of some rigorous results

In the exact description of series arrangements, as sketched in Fig. 1 for the general case, the output count rated R is written in the form

$$R = \rho\, T_2(\theta_2)\, T_1(\theta_1,\theta_2) \;, \tag{11}$$

where the input is assumed to form a Poisson process of rate ρ.

The transmission factor T_2, which describes the behaviour of the second dead time alone (i.e. for $\alpha = 0$), is known to be of the form [4]

$$T_2(\theta_2) = 1 - x + (1 - \tfrac{1}{2}\theta_2)\, x^2 - (1 - \theta_2 + \tfrac{1}{6}\theta_2^2)\, x^3 \pm \dots \;, \tag{12}$$

if we use again a series expansion of third order.

The transmission T_1 accounts for the additional effect produced by the first dead time on the output. We write it, since a linear term is known to be absent, as

$$T_1(\theta_1,\theta_2) = 1 + a_2\, x^2 + a_3\, x^3 \pm \dots \;. \tag{13}$$

The general expressions for the coefficients a_2 and a_3 have recently become available [5] and we shall make use of them in what follows. As the exact form of these coefficients depends on the range in which α lies, the description has to be subdivided accordingly.

a) *For* $\alpha \leq 1/3$

For the range $\alpha \leq 1/3$ the required coefficients are

$$a_2 = \frac{1}{2}(2\theta_2 - 1)\,\alpha^2 \,, \tag{14a}$$

$$a_3 = \frac{1}{6}(2\theta_2-1)\left[(\theta_2-1)(3+\alpha) + 2(2\,\theta_1-1)\alpha\right]\alpha^2 \,. \tag{14b}$$

This leads for the transmission factor T_1 to the explicit form

$$T_1(\theta_1,\theta_2) \cong 1 + \frac{1}{2}(2\theta_2-1)\,\alpha^2x^2 + \frac{1}{6}(2\theta_2-1)\left[(\theta_2-1)(3+\alpha) + 2(2\theta_1-1)\alpha\right]\alpha^2x^3 \,. \tag{15}$$

With the expressions (12) and (15) at hand, we can use (11) to obtain the output rate R. The result for the product $T_2\,T_1$, after some rearrangement, can be written in the form

$$\frac{R}{\rho} \cong 1 - x + \left[1 - \frac{1}{2}\alpha^2 - \frac{1}{2}\theta_2\,(1-2\alpha^2)\right]x^2$$

$$- \frac{1}{6}\{(2\theta_2-1)\left[6 + 3\alpha - 4\theta_1\alpha - \theta_2(3+\alpha)\right]\alpha^2 + 6 - 6\theta_2 + \theta_2^2\}\,x^3 \,. \tag{16}$$

This expression is independent of any simplifying model and therefore rigorous (to third order in x) for any series arrangement of two generalized dead times.

Let us now consider the special cases which correspond to the series arrangement of dead times of the traditional type. The results are assembled in Table 3 in terms of the coefficients A_2 and A_3 appearing in the general expansion formula

$$\frac{R}{\rho} = 1 - x + A_2\,x^2 - A_3\,x^3 \pm \ \dots \,. \tag{17}$$

A comparison of Table 3 with Tables 1 and 2 reveals that it is only for the sequence N,E of the first model and for the sequence E,N of the second model that the results are in agreement up to third order. In all the other cases even the second-order terms are incorrect for both models.

Table 3 - Values of the coefficients appearing in (17). The output count rates correspond to the indicated sequence of dead-time types and assume (for N,N and E,E) that $\alpha \le 1/3$.

Type sequence	A_2	A_3
N,N	$1 - \frac{1}{2}\alpha^2$	$1 - \alpha^2 - \frac{1}{2}\alpha^3$
E,N	$1 - \frac{1}{2}\alpha^2$	$1 - \alpha^2 + \frac{1}{6}\alpha^3$
N,E	$\frac{1}{2}(1 + \alpha^2)$	$\frac{1}{6}(1 + 3\alpha^2 + 2\alpha^3)$
E,E	$\frac{1}{2}(1 + \alpha^2)$	$\frac{1}{6}(1 + 3\alpha^2 - 2\alpha^3)$

b) *For* $1/3 \le \alpha \le 1/2$

According to [5], the coefficients to be used in (13) for the range $1/3 \le \alpha \le 1/2$ are

$$a_2 = \frac{1}{2}(2\theta_2-1)\alpha^2 , \tag{18a}$$

$$a_3 = \frac{1}{6}\{(2\theta_2-1)\alpha^2 \left[(\theta_2-1)(3+\alpha) + 2(2\theta_1-1)\alpha \right] - (\theta_1+\theta_2-1)^2 (3\alpha-1)^3\} . \tag{18b}$$

Therefore, the transmission factor T_1 is now

$$T_1(\theta_1,\theta_2) \cong 1 + \frac{1}{2}(2\theta_2-1)\alpha^2 x^2$$
$$+ \frac{1}{6}\{(2\theta_2-1)\alpha^2 \left[(\theta_2-1)(3+\alpha) + 2(2\theta_1-1)\alpha \right] - (\theta_1+\theta_2-1)^2 (3\alpha-1)^3\} x^3 . \tag{19}$$

For the product $T_2\, T_1$, this leads us to

$$\frac{R}{\rho} \cong 1 - x + \left[1 - \frac{1}{2}\alpha^2 - \frac{1}{2}\theta_2 (1-2\alpha^2) \right] x^2 - \frac{1}{6}\{(2\theta_2-1)\left[6 + 3\alpha - 4\theta_1\alpha - \theta_2(3+\alpha) \right] \alpha^2$$
$$+ (\theta_1+\theta_2-1)^2 (3\alpha-1)^3 + 6 - 6\theta_2 + \theta_2^2\} x^3 . \tag{20}$$

This could be rearranged in a number of other ways, but little would be gained.

If the classical series arrangements are again expressed by the development (17), the respective coefficients are those listed in Table 4.

Table 4 - Values of the coefficients A_2 and A_3 as in Table 3, but for $1/3 \leq \alpha \leq 1/2$.

Type sequence	A_2	A_3
N,N	$1 - \frac{1}{2}\alpha^2$	$\frac{1}{6}(5 + 9\alpha - 33\alpha^2 + 24\alpha^3)$
E,N	$1 - \frac{1}{2}\alpha^2$	$1 - \alpha^2 + \frac{1}{6}\alpha^3$
N,E	$\frac{1}{2}(1 + \alpha^2)$	$\frac{1}{6}(1 + 3\alpha^2 + 2\alpha^3)$
E,E	$\frac{1}{2}(1 + \alpha^2)$	$\frac{1}{6}\alpha(9 - 24\alpha + 25\alpha^2)$

As expected, the only coefficients differing from those given in Table 3 are A_3 for the arrangements N,N and E,E.

c) *For* $1/2 \leq \alpha \leq 1$

Following [5], the coefficients applicable in (13) for the range $1/2 \leq \alpha \leq 1$ are

$$a_2 = \frac{1}{2}\{(2\theta_2-1)\,\alpha^2 - (\theta_1+\theta_2-1)\,(2\alpha-1)^2\}\ , \tag{21a}$$

$$a_3 = \frac{1}{6}\{(2\theta_2-1)\,\alpha^2\left[(\theta_2-1)(3+\alpha) + 2(2\theta_1-1)\alpha\right] - (\theta_1+\theta_2-1)^2\,(3\alpha-1)^3 + (\theta_1+\theta_2-1)\left[5-\theta_1-7\theta_2 - (7-8\theta_1-8\theta_2)\alpha\right](2\alpha-1)^2\}\ . \tag{21b}$$

As is well known from previous studies, it is this range which gives rise to the most complicated formulae, and the present case is no exception. To shorten the presentation, we simply quote the formula for the output rate which is

$$\frac{R}{\rho} \cong 1 - x + \frac{1}{2}\left[2 - \theta_2 + (2\theta_2-1)\alpha^2 - (\theta_1+\theta_2-1)\,(2\alpha-1)^2\right]x^2 - \frac{1}{6}\{(2\theta_2-1)\left[6 - 3\theta_2 + (3 - 4\theta_1-\theta_2)\,\alpha\right]\alpha^2 - (\theta_1+\theta_2-1)\,\{(2\alpha-1)^2\left[8 - \theta_1 - 7\theta_2 - (7 - 8\theta_1 - 8\theta_2)\,\alpha\right] - (3\alpha-1)^3\,(\theta_1+\theta_2-1)\} + 6 - 6\theta_2 + 6\theta_2^2\}\,x^3\ . \tag{22}$$

Table 5 - Coefficients as given in Tables 3 and 4, but for $1/2 \leq \alpha \leq 1$.

Type sequence	A_2	A_3
N,N	$\frac{1}{2}(3 - 4\alpha + 3\alpha^2)$	$\frac{1}{6}(13 - 30\alpha + 27\alpha^2 - 4\alpha^3)$
E,N	$1 - \frac{1}{2}\alpha^2$	$1 - \alpha^2 + \frac{1}{6}\alpha^3$
N,E	$\frac{1}{2}(1 + \alpha^2)$	$\frac{1}{6}(1 + 3\alpha^2 + 2\alpha^3)$
E,E	$\frac{1}{2}\alpha(4 - 3\alpha)$	$\alpha^2(2 - \frac{11}{6}\alpha)$

For the series arrangements of traditional dead times this leads for the expansion (17) to the coefficients listed in Table 5. As expected, the coefficients listed for the arrangements E,N and N,E agree with those in the two previous tables and are thus independent of the range of α. The other coefficients can be shown to agree at the borders. Specifically

- for N,N at	$\alpha = 1/3$:	$A_2 = 17/18$,	$A_3 = 47/54$
	$\alpha = 1/2$:	$A_2 = 7/8$,	$A_3 = 17/24$,
- for E,E at	$\alpha = 1/3$:	$A_2 = 5/9$,	$A_3 = 17/81$
	$\alpha = 1/2$:	$A_2 = 5/8$,	$A_3 = 13/48$.

6. Discussion

To clarify the behaviour of the exact expansion coefficients A_2 and A_3, the data of Tables 3 to 5 is represented in graphical form in Figs. 4 and 5. To these graphs are added the corresponding coefficients from the models described in sections 3 and 4.

The obvious conclusion from this comparison is that the predictions of both models are quite poor. This is particularly so for actual series arrangements, i.e. when α is not too close to one of its limiting values 0 or 1, which correspond in fact to the single dead-time situation for which the model has been tailored to fit exactly.

The only exceptions concern the series arrangements N,E (for the first model) and E,N (for the second model), for which the agreement is perfect. This is neither a real surprise nor a pure coincidence, but rather a feature built into the models. It can be

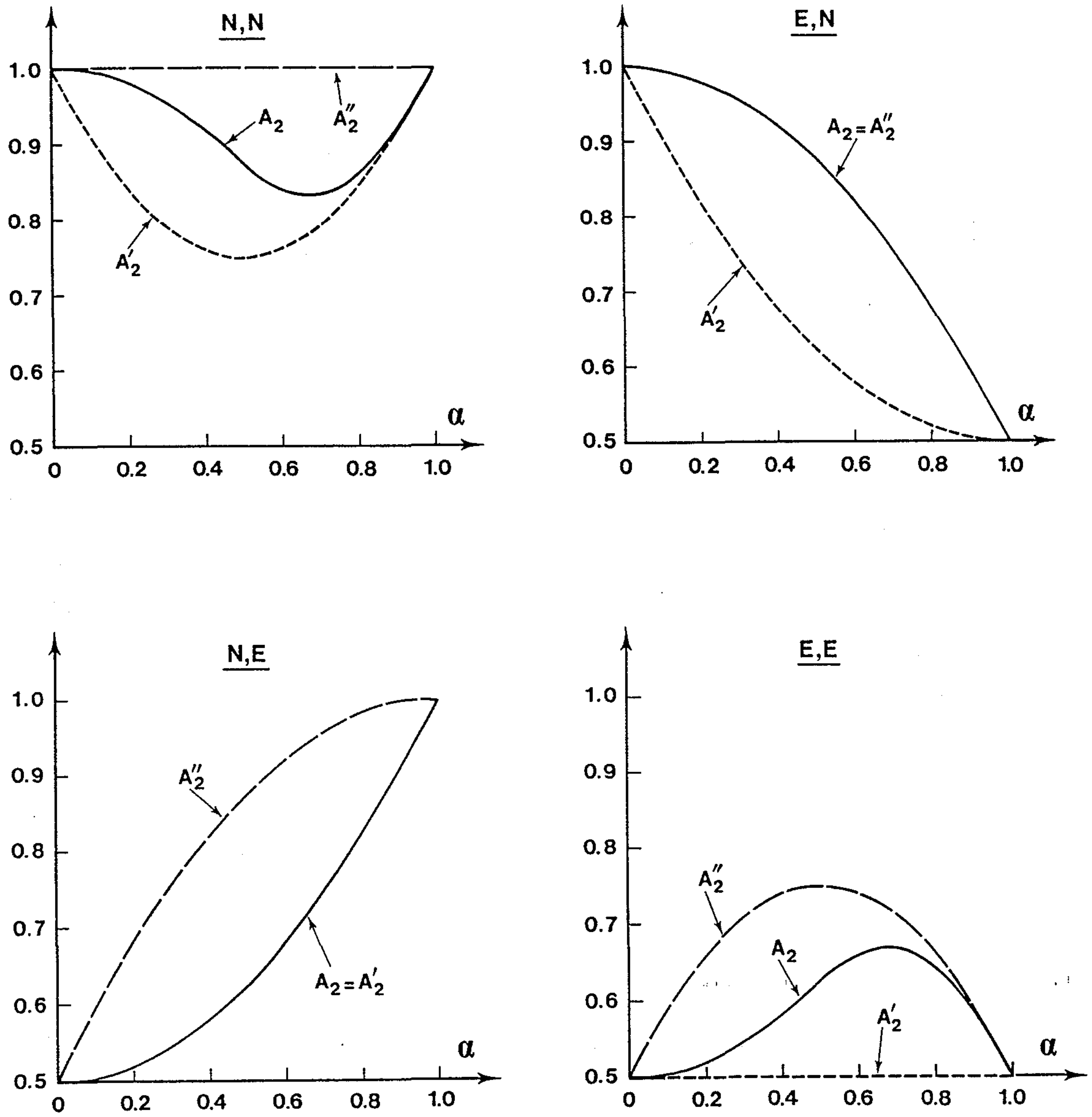

Fig. 4 - Graphical representation of the expansion coefficients of second order appearing in Tables 1 to 5. Note that A_2 corresponds to the exact solution, whereas A'_2 and A''_2 result from the two models discussed in the text. Each of the four plots is for a specific arrangement of dead times.

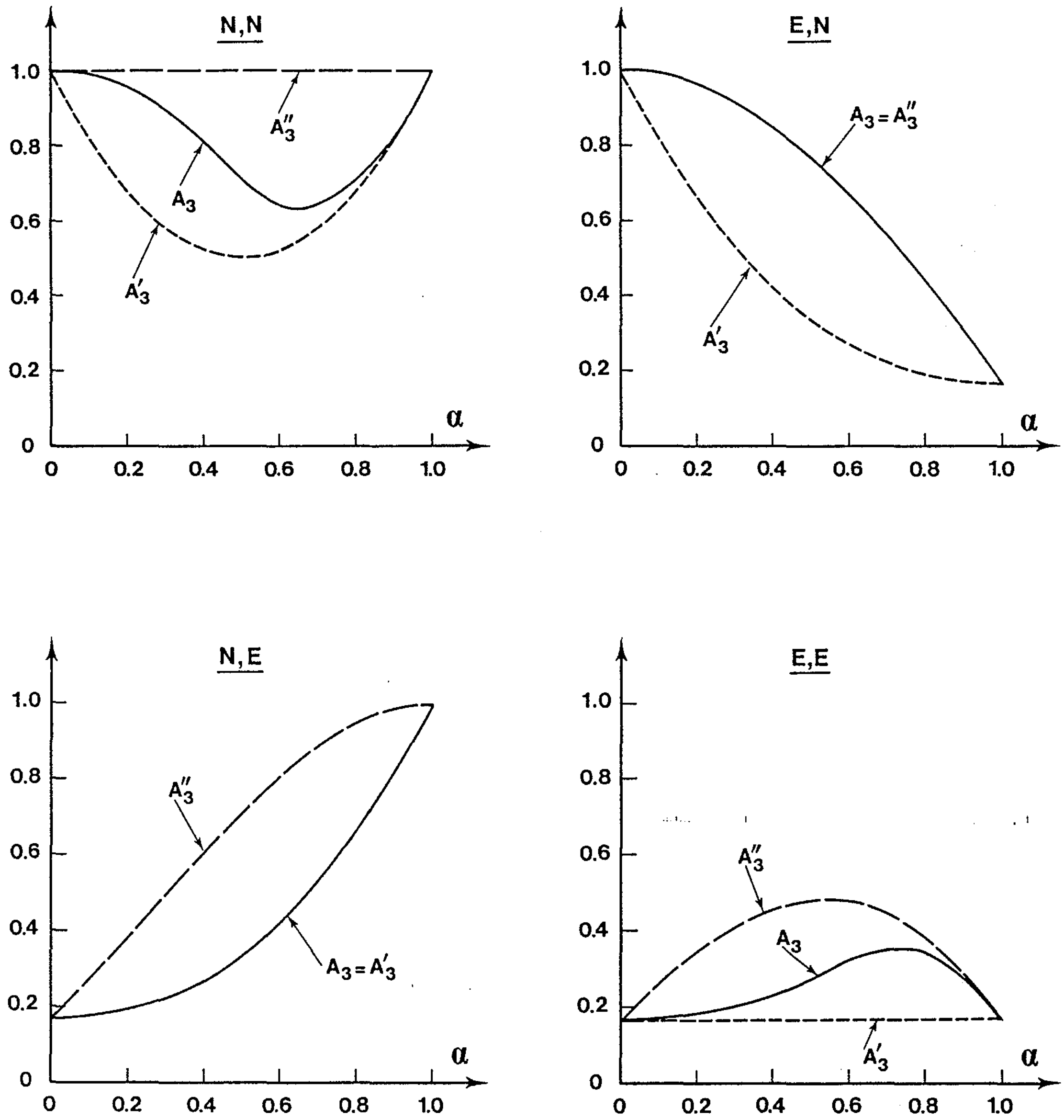

Fig. 5 - Analogous to Fig. 4, but for the expansion coefficients A_3, A'_3 and A''_3 of third order.

best seen by starting from the two known exact expressions for the output count rates [1]

$$R(N,E) = \frac{e^{-(1-\alpha)x}}{1+\alpha x} \quad \text{and} \tag{23}$$

$$R(E,N) = \frac{\rho\, e^{-\alpha x}}{1+(1-\alpha)\, x\, e^{-\alpha x}} \ . \tag{24}$$

Our basic model requires that the output rate can be written in the form

$$R = \alpha\, T_1'\, T_2' \ .$$

For the case N,E, and by applying the rules of the first model, we find

$$T_1' = \frac{1}{1+\alpha x} \quad \text{and} \quad T_2' = e^{-\rho_2\tau_2} = e^{-(1-\alpha)x} \ ,$$

since $\rho_2 = \rho$ and $\tau_2 = (1-\alpha)\tau$. This is clearly identical with (23).

For the case E,N and with the second model, we have

$$T_1' = e^{-\alpha x} \quad \text{and} \quad T_2' = \frac{1}{1+\rho_2\tau_2} = [\,1+\rho\cdot e^{-\alpha x}\,(1-\alpha)\tau\,]^{-1} \ ,$$

as now $\rho_2 = \rho T'_1$ and $\tau_2 = (1-\alpha)\tau$. Again the product confirms (24) exactly.

From a comparison of the respective expansion coefficients we conclude that the simple models considered in this report cannot be taken seriously. They are so constructed that each leads to correct results for a specific arrangement of two dead times (namely for the simple cases N,E or E,N), but none allows a valid extension to other cases. The supposedly general formulae (7), (8) and (10) therefore have no real foundation and should not be used.

Must we conclude, therefore, that all the effort spent above is for nothing? This would certainly be too pessimistic, as some of the results derived remain exact and useful.

First of all, there are the expansions (up to third order in x) for the output count rate R written as

$$R \cong \rho(1 - x + A_2\, x^2 - A_3\, x^3) \ .$$

The general form of the coefficients A_2 and A_3, for arbitrary values of the type parameters θ_1 and θ_2, are given in (16), (20) and (22) for the various domains of the parameter α. For the specific situations where both dead times are of the traditional type, the respective coefficients are listed in Tables 3 to 5.

Our initial objective was to find "general" and "simple" expressions for dead times in series. Do we have to abandon this idea? Not really. While it must be admitted that the off-hand construction of models has come to a deadlock, it is possible to obtain simple results from the exact expressions, provided we are willing to accept some restrictions.

If we limit ourselves to second order in x and to $\alpha \leq 1/2$, we readily find from (16) or (20)

$$A_2 = 1 - \frac{1}{2}\alpha^2 - \frac{1}{2}\theta_2(1 - 2\alpha^2) \ . \tag{25}$$

This shows that at this level θ_1 has no influence on the output.

If we are interested only in classical dead times, an additional simplification is possible. Putting

$$\eta = \begin{cases} 1 \ \text{ for E,} & \text{i.e. } \theta_2 = 1 \\ -1 \ \text{ " N,} & \text{i.e. } \theta_2 = 0 \end{cases} \tag{26}$$

the above relation can be brought into the simple form (always for $\alpha \leq 1/2$)

$$A_2 = \frac{1}{4}\left[3 - \eta\,(1\text{-}2\alpha^2)\right] \ . \tag{27}$$

It is easy to verify that (27) is capable of reproducing the coefficients A_2 listed in Tables 3 and 4. We may note, in passing, that this is already much more than what our models could achieve, as they failed for second order in three out of four arrangements.

Simplifications can also be made at third order if we restrict attention to $\alpha \leq 1/3$ and to the classical types. In particular, if θ_1 is assumed to have the value 1/2 (for instance out of our ignorance), one can find the expression

$$A_3 = \frac{1}{12}\{7 - (3+\alpha)\,\alpha^2 - \eta\left[5 - (9+\alpha)\,\alpha^2\right]\} \ , \tag{28a}$$

with η defined as above.

However, since accurate expressions are available in (16), (20) and (22), we should not expect to see the above approximations used frequently. In a way, this just confirms the popular wisdom that there is nothing more practical than a general solution and that one should be cautious in using shortcuts.

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