Measurement of a dead time by correlation techniques

Part I: For an extended dead time

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Abstract

After a short review of modulo 2 counting, this technique is applied to the measurement of a dead time which does not lend itself to the usual measuring methods. The approach is based on the known exact form of the Poisson law modified by the presence of an extended dead time.

1. Introduction

The use of correlation methods in nuclear physics, and in particular for problems related to radioactivity, is a rather well-established technique. For a short review, see e.g. [1] or [2]. The approach has been used since the early sixties, but infrequently and by a small number of laboratories. The reasons for the somewhat limited success of the method seem to be both experimental and theoretical, but the present limitations may be overcome by new developments. In a way, this report is a first step in this direction.

In the traditional application of the "correlation method" to activity measurements, the quantity measured is normally a covariance based on the number of pulses registered in two counting channels within a given time interval. Whenever the two series of events are correlated, for instance by the presence of pulses which have a common origin ("coincidences"), there exists a measurable correlation. In practice, the covariance is obtained by using a very large number of successive short measuring cycles. For the special situation where only a single series of pulses is available, the quantity measured and analyzed is a variance. In both cases the use of an on-line computer is necessary, but this is no longer an obstacle to the use of the technique. Much more serious is the fact that the corrections which account for dead time and parent-daughter decay are still known only essentially to first order [3]. This prevents an accurate analysis of correlation data obtained at high count rates.

A possible way to overcome these difficulties is based on a variant of the original version proposed by Landaud and Mabboux as early as 1960 [4]. It concerns a single counting channel and takes advantage of the simple characteristics of modulo 2 counting.

If the two states corresponding to a modulo 2 counter are designated, rather arbitrarily, by +1 and −1, the series of incoming events can be represented by an associated random function $x(t_o)$ which changes state at the arrival of each pulse. This is shown in Fig. 1.

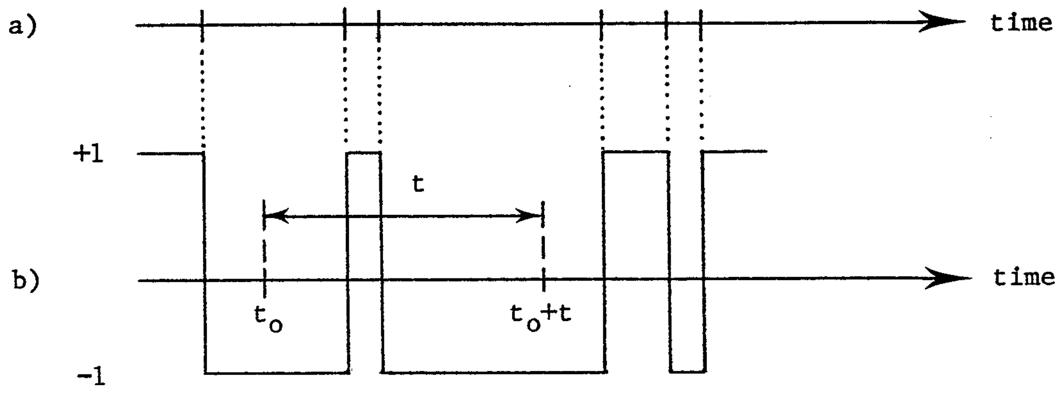

Fig. 1: Schematic representation of the arrival of random events (in a) and of the corresponding random function $x(t_o)$, with values ±1 (in b).

This type of "random telegraph signal" can readily be used to form a correlation function. For a given time interval t, the correlation function R(t) is defined by the expectation value

$$R(t) \quad = \quad E\{x(t_o)\ x(t_o+t)\} \ , \tag{1}$$

where t_o is an arbitrary starting time for the interval (see Fig. 1). If we realize that the product in (1) changes sign for each pulse arrival, we are led to

$$R(t) \quad = \quad \sum_{k=0}^{\infty} (-1)^k\ W_k(t) \ , \tag{2}$$

where $W_k(t)$ is the probability for observing k events within an interval of duration t. As in modulo 2 counting only the parities of k are relevant, the correlation function is also generally given by the relation

$$R(t) \quad = \quad \text{Prob}(k \text{ even}) - \text{Prob}(k \text{ odd}) \ . \tag{3a}$$

By putting

$$\text{Prob}(k \text{ odd}) \ \equiv \ \Pi(t) \ ,$$

which will be called parity function for short, one arrives, since any value of k is either even or odd, at the equivalent form

$$R(t) \quad = \quad 1 - 2\ \Pi(t) \ . \tag{3b}$$

This shows that, in the context of modulo 2 counting, determining a correlation function or a parity function is fully equivalent and requires the same measurements.

Since we are dealing here with original Poisson processes and distorted variants of them, it is clearly of interest to treat first the case of a simple Poisson process of rate ρ. It has long been known [5] that in this case the parity function and the correlation function take respectively the exact forms

$$\Pi(t) = \frac{1}{2}(1 - e^{-2\rho t}) \qquad \text{and} \tag{4a}$$

$$R(t) = e^{-2\rho t} . \tag{4b}$$

A derivation of this basic result is given in the Appendix.

2. Effect of a dead time

Obviously, it is of little consequence whether we deal with $\Pi(t)$ or with $R(t)$ in what follows. In view of the simple form of (4b) let us choose $R(t)$.

It was quickly recognized that the insertion of a dead time τ in the original Poisson process modifies the simple form (4b) of the correlation function. The problem was treated by Landaud [6] who found an approximation expressed by

$$R(t) = e^{-2\rho(1+\rho\tau)t} . \tag{5}$$

Simple checks of some limiting cases ($t = 0$, $t \to \infty$ or $\rho\tau \to 0$) lead to the expected results, but for values of t which are of the order of the dead time τ, the expression (5) does not look trustworthy. Indeed, one would expect some structure showing up which reflects the minimum allowed distance between pulses. As will be shown later even the first-order term in a series expansion for (5) is incorrect.

It is worthwhile to try to generalize (4b) for several reasons. In contrast to the situation in the sixties, rigorous expressions are now available for the modified Poisson probabilities $W_k(t)$ that k events occur in time t. This is true for both types of dead time. Provided that such formulae allow determinations of $R(t)$, a comparison with the directly measured value can be made. By expressing the original count rate ρ in terms of the measured one, the dead time becomes the sole unknown. An appropriate arrangement of the formulae should make it possible to use the measured value of $R(t)$ for a determination of the value of the inaccessible dead time τ, transforming a disturbing effect into a useful measuring technique. This is roughly the program we have in mind. Its realization depends, of course, on whether we succeed in carrying out the necessary algebra. The developments to be presented below show that this can be done.

For the case of an extended dead time the rearrangements are surprisingly simple, which is why we tackle this case first. The formal derivations necessary for a non-extended dead time are more involved and will be described in Part II.

3. Statistics for an extended dead time and evaluation of $\Pi(t)$

For a rigorous description of the situation it is obvious that we must return to the exact expressions which describe the statistics of a Poisson process of rate ρ which has been distorted by a dead time τ of the extended type.

The derivation of the required basic formula has a rather long and tortured history (for a summary see [7]); the final clarification is due to Libert [8].

Since in our experimental arrangement the periods of measurement begin independently of the arrival of pulses, we have what is described as an equilibrium process. In this case the probability of observing k events in a time interval t is given by

$$W_k(t) = \sum_{j=k}^{K+1} \frac{(-1)^{j-k}}{k!\,(j-k)!} (T_{j-1}\, e^{-x})^j , \tag{6}$$

where

$$T_n = \mu - nx ,$$

$$\text{with } \mu = \rho t \quad \text{and} \quad x = \rho\tau ,$$

while K is the largest integer below t/τ.

Since $W_k(t)$ is a function of the parameter K which, in turn, is related to the duration t of a time interval by

$$K\tau < t \leqslant (K+1)\tau ,$$

it is useful first to list some special cases of (6) in tabular form in ascending K. Table 1 shows this for values of K up to 3.

K	k	$W_k(t)$
0	0	$1 - \mu e^{-x}$
	1	μe^{-x}
1	0	$1 - \mu e^{-x} + \frac{1}{2} (\mu-x)^2\ e^{-2x}$
	1	$\mu e^{-x} - (\mu-x)^2\ e^{-2x}$
	2	$\frac{1}{2} (\mu-x)^2\ e^{-2x}$
2	0	$1 - \mu e^{-x} + \frac{1}{2} (\mu-x)^2\ e^{-2x} - \frac{1}{6} (\mu-2x)^3\ e^{-3x}$
	1	$\mu e^{-x} - (\mu-x)^2\ e^{-2x} + \frac{1}{2} (\mu-2x)^3\ e^{-3x}$
	2	$\frac{1}{2} (\mu-x)^2\ e^{-2x} - \frac{1}{2} (\mu-2x)^3\ e^{-3x}$
	3	$\frac{1}{6} (\mu-2x)^3\ e^{-3x}$
3	0	$1 - \mu e^{-x} + \frac{1}{2} (\mu-x)^2\ e^{-2x} - \frac{1}{6} (\mu-2x)^3\ e^{-3x} + \frac{1}{24} (\mu-3x)^4\ e^{-4x}$
	1	$\mu e^{-x} - (\mu-x)^2\ e^{-2x} + \frac{1}{2} (\mu-2x)^3\ e^{-3x} - \frac{1}{6} (\mu-3x)^4\ e^{-4x}$
	2	$\frac{1}{2} (\mu-x)^2\ e^{-2x} - \frac{1}{2} (\mu-2x)^3\ e^{-3x} + \frac{1}{4} (\mu-3x)^4\ e^{-4x}$
	3	$\frac{1}{6} (\mu-2x)^3\ e^{-3x} - \frac{1}{6} (\mu-3x)^4\ e^{-4x}$
	4	$\frac{1}{24} (\mu-3x)^4\ e^{-4x}$

Table 1 - Explicit form of the probabilities $W_k(t)$ given in (6), for measurement times t not exceeding four times the dead time τ.

From the formulae of Table 1 it is easy to obtain the values of the parity function $\Pi(t)$ for odd k. These are shown in Table 2.

K	$\Pi(t)$ = Prob(k odd)
0	$\mu\, e^{-x}$
1	$\mu\, e^{-x} - (\mu-x)^2\, e^{-2x}$
2	$\mu\, e^{-x} - (\mu-x)^2\, e^{-2x} + \frac{2}{3}\,(\mu-2x)^3\, e^{-3x}$
3	$\mu\, e^{-x} - (\mu-x)^2\, e^{-2x} + \frac{2}{3}\,(\mu-2x)^3\, e^{-3x} - \frac{1}{3}\,(\mu-3x)^4\, e^{-4x}$

Table 2 - Expressions for the parity function $\Pi(t)$ which gives the probability that k is odd in t, for $K \leqslant 3$.

It is not too difficult to deduce by induction from these results the general formula

$$\Pi(t) = \sum_{k=0}^{K} \alpha_k \left[(\mu-kx)\, e^{-x}\right]^{k+1} , \qquad (7a)$$

$$\text{with} \quad \alpha_k = \sum_{j=0}^{[k/2]} \frac{(-1)^k}{(1+2j)!\,(k-2j)!} .$$

The expression for α_k may be simplified. Using the relation

$$\sum_{j=0}^{[k/2]} \binom{k}{2j} \frac{1}{1+2j} = \frac{2^k}{k+1} ,$$

which follows from an identity given in [9], α_k can be brought into the form

$$\alpha_k = \frac{(-2)^k}{(k+1)!} . \qquad (7b)$$

Evaluation of the first coefficients yields

$\alpha_o = 1,$ $\qquad \alpha_1 = -1 ,$

$\alpha_2 = 2/3 ,$ $\qquad \alpha_3 = -1/3 ,$

$\alpha_4 = 2/15 ,$ $\qquad \alpha_5 = -2/45 .$

4. Evaluation of the correlation function

By substituting (7a) into (3b) we readily obtain for the correlation function

$$R(t) = 1 - 2 \sum_{k=0}^{K} \alpha_k \left[(\mu - kx)\, e^{-x}\right]^{k+1} . \tag{8}$$

We now wish to express the quantities μ and x appearing in (8) in terms of the experimentally measured mean value m_1. To do this it is helpful to count the measurement time in units of the dead time τ, the numerical value of which we want to determine, so we put $t = \nu\tau$.

Since the measured count rate is given by $r = \rho\, e^{-x}$, it follows that

$$\mu\, e^{-x} = \rho t\, e^{-x} = rt = m_1$$

and likewise

$$x\, e^{-x} = \frac{\tau}{t} m_1 = \frac{m_1}{\nu} .$$

The correlation function (8) can therefore be written as

$$R(t) = 1 - 2m_1 \sum_{j=0}^{K} \alpha_j\, m_1^j \left(1 - \frac{j}{\nu}\right)^{j+1} . \tag{9}$$

Expanding $R(t)$ as a power series in m_1 gives

$$R(t) = 1 - 2m_1 + \sum_{k=2}^{K+1} b_k\, m_1^k . \tag{10}$$

Comparison with (9) yields the coefficients b_k of lowest order

- for $k = 2$:

$$b_2 = -2\alpha_1 \left(1 - \frac{1}{\nu}\right)^2 = 2 \left(1 - \frac{1}{\nu}\right)^2 , \tag{11}$$

  provided $K \geq 1$, i.e. $\nu \geq 1$;

- for $k = 3$:

$$b_3 = -2\alpha_2 \left(1 - \frac{2}{\nu}\right)^3 = -\frac{4}{3} \left(1 - \frac{2}{\nu}\right)^3 , \tag{12}$$

  provided $K \geq 2$, i.e. $\nu \geq 2$;

- for $k = 4$:

$$b_4 = -2\alpha_3 \left(1 - \frac{3}{\nu}\right)^4 = \frac{2}{3} \left(1 - \frac{3}{\nu}\right)^4 , \tag{13}$$

  provided $K \geq 3$, i.e. $\nu \geq 3$;

- for $k = 5$:

$$b_5 = -2\alpha_4 (1 - \frac{4}{\nu})^5 = -\frac{4}{15} (1 - \frac{4}{\nu})^5 , \tag{14}$$

provided $K > 4$, i.e. $\nu > 4$.

Whenever the proviso is not respected, we have to put $b_k = 0$.

The influence of the dead time is best seen by comparing the measured correlation with the one that would correspond to an undisturbed Poisson process with the same count rate. In this case, i.e. for $\tau = 0$ and mean value m_1, the relation (4) allows us to expect for the correlation

$$R_o(t) = e^{-2m_1} \cong 1 - 2m_1 + 2m_1^2 - \frac{4}{3} m_1^3 + \frac{2}{3} m_1^4 - \frac{4}{15} m_1^5 . \tag{15}$$

Note that $R_o(t)$ requires for its evaluation only the knowledge of the measured value m_1.

We can now form the difference

$$\Delta(t) = R_o(t) - R(t) = \sum_{k=2}^{\infty} c_k m_1^k . \tag{16}$$

A comparison of (15) with (10) explains the absence of c_o and c_1 in (16). To find general expressions for the first coefficients c_k, term by term differences are formed using the values b_k, given in (11) to (14), and the corresponding coefficients in (15).

Up to fifth order, the new coefficients c_k are

$$c_2 = \begin{cases} 2 & , \text{ for } \nu \leq 1 , \\ 2 - b_2 = 2[1 - (1 - \frac{1}{\nu})^2] = \frac{2}{\nu^2}(2\nu - 1) & , \text{ " } \nu \geq 1 ; \end{cases} \tag{17}$$

$$c_3 = \begin{cases} -\frac{4}{3} & , \text{ for } \nu \leq 2 , \\ -\frac{4}{3} - b_3 = -\frac{4}{3}[1 - (1 - \frac{2}{\nu})^3] = \frac{-8}{3\nu^3}(3\nu^2 - 6\nu + 4) & , \text{ " } \nu \geq 2 ; \end{cases} \tag{18}$$

$$c_4 = \begin{cases} \frac{2}{3} & , \text{ for } \nu \leq 3 , \\ \frac{2}{3} - b_4 = \frac{2}{3}[1 - (1 - \frac{3}{\nu})^4] & \\ \quad = \frac{2}{\nu^4}(4\nu^3 - 18\nu^2 + 36\nu - 27) & , \text{ " } \nu \geq 3 ; \end{cases} \tag{19}$$

$$c_5 = \begin{cases} -\dfrac{4}{15} & , \text{ for } \nu \leqslant 4 , \\ -\dfrac{4}{15} - b_5 = -\dfrac{4}{15}\left[1 - (1 - \dfrac{4}{\nu})^5\right] & \\ = \dfrac{-16}{15\nu^5}(5\nu^4 - 40\nu^3 + 160\nu^2 - 320\nu + 256) & , \text{ " } \nu \geqslant 4 . \end{cases} \tag{20}$$

For convenience, some of the coefficients given by (17) to (20) are shown in numerical form in Table 3.

ν	c_2	c_3	c_4	c_5
$\leqslant$ 1.0	2.0000	- 1.3333	0.6667	- 0.2667
1.1	1.9835	- 1.3333	0.6667	- 0.2667
1.2	1.9444	- 1.3333	0.6667	- 0.2667
1.3	1.8935	- 1.3333	0.6667	- 0.2667
1.4	1.8367	- 1.3333	0.6667	- 0.2667
1.5	1.7778	- 1.3333	0.6667	- 0.2667
1.6	1.7188	- 1.3333	0.6667	- 0.2667
1.7	1.6609	- 1.3333	0.6667	- 0.2667
1.8	1.6049	- 1.3333	0.6667	- 0.2667
1.9	1.5512	- 1.3333	0.6667	- 0.2667
2.0	1.5000	- 1.3333	0.6667	- 0.2667
2.2	1.4050	- 1.3323	0.6667	- 0.2667
2.4	1.3194	- 1.3272	0.6667	- 0.2667
2.6	1.2426	- 1.3169	0.6667	- 0.2667
2.8	1.1735	- 1.3022	0.6667	- 0.2667
3.0	1.1111	- 1.2840	0.6667	- 0.2667
3.5	0.9796	- 1.2284	0.6664	- 0.2667
4.0	0.8750	- 1.1667	0.6641	- 0.2667
4.5	0.7901	- 1.1047	0.6584	- 0.2667
5.0	0.7200	- 1.0453	0.6496	- 0.2666
5.5	0.6612	- 0.9897	0.6382	- 0.2663
6.0	0.6111	- 0.9383	0.6250	- 0.2656
6.5	0.5680	- 0.8909	0.6106	- 0.2644
7.0	0.5306	- 0.8474	0.5956	- 0.2628
7.5	0.4978	- 0.8075	0.5803	- 0.2608
8.0	0.4688	- 0.7708	0.5649	- 0.2583

Table 3 - Numerical values of the coefficients c_2 to c_5 appearing in (16), for $\nu \leqslant 8$.

If m_1 is not sufficiently small compared to unity, higher values of k may be needed in (16). The coefficients c_k are given by the general formula

$$c_k = \frac{(-2)^k}{k!}\left[1 - \left(1 - \frac{k-1}{\nu}\right)_+^k\right], \tag{21}$$

where
$$(\alpha)_+ \equiv \begin{cases} 0 & \text{for } \alpha \leqslant 0 \\ \alpha & \text{" } \alpha > 0 . \end{cases} \tag{22}$$

Note that for $\nu \leqslant 1$ only a lower limit can be determined for the unknown dead time τ.

5. Measurement of the dead time

a) An approximate series expansion

The practical determination of the dead time may be based on the relation (16), in which R(t) is replaced by the experimental value R_{exp}.

For obvious reasons the experimental conditions must be chosen so that the mean value m_1 is small compared to unity, and the measurement time t (per cycle) should be of the same order as the unknown dead time τ.

A numerical example is useful to illustrate the procedure. Let us assume that the measured values are

$$t = 5.1\ \mu s, \qquad m_1 = 0.0973 \qquad \text{and} \qquad \Pi_{exp} = 0.0957. \tag{23}$$

With (3b) this corresponds to $R_{exp} = 0.8086$ and by means of (16) one finds for the experimental difference

$$\Delta_{exp} = e^{-2m_1} - R_{exp} = 0.014\ 56 .$$

If we restrict consideration to the first term in (16), a rough estimate of c_2 (in fact a lower limit) may be obtained from

$$c_2' = \Delta_{exp}/m_1^2 \cong 1.5 .$$

From Table 3 it follows that this implies $\nu \leqslant 2.0$.

For c_2 given by (17) the following table shows values of $\Delta(t)$ calculated by (16) for particular values of ν.

ν	$c_2 m_1^2$	$\Delta(t)$
1.60	0.016 27	0.015 10
1.65	0.016 00	0.014 83
1.70	0.015 72	0.014 55
1.75	0.015 46	0.014 29
1.80	0.015 19	0.014 02

In the range considered, c_3 and c_4 are constant and the contribution of c_5 is negligible. Comparison with Δ_{exp} leads to $\nu \cong 1.70$, which gives for the dead time, assumed extended, the value

$$\tau = t/\nu \cong 3.0 \ \mu s.$$

b) An exact evaluation

It is worthwhile to realize that, in the present case of an extended dead time, the series development just described is not necessary and that it can be replaced by a rigorous method. This solves the possible problems arising from poor convergence and truncation.

For this purpose let us start from (7a). We note that the parity function can also be expressed in the form

$$\Pi(t) = \sum_{k=0}^{K} \alpha_k \left[\left(1 - \frac{k}{\nu}\right) m_1\right]^{k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} \left[\left(1 - \frac{k\tau}{t}\right)_+ m_1\right]^{k+1} , \qquad (24)$$

where (22) guarantees that the sum in (24) is actually finite since for $k > K$ all contributions vanish. As the measured parity Π_{exp}, the mean value m_1 and the time interval t are all known from direct measurements, the dead time is the only unknown quantity and can therefore be determined from (24). If t does not exceed τ by a factor of more than about three, the numerical evaluation is simple and can readily be performed. On the other hand, (24) may be programmed and then yields τ directly.

In what follows this is illustrated numerically using again the data in (23). For the chosen trial values of τ we obtain the parities listed below.

τ (μs)	$\Pi(t)$, calculated from (24)
2.90	0.095 54
2.95	0.095 62
3.00	0.095 69
3.05	0.095 77
3.10	0.095 84

Comparison with the experimental value $\Pi_{exp} = 0.0957$ readily leads to $\tau \cong 3.0$ μs as the best estimate of the value of the inaccessible dead time (here assumed extended). This result agrees with that given in a) above.

The experimental uncertainty of the value of the dead time, determined by means of the correlation method, is essentially due to the limited precision of Π_{exp}. Its evaluation will be discussed when the results of Part II concerning non-extended dead times are available.

Appendix

Derivation of R(t) for a Poisson process

For a Poisson process with count rate ρ, the probability of observing k events in a time interval t is known to be given by

$$\mathrm{Prob}(k) = (\rho t)^k \frac{e^{-\rho t}}{k!}, \text{ with } k = 0, 1, \ldots . \tag{A1}$$

From this we obtain the probability for an even number of events as

$$\mathrm{Prob}(k \text{ even}) = e^{-\rho t} \sum_{r=0}^{\infty} \frac{(\rho t)^{2r}}{(2r)!} . \tag{A2}$$

For the evaluation of this sum let us have recourse to the identity

$$2 \sum_{r=0}^{\infty} \frac{(\rho t)^{2r}}{(2r)!} = \sum_{k=0}^{\infty} \frac{(\rho t)^k}{k!} + \sum_{k=0}^{\infty} \frac{(-\rho t)^k}{k!} , \tag{A3}$$

which is obviously true, since for k even both terms on the right-hand side are equal whereas they cancel for k odd. Thus we can now also write

$$\mathrm{Prob}(k \text{ even}) = e^{-\rho t} \frac{1}{2} (e^{\rho t} + e^{-\rho t}) = \frac{1}{2} (1 + e^{-2\rho t}) . \tag{A4a}$$

This result implies that, still for a Poisson process,

$$\mathrm{Prob}(k \text{ odd}) = \frac{1}{2} (1 - e^{-2\rho t}) . \tag{A4b}$$

For the correlation function R(t) this leads to

$$R(t) = 1 - 2\,\mathrm{Prob}(k \text{ odd}) = e^{-2\rho t} . \tag{A5a}$$

Since in our applications t is a time delay, it can also be negative. The corresponding general formula is then given by

$$R(t) = e^{-2\rho |t|} , \tag{A5b}$$

in full agreement with an earlier result [5].

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