Some series expansions of T_1

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Abstract

For the series arrangement of two dead times, the effect of the (smaller) first element on the observed output count rate is quantitatively described by the transmission factor T_1. We give explicit power series for the three arrangements "E,E", "N,E" and "E,N" of two dead times of traditional type. Regularly the expressions turn out to be simpler for $1/T_1$ than for T_1.

1. Introduction

The series arrangement of two dead times - when each of them is of the classical type, hence either extended (E) or non-extended (N) -, gives rise to four different combinations. As the sequence "N,N", which turns out to be the most complicated one, has been treated before [1], there remain three cases to be considered here.

The relevant formulae for the output count rates (or equivalent quantities) have been well known for long (see e.g. [2]); the emphasis in this report will be laid on an overall view of some approximate series expansions. In what follows we shall always assume that the input events form a Poisson process with count rate ρ (cf. Fig. 1). It is practical to consider the influence of the first (shorter) dead time τ_1 as an additional distortion which modifies somewhat the main counting losses due to the second (longer) dead time τ_2. This can be done by the use of the transmission factors T.

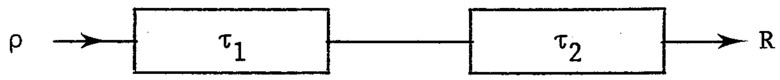

Fig. 1 - Schematic arrangement of two dead times in series (of any type), with ratio $\tau_1/\tau_2 \equiv \alpha \leqslant 1$. Input and output count rates are denoted by ρ and R.

They may be defined as follows. If τ_1 is absent (i.e. for $\alpha = 0$), we write for the output count rate

$$R = R_o = \rho T_2 , \qquad (1a)$$

whereas in the general situation we have

$$R = R_o T_1 = \rho T_2 T_1 . \qquad (1b)$$

Hence, while T_2 depends only on τ_2 (for a given input rate), T_1 accounts for the additional influence of τ_1 on the output rate R. Its numerical value is a function of α and of the types involved; it turns out that T_1 can be smaller or larger than unity.

2. The transmission factor $T_1(E,E)$

For the series arrangement of two extended dead times, the transmission factor is known to be given by (cf. [2], eq. 53)

$$_K T_1(E,E) = e^{-\alpha x} \sum_{k=0}^{K} \{[-(1-k\alpha)x]^k/k!\} \, e^{(1-k\alpha)x}$$

$$= e^{(1-\alpha)x} \sum_{k=0}^{K} \{\frac{1}{k!} [-(1-k\alpha)x]^k\} \, e^{-k\alpha x} , \qquad (2)$$

where, as previously, $x = \rho\tau_2$ and K is defined by the inequality $K\tau_1 < \tau_2 \leqslant (K+1)\tau_1$. Thus, K is the largest integer below τ_2/τ_1.

A derivation of (2) can be found in [3].

Since (2) depends on K, different formulae are applicable for the various ranges of α. The situation is therefore similar to the one we met for the series arrangements of type "N,N".

The easiest case corresponds to K = 1, i.e. for the range $1/2 \leqslant \alpha \leqslant 1$, where (2) readily leads to

$$_1 T_1(E,E) = e^{(1-\alpha)x} \, [1 - (1-\alpha)x \, e^{-\alpha x}] . \qquad (3)$$

For K > 1 it is practical to develop first from (2) a recursion formula, for instance of the type

$$_K T_1(E,E) = {}_{K-1} T_1(E,E) + T_1^{(K)} ,$$

with $$T_1^{(K)} = \frac{1}{K!} e^{(1-\alpha)x} \, [-(1-K\alpha)x]^K \, e^{-K\alpha x} , \qquad (4a)$$

or likewise, for $K \geqslant 1$,

$$ {}_K T_1(E,E) = \sum_{k=0}^{K} T_1^{(k)} , $$

with
$$ T_1^{(k)} = \frac{(-1)^k}{k!} \left[(1-k\alpha)x\right]^k e^{[1-(k+1)\alpha]x} . \qquad (4b) $$

3. Series developments

If we limit ourselves to power expansions in x of fourth order, for instance, then the explicit development of the terms appearing in (4b) yields

$$ T_1^{(0)} = e^{(1-\alpha)x} $$
$$ \cong 1 + (1-\alpha)\, x + \frac{1}{2}(1-\alpha)^2\, x^2 + \frac{1}{6}(1-\alpha)^3\, x^3 + \frac{1}{24}(1-\alpha)^4\, x^4 , $$

$$ T_1^{(1)} = -(1-\alpha)x\, e^{(1-2\alpha)x} $$
$$ \cong -(1-\alpha)\, x - (1-\alpha)(1-2\alpha)\, x^2 - \frac{1}{2}(1-\alpha)(1-2\alpha)^2\, x^3 - \frac{1}{6}(1-\alpha)(1-2\alpha)^3\, x^4 , $$

$$ T_1^{(2)} = \frac{1}{2}(1-2\alpha)^2 x^2\, e^{(1-3\alpha)x} \qquad (5) $$
$$ \cong \frac{1}{2}(1-2\alpha)^2\, x^2 + \frac{1}{2}(1-2\alpha)^2(1-3\alpha)\, x^3 + \frac{1}{4}(1-2\alpha)^2(1-3\alpha)^2\, x^4 , $$

$$ T_1^{(3)} = -\frac{1}{6}(1-3\alpha)^3 x^3\, e^{(1-4\alpha)x} $$
$$ \cong -\frac{1}{6}(1-3\alpha)^3\, x^3 - \frac{1}{6}(1-3\alpha)^3(1-4\alpha)\, x^4 , $$

$$ T_1^{(4)} = \frac{1}{24}(1-4\alpha)^4 x^4\, e^{(1-5\alpha)x} $$
$$ \cong \frac{1}{24}(1-4\alpha)^4\, x^4 . $$

According to (4b) the expressions for ${}_K T_1(E,E)$ are simply obtained by summation of the terms given in (5); they are assembbled (up to fourth order) in Table 1. Likewise we tabulate the first coefficients for the corresponding reciprocal series in Table 2.

Table 1 - Series expansions of the transmission factors
$_KT_1(E,E) \cong 1 + a_2x^2 + a_3x^3 + a_4x^4$

K	a_2	a_3	a_4
1	$-\frac{1}{2}(1-\alpha)(1-3\alpha)$	$-\frac{1}{6}(1-\alpha)(2-10\alpha+11\alpha^2)$	$-\frac{1}{24}(1-\alpha)(3-21\alpha+45\alpha^2-31\alpha^3)$
2	$\frac{1}{2}\alpha^2$	$\frac{1}{6}(1-9\alpha+27\alpha^2-25\alpha^3)$	$\frac{1}{24}(3-36\alpha+156\alpha^2-284\alpha^3+185\alpha^4)$
3	$\frac{1}{2}\alpha^2$	$\frac{1}{3}\alpha^3$	$-\frac{1}{24}(1-16\alpha+96\alpha^2-256\alpha^3+247\alpha^4)$
$\geqslant 4$	$\frac{1}{2}\alpha^2$	$\frac{1}{3}\alpha^3$	$\frac{3}{8}\alpha^4$

Table 2 - Series expansions of the reciprocal transmission factors
$_KT_1^{-1}(E,E) \cong 1 + b_2x^2 + b_3x^3 + b_4x^4$

K	b_2	b_3	b_4
1	$\frac{1}{2}(1-\alpha)(1-3\alpha)$	$\frac{1}{6}(1-\alpha)(2-10\alpha+11\alpha^2)$	$\frac{1}{24}(1-\alpha)(9-63\alpha+135\alpha^2-85\alpha^3)$
2	$-\frac{1}{2}\alpha^2$	$-\frac{1}{6}(1-9\alpha+27\alpha^2-25\alpha^3)$	$-\frac{1}{24}(3-36\alpha+156\alpha^2-284\alpha^3+179\alpha^4)$
3	$-\frac{1}{2}\alpha^2$	$-\frac{1}{3}\alpha^3$	$\frac{1}{24}(1-16\alpha+96\alpha^2-256\alpha^3+253\alpha^4)$
$\geqslant 4$	$-\frac{1}{2}\alpha^2$	$-\frac{1}{3}\alpha^3$	$-\frac{1}{8}\alpha^4$

Again some simple checks can be made at the borderlines of the domains. Thus

$$\text{- for } \alpha = \frac{1}{2}: \quad {}_1T_1 = {}_2T_1 \cong 1 + \frac{1}{8}x^2 + \frac{1}{48}x^3 + \frac{1}{384}x^4$$
$$\text{and} \quad {}_1T_1^{-1} = {}_2T_1^{-1} \cong 1 - \frac{1}{8}x^2 - \frac{1}{48}x^3 + \frac{5}{384}x^4 , \tag{6a}$$

$$\text{- for } \alpha = \frac{1}{3}: \quad {}_2T_1 = {}_3T_1 \cong 1 + \frac{1}{18}x^2 + \frac{1}{81}x^3 + \frac{1}{243}x^4$$
$$\text{and} \quad {}_2T_1^{-1} = {}_3T_1^{-1} \cong 1 - \frac{1}{18}x^2 - \frac{1}{81}x^3 + \frac{1}{972}x^4 , \tag{6b}$$

$$\text{- for } \alpha = \frac{1}{4}: \quad {}_3T_1 = {}_4T_1 \cong 1 + \frac{1}{32}x^2 + \frac{1}{192}x^3 + \frac{3}{2\,048}x^4$$
$$\text{and} \quad {}_3T_1^{-1} = {}_4T_1^{-1} \cong 1 - \frac{1}{32}x^2 - \frac{1}{192}x^3 + \frac{1}{2\,048}x^4 . \tag{6c}$$

This lends support to the correctness of the coefficients listed in Tables 1 and 2.

4. Limiting forms

For the sake of curiosity one might wonder what the series development of ${}_KT_1(E,E)$, or of its reciprocal, looks like if we may assume that $K \gg 1$. For this purpose we have evaluated the transmission factor up to order 8 in x (assuming $K \geqslant 8$), with the result that

$$T_1(E,E) \cong 1 + \frac{1}{2}(\alpha x)^2 + \frac{1}{3}(\alpha x)^3 + \frac{3}{8}(\alpha x)^4 + \frac{11}{30}(\alpha x)^5 + \frac{53}{144}(\alpha x)^6 + \frac{103}{280}(\alpha x)^7 + \frac{2\,119}{5\,760}(\alpha x)^8 , \tag{7}$$

and likewise for its reciprocal

$$T_1^{-1}(E,E) \cong 1 - \frac{1}{2}(\alpha x)^2 - \frac{1}{3}(\alpha x)^3 - \frac{1}{8}(\alpha x)^4 - \frac{1}{30}(\alpha x)^5 - \frac{1}{144}(\alpha x)^6 - \frac{1}{840}(\alpha x)^7 - \frac{1}{5\,760}(\alpha x)^8 . \tag{8}$$

The development (8) confirms our earlier observation (made in [1]) that the series expansion is simpler for $1/T_1$ than for T_1, and it is not too hard to see that apparently

$$_K T_1^{-1}(E,E) \cong 1 - \sum_{k=2}^{K} \frac{(\alpha x)^k}{k(k-2)!} , \qquad (9)$$

for a development up to order x^K.

The general expression corresponding to (7) is somewhat more complicated, but can be found to be given by

$$_K T_1(E,E) \cong 1 + \sum_{k=2}^{K} \frac{(-\alpha x)^k}{k!} \sum_{j=0}^{k} \binom{k}{j} (-j)^j (j+1)^{k-j} . \qquad (10)$$

5. Series arrangements of different types

Dead times of the types "N,E" and "E,N" give rise to transmission factors which have a simple structure and are valid for the whole domain of the ratio α (see [2], eqs. 55 and 57). Again we are primarily interested here in their series developments. The basic formulae for both cases have been derived in [4].

a) The case "N,E"

For this series arrangement the exact formula for the transmission factor is known to be

$$T_1(N,E) = \frac{e^{\alpha x}}{1 + \alpha x} . \qquad (11)$$

A straightforward expansion, for instance to order 6, leads to the expression

$$T_1(N,E) \cong 1 + \frac{1}{2}(\alpha x)^2 - \frac{1}{3}(\alpha x)^3 + \frac{3}{8}(\alpha x)^4 - \frac{11}{30}(\alpha x)^5 + \frac{53}{144}(\alpha x)^6 . \qquad (12)$$

For the reciprocal this gives

$$T_1^{-1}(N,E) \cong 1 - \frac{1}{2}(\alpha x)^2 + \frac{1}{3}(\alpha x)^3 - \frac{1}{8}(\alpha x)^4 + \frac{1}{30}(\alpha x)^5 - \frac{1}{144}(\alpha x)^6 . \qquad (13)$$

The corresponding general formulae are

$$T_1(N,E) = 1 + \sum_{k=2}^{\infty} (-\alpha x)^k \sum_{j=2}^{k} \frac{(-1)^j}{j!} \quad (14)$$

and

$$T_1^{-1}(N,E) = 1 - \sum_{k=2}^{\infty} \frac{(-\alpha x)^k}{k(k-2)!}\,, \quad (15)$$

as can be easily verified. Again T_1^{-1} leads to a simpler expansion.

b) The case "E,N"

For this arrangement the exact formula for the transmission factor is

$$T_1(E,N) = \frac{1 + x}{(1-\alpha)x + e^{\alpha x}}\,, \quad (16)$$

and a series development to order 6 leads to

$$\begin{aligned} T_1(E,N) \cong\ & 1 - \frac{1}{2}\alpha^2 x^2 + \frac{1}{6}\alpha^2 (3-\alpha) x^3 - \frac{1}{24}\alpha^2 (12-4\alpha-5\alpha^2) x^4 \\ & + \frac{1}{120}\alpha^2 (60-20\alpha-55\alpha^2+19\alpha^3) x^5 \\ & - \frac{1}{720}\alpha^2 (360-120\alpha-510\alpha^2+234\alpha^3+41\alpha^4) x^6 \,. \end{aligned} \quad (17)$$

The analogous expression for the reciprocal turns out to be

$$\begin{aligned} T_1^{-1}(E,N) \cong\ & 1 + \frac{1}{2}\alpha^2 x^2 - \frac{1}{6}\alpha^2 (3-\alpha) x^3 + \frac{1}{24}\alpha^2 (12-4\alpha+\alpha^2) x^4 \\ & - \frac{1}{120}\alpha^2 (60-20\alpha+5\alpha^2-\alpha^3) x^5 \\ & + \frac{1}{720}\alpha^2 (360-120\alpha+30\alpha^2-6\alpha^3+\alpha^4) x^6 \,. \end{aligned} \quad (18)$$

For the reciprocal series this corresponds to the general expansion formula

$$T_1^{-1}(E,N) = 1 + \sum_{k=2}^{\infty} (-x)^k \sum_{j=2}^{k} \frac{(-\alpha)^j}{j!}\,, \quad (19)$$

whereas no simple formula has been found for the series $T_1(E,N)$.

For the special choice $\alpha = 1$ the above developments yield

$$T_1(E,N) = T_1^{-1}(N,E) \cong 1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{30}x^5 - \frac{1}{144}x^6$$

and (20)

$$T_1(N,E) = T_1^{-1}(E,N) \cong 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{8}x^4 - \frac{11}{30}x^5 + \frac{53}{144}x^6 ,$$

which agrees with known [5] exact relations and also with the fact that, for $\alpha = 1$,

$$T_1(E,N) = (1+x)\,e^{-x} = 1 - \sum_{k=2}^{\infty} \frac{(-x)^k}{k(k-2)!} . \qquad (21)$$

We would like to mention that some series developments (to second order in x) have also been indicated by Funck [6] for all four series arrangements, assuming, in general, that $1/2 \leqslant \alpha \leqslant 1$. They can be easily recovered, and thus confirmed, from the expansions given above. In view of the uncertainties resulting from an experimental measurement of type and length of the first dead time τ_1 it will be advantageous, however, to apply the appropriate transmission factor T_1 for a smaller value of the dead-time ratio α.

References

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