Explicit evaluation of the transmission factor $T_1(\theta,E)$

Part II: For arbitrary dead-time ratios

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Abstract

We derive general formulae for the first coefficients which appear when the transmission factor T_1 for an arrangement of two dead times is expanded in a power series. This quantity T_1 describes the additional effect on the output count rate produced by a first dead time of general type preceding an extended second dead time.

1. Introduction

The basic evaluation of the transmission factor $T_1(\theta,E)$ in the form of a power-series expansion

$$T_1(\theta,E) \cong 1 + a_2 x^2 + a_3 x^3 + a_4 x^4 \qquad (1)$$

has been performed in [1], assuming that $k \leq K$ for a_k, where K is the largest integer below τ_2/τ_1, the ratio of the two dead times involved in a series arrangement. The quantity $x = \rho\tau_2$, where ρ is the input count rate, has to be assumed to be smaller than unity for a meaningful development (1).

As the relevant experimental situation is equivalent to the one described previously [1], it will not be repeated here. A clear understanding of what follows requires that the reader have a copy of the first part of this report at hand, to which we shall often refer. Both parts use the same notation and belong together.

The expressions derived in [1] for the coefficients a_k are therefore only applicable to those cases where $\alpha = \tau_1/\tau_2$ is sufficiently small. In particular, this excludes the situation where the first dead time is comparable to the second one. In order to fill this gap and render all values of α (between 0 and 1) feasible for useful measurements, the previous evaluation of $T_1(\theta,E)$ has to be somewhat modified. The generalization takes account of the finite value of K which effectively applies to a given experimental situation. This will result in a tabulation of the coefficients ${}_K a_k$, for $2 \leqslant k \leqslant 4$ and $K \geqslant 1$, similar to the ones listed recently in [2] and [3] for the series arrangements "N,N" and "E,E", or in an equivalent presentation.

2. Outline of the traditional approach

In the evaluation of $T_1(\theta,E)$, as described in [1], we were led to determine first a quantity which we called the loss L. It is given in (I/13)* as

$$L = \frac{1}{\theta} \sum_{j=1}^{J} p_j + \left(\frac{1-\theta}{\theta}\right) \sum_{j=1}^{J} q_j ,$$

in which two new quantities p_j and q_j appear.

Their definitions, stated in (I/14) and (I/16) as

$$p_j = \int_{j\alpha\tau}^{\tau} A_j \, dt \qquad \text{and} \qquad q_j = \int_{j\alpha\tau}^{\tau} B_j \, dt ,$$

show that the upper limit J has to be identified with K (explained in the introduction), because A_j (and thus p_j) would vanish for $j > K$, and likewise for B_j, as can be seen from (I/8).

Therefore, we now write for the loss more explicitly

$$L_K = \frac{1}{\theta} \sum_{j=1}^{K} p_j + \left(\frac{1-\theta}{\theta}\right) \sum_{j=1}^{K} q_j . \tag{2}$$

Obviously, it would be possible to evaluate successively L_K, for K = 1, 2, ..., as this has been done in [1] for $L = L_4$. Subsequently, these expressions could be used in the formula

$${}_K T_1(\theta,E) = \frac{r}{\rho} e^x (1 - L_K) , \tag{3}$$

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* The abbreviation (I/n) refers to equation (n) in [1].

which is equivalent to (I/23). From its series development

$$ {}_{K}T_1(\theta,E) \cong 1 + {}_{K}a_2\, x^2 + {}_{K}a_3\, x^3 + {}_{K}a_4\, x^4 \qquad (4)$$

we could finally obtain the coefficients ${}_{K}a_k$, valid for a series arrangement "θ,E", in which we are mainly interested.

Whereas in this way it would certainly be possible to determine ${}_{K}a_k$, at least in principle, the practical evaluation would turn out to be very cumbersome and therefore seriously subject to computational errors. It therefore seems preferable to follow another strategy which is both simpler and safer.

The basic idea is to proceed in reversed order, taking advantage of the fact that L_4 is already known from [1]. For the practical execution of this approach we first have to derive some recursion formulae.

3. Some useful recursion formulae

From the explicit expressions given in (I/18) and (I/20) we note that the lowest powers of x appearing in their series expansions are

$$ x^j \quad \text{for} \quad p_j \qquad \text{and} \qquad x^{j+1} \quad \text{for} \quad q_j \, . \qquad (5)$$

Hence, if we limit ourselves, as usual, to the fourth order in x, we have in fact in (2)

$$ L_4 \cong \frac{1}{\theta}(p_1 + p_2 + p_3 + p_4) + \left(\frac{1-\theta}{\theta}\right)(q_1 + q_2 + q_3) \, ,$$

and likewise (till x^4)

$$ L_3 \cong \frac{1}{\theta}(p_1 + p_2 + p_3) + \left(\frac{1-\theta}{\theta}\right)(q_1 + q_2 + q_3) \, .$$

Hence, one can also write

$$ L_3 \cong L_4 - \frac{1}{\theta} p_4 \cong L_4 + \frac{1}{24}(1-4\alpha)^4\, \theta^3\, x^4 \, , \qquad (6)$$

where use has been made of (I/18) for p_4.

This now allows us to evaluate ${}_3T_1(\theta,E)$ in a simple way from the known result $T_1(\theta,E) \equiv {}_4T_1(\theta,E)$, for substitution of (6) in (3) leads to

$$
\begin{aligned}
{}_3T_1(\theta,E) &\cong \frac{r}{\rho}\, e^x \left[1 - L_4 - \frac{1}{24}(1-4\alpha)^4\, \theta^3\, x^4\right] \\
&= {}_4T_1(\theta,E) - \frac{1}{24}(1-4\alpha)^4\, \theta^3\, x^4 \,, \qquad (7)
\end{aligned}
$$

always up to x^4, with $\frac{r}{\rho} e^x$ taken from (I/24).

For the coefficients appearing in the power series (4) it follows from (7) that ${}_3a_2 = {}_4a_2$ and ${}_3a_3 = {}_4a_3$, thus they are equal to the values given in (I/25), whereas

$$
\begin{aligned}
{}_3a_4 &= {}_4a_4 - \frac{\theta^3}{24}(1 - 4\alpha)^4 \\
&= \frac{1}{24}\left[(9 - 11\theta + 11\theta^2)\,\alpha^4 - \theta^3(1 - 4\alpha)^4\right] . \qquad (8)
\end{aligned}
$$

These results for $K = 3$ have been particularly simple to derive, but the approach can also be applied with advantage to $K = 2$ and even $K = 1$, as we are going to show.

Starting from the case $K = 3$, we have for the loss

$$L_2 = L_3 - \frac{1}{\theta}\left[p_3 + (1-\theta)\, q_3\right] . \qquad (9)$$

By means of (I/18) and (I/20) this leads for the difference (up to order x^4) to

$$L_2 - L_3 \cong -\frac{1}{6}(1-3\alpha)^3\,(\theta^2 x^3 - 3\alpha\theta^3 x^4) + \frac{1}{24}(1-\theta)\,\theta^2\,(1-3\alpha)^4\, x^4 \,,$$

which can be rearranged to

$$L_3 - L_2 \cong \frac{1}{24}\,\theta^2\,(1-3\alpha)^3\,\left\{4x^3 - \left[1-\theta - 3\alpha(1-5\theta)\right]\, x^4\right\} . \qquad (10)$$

According to (3) one can write

$$ {}_2T_1(\theta,E) = \frac{r}{\rho} e^x (1 - L_2) $$

$$ = {}_3T_1(\theta,E) + \frac{r}{\rho} e^x (L_3 - L_2) \; . \qquad (11) $$

Hence, with (I/24) and (10) the corrective term in (11) is given (up to order x^4) by

$$ \frac{r}{\rho} e^x (L_3 - L_2) $$

$$ \cong [1 + (1-\alpha)\, x] \frac{1}{24} \theta^2(1-3\alpha)^3 \{4 - [1-\theta-3(1-5\theta)\alpha]x\} x^3 $$

$$ = \frac{1}{24} \theta^2 (1-3\alpha)^3 \{4 + [3+\theta-(1+15\theta)\alpha]x\} x^3 \; . $$

Therefore (11) now leads us to

$$ {}_2T_1(\theta,E) \cong {}_3T_1(\theta,E) + \frac{1}{6} \theta^2(1-3\alpha)^3 x^3 $$

$$ + \frac{1}{24} \theta^2 [3+\theta - (1+15\theta)\alpha] (1-3\alpha)^3 x^4 \; . \qquad (12) $$

This yields for the coefficients

$$ {}_2a_3 = {}_3a_3 + \frac{1}{6} \theta^2 (1-3\alpha)^3 $$

$$ = - \frac{1}{6} [2(1-2\theta) \alpha^3 - \theta^2(1-3\alpha)^3] \; , \qquad (13) $$

since ${}_3a_3 = {}_4a_3$, and

$$ {}_2a_4 = {}_3a_4 + \frac{1}{24} \theta^2 [3 + \theta - (1+15\theta)\alpha] (1-3\alpha)^3 $$

$$ = \frac{1}{24} \{(9-11\theta+11\theta^2)\alpha^4 - \theta^3(1-4\alpha)^4 $$

$$ + \theta^2 [3+\theta - (1+15\theta)\alpha] (1-3\alpha)^3\} \; , \qquad (14) $$

where ${}_3a_4$ has been taken from (8).

Finally, there remains the case K = 1, which requires the longest calculations. We start from

$$L_1 = L_2 - \frac{1}{\theta}\left[p_2 + (1-\theta)\, q_2\right] . \tag{15}$$

Use of (I/18) and (I/20) gives for the difference (up to x^4)

$$L_1 - L_2 \cong \frac{1}{2}(1-2\alpha)^2\left[\theta x^2 - 2\alpha\theta^2 x^3 + 2\alpha^2\theta^3 x^4\right]$$
$$- \frac{1}{6}(1-\theta)\ \theta(1-2\alpha)^3\left[x^3 - \frac{1}{4}(1-2\alpha+8\theta\alpha)\ x^4\right] .$$

By grouping the powers of x and of α this can be brought into the form

$$L_1 - L_2 \cong \frac{1}{24}\theta(1-2\alpha)^2\left\{12x^2 - 4\left[1-\theta - 2(1-4\theta)\alpha\right] x^3\right.$$
$$\left. + \left[1-\theta - 4(1-3\theta+2\theta^2)\alpha + 4(1-5\theta+10\theta^2)\alpha^2\right] x^4\right\} . \tag{16}$$

Similarly to (11) we can also write

$${}_1T_1(\theta,E) = \frac{r}{\rho}e^x(1-L_1) = \frac{r}{\rho}e^x\left[1 - L_2 - (L_1-L_2)\right]$$
$$= {}_2T_1(\theta,E) - \frac{r}{\rho}e^x(L_1-L_2) . \tag{17}$$

For the evaluation of the second term in (17) up to x^4 we need the expansion of $(r/\rho)e^x$ to second order; from (I/24) this is

$$\frac{r}{\rho}e^x \cong 1 + (1-\alpha)\, x + \frac{1}{2}\left[1 - 2\alpha + (2-\theta)\alpha^2\right] x^2 .$$

Since L_1-L_2 is known from (16), the necessary multiplication can be performed. After arrangement of the terms this yields

$$\frac{r}{\rho}e^x(L_1 - L_2) \cong \frac{1}{24}\theta\,(1-2\alpha)^2\left\{12x^2 + 4\left[2+\theta - (1+8\theta)\alpha\right] x^3\right.$$
$$\left. + \left[3(1+\theta) - 4(1+6\theta+2\theta^2)\alpha + 2(4+3\theta+20\theta^2)\alpha^2\right] x^4\right\} . \tag{18}$$

Hence, with the coefficients for the expansion of ${}_2T_1(\theta,E)$ taken from (13) and (14), substitution into (17) yields for the coefficients with K = 1

$$ {}_1a_2 = {}_2a_2 - \frac{1}{24}\theta(1-2\alpha)^2\,12 = \frac{1}{2}\left[\alpha^2 - \theta(1-2\alpha)^2\right], \tag{19} $$

$$ {}_1a_3 = {}_2a_3 - \frac{1}{24}\theta\,(1-2\alpha)^2\,4\left[2+\theta - (1+8\theta)\alpha\right](1-2\alpha)^2 $$

$$ = -\frac{1}{6}\left\{2(1-2\theta)\alpha^3 - \theta^2(1-3\alpha)^3 + \theta\left[2+\theta - (1+8\theta)\alpha\right](1-2\alpha)^2\right\} \tag{20} $$

and finally also

$$ {}_1a_4 = {}_2a_4 - \frac{1}{24}\theta\,(1-2\alpha)^2\left[3(1+\theta) - 4(1+6\theta+2\theta^2)\alpha + 2(4+3\theta+20\theta^2)\alpha^2\right] $$

$$ \begin{aligned} = \frac{1}{24}\Big\{&(9-11\theta+11\theta^2)\alpha^4 - \theta^3(1-4\alpha)^4 \\ &+ \theta^2\left[3+\theta - (1+15\theta)\alpha\right](1-3\alpha)^3 \\ &- \theta\left[3(1+\theta) - 4(1+6\theta+2\theta^2)\alpha + 2(4+3\theta+20\theta^2)\alpha^2\right](1-2\alpha)^2\Big\}. \end{aligned} \tag{21} $$

4. Various checks

Some comparisons with known results may be useful. While those for $\theta = 0$ are not very useful, the case $\theta = 1$ is more informative. This corresponds to the arrangement "E,E" and

- for K = 3: $$ a_4 = -\frac{1}{24}(1-16\alpha+96\alpha^2-256\alpha^3+247\alpha^4)\ ; \tag{22} $$

- for K = 2: $$ \begin{aligned} a_3 &= \frac{1}{6}(1-9\alpha+27\alpha^2-25\alpha^3)\ , \\ a_4 &= \frac{1}{24}(3-36\alpha+156\alpha^2-284\alpha^3+185\alpha^4)\ ; \end{aligned} \tag{23} $$

- for K = 1: $$ \begin{aligned} a_2 &= -\frac{1}{2}(1-4\alpha+3\alpha^2)\ , \\ a_3 &= -\frac{1}{6}(2-12\alpha+21\alpha^2-11\alpha^3)\ , \\ a_4 &= -\frac{1}{24}(3-24\alpha+66\alpha^2-76\alpha^3+31\alpha^4)\ . \end{aligned} \tag{24} $$

All these results agree with the coefficients given in Table 1 of [1].

Considering the long chain of reasoning which was used in the previous section – starting from $L = L_4$, for which an expression was available from [1], we obtained successively formulae for L_3, L_2 and L_1, and from them the corresponding expansion coefficients for T_1 – it would no doubt be useful to have a serious check available for the last results (with K = 1). Indeed, it is not difficult to imagine that an error might have crept in somewhere and remained undetected.

Such a control is indeed possible since L_1 can also be obtained directly by evaluating

$$L_1 = \frac{1}{\rho}\left[p_1 + (1-\theta)\,q_1\right] . \tag{25}$$

Substitution of (I/18) and (I/20) gives (up to x^4)

$$\begin{aligned} L_1 \cong\ & (1-\alpha)\left[x - \alpha\theta x^2 + \frac{1}{2}\alpha^2\theta^2x^3 - \frac{1}{6}\alpha^3\theta^3x^4\right] \\ & - \frac{1}{2}(1-\theta)\,(1-\alpha)^2\left\{x^2 - \frac{1}{3}(1-\alpha+3\alpha\theta)\,x^3\right. \\ & \left. + \frac{1}{12}\left[(1-\alpha)^2 + 4\alpha(1-\alpha)\theta + 6\alpha^2\theta^2\right]x^4\right\} . \end{aligned}$$

This can equally be written in the form

$$\begin{aligned} L_1 =\ & (1-\alpha)\left\{x - \frac{1}{2}\left[1-\alpha - (1-3\alpha)\theta\right]x^2\right. \\ & + \frac{1}{6}\left[(1-\alpha)^2 - (1-\alpha)\,(1-4\alpha)\theta - 3\alpha(1-2\alpha)\theta^2\right]x^3 \\ & \left. - \frac{1}{24}\left[(1-\alpha)^3 - (1-\alpha)^2(1-5\alpha)\theta - 2\alpha(1-\alpha)(2-5\alpha)\theta^2 - 2\alpha^2(3-5\alpha)\theta^3\right]x^4\right\} . \end{aligned}$$

After some lengthy algebra this can also be arranged by powers of α. The result is

$$\begin{aligned} L_1 =\ & (1-\alpha)\,x - \frac{1}{2}\left[1 - \theta - 2(1-2\theta)\alpha + (1-3\theta)\alpha^2\right]x^2 \\ & + \frac{1}{6}\left[1 - \theta - 3(1-\theta)\alpha^2 + 3(1-3\theta+3\theta^2)\alpha^2 - (1-4\theta+6\theta^2)\alpha^3\right]x^3 \\ & - \frac{1}{24}\left[1 - \theta - 4(1-\theta)^2\alpha + 6(1-\theta)^3\alpha^2\right. \\ & \left. - 4(1-4\theta+6\theta^2-4\theta^3)\alpha^3 + (1-5\theta+10\theta^2-10\theta^3)\alpha^4\right]x^4 . \end{aligned} \tag{26}$$

According to (3) we have the relation

$$ {}_1T_1(\theta,E) = \frac{r}{\rho} e^x (1-L_1) , $$

where L_1 is now given by (23) while the series expansion for $(r/\rho)e^x$ can be taken from (I/24). Again the necessary multiplication is elementary, but tedious to perform. We confine ourselves to giving the result which can be brought, for example, into the form

$$ \begin{aligned} {}_1T_1(\theta,E) \cong 1 &- \frac{1}{2}\left[\theta - 4\theta\alpha - (1-4\theta)\alpha^2\right] x^2 \\ &- \frac{1}{6}\left[2\theta - 3\theta(3+\theta)\alpha + 3\theta(4+3\theta)\alpha^2 + (2-8\theta-5\theta^2)\alpha^3\right] x^3 \\ &- \frac{1}{24}\left[3\theta - 8\theta(2+\theta)\alpha + 6\theta(6+4\theta+\theta^2)\alpha^2 \right. \\ &\left. \quad - 4\theta(12+3\theta+4\theta^2)\alpha^3 - (9-43\theta+14\theta^2-11\theta^3)\alpha^4\right] x^4 . \end{aligned} \tag{27} $$

It can be verified by some additional rearrangements that (27) is indeed in agreement with the coefficients ${}_1a_k$ as given in (19) to (21). Obviously, also the checks for the cases $\theta = 0$ or 1 are in line with the developments which are known for a long time to hold for the corresponding series arrangements "N,E" and "E,E" of two traditional dead times.

5. The general coefficients for the arrangement "θ,E"

We now try to write the expansion coefficients a_k in a way which is valid for any value of the dead-time ratio α and where the quantity K can be dropped. This can be readily achieved if we use the expressions derived in section 3 (rather than the ones obtained in section 4 for checking purposes). In addition, it is practical to define the operation $(\ldots)_+$ by demanding that

$$ (z)_+ \equiv \begin{cases} z & \text{for } z > 0 \\ 0 & \text{"} \quad z < 0 . \end{cases} \tag{28} $$

This is in essence equivalent to the unit-step function which we used previously in similar contexts, but (28) has the advantage of notational simplicity. With this convention the first coefficients appearing in a series development (1) of $T_1(\theta,E)$ can now be written quite generally (for any value of α) as

$$a_2 = \frac{1}{2} \{\alpha^2 - \theta(2\alpha-1)_+^2\} , \tag{29}$$

$$a_3 = -\frac{1}{6} \{2(1-2\theta)\alpha^3 + \theta^2(3\alpha-1)_+^3 + \theta \left[2+\theta - (1+8\theta)\alpha\right] (2\alpha-1)_+^2\} , \tag{30}$$

$$\begin{aligned} a_4 = \frac{1}{24} \{ & (9-11\theta+11\theta^2)\alpha^4 - \theta^3(4\alpha-1)_+^4 - \theta^2 \left[3+\theta - (1+15\theta)\alpha\right] (3\alpha-1)_+^3 \\ & - \theta \left[3(1+\theta) - 4(1+6\theta+2\theta^2)\alpha + 2(4+3\theta+20\theta^2)\alpha^2\right] (2\alpha-1)_+^2\} . \end{aligned} \tag{31}$$

Use of the convention (28) allows us to drop the index K used in section 3. The expressions (29) to (31) clearly generalize the formula (I/25) given previously, which can be readily recovered from the new expressions by putting $\alpha \leqslant 1/4$, as in this case only the first terms in (29) to (31) remain whereas all the others vanish by virtue of (28).

It is instructive to see how the general formulae become progressively simpler as α diminishes, reaching their final expression for $\alpha \leqslant 1/k$.

The above results - in addition to their intrinsic interest - form an important intermediate step in our long-term effort for obtaining coefficients which are valid in the most general situation, namely when both dead times forming a series arrangement are of the generalized type.

References

[1] J.W. Müller: "Explicit evaluation of the transmission factor $T_1(\theta,E)$. Part I: For small dead-time ratios", Rapport BIPM-87/5 (1987)

[2] id.: "The transmission factor $T_1(N,N)$ revisited", Rapport BIPM-88/12 (1988)

[3] id.: "Some series expansions of T_1", Rapport BIPM-88/13 (1988)

(December 1988)