Generalized normalizing transformation for a distorted Poisson distribution

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Abstract

We evaluate the asymptotic form of the normalizing transform applicable to counting data resulting from a Poisson process which has been distorted by a dead time of the generalized type. For the two limiting cases, which correspond to the traditional types, the recently derived expressions are recovered and recognized as exact.

1. Introduction

The transformation of random variables is a subject dealt with in detail in most textbooks on mathematical statistics, as it provides a basic and useful tool for the handling of many practical situations. The closely related field of so-called normalizing transformations, on the other hand, seems to be clearly less favoured by the writers of introductory texts, although, of course, the specialized literature is far from being scarce. Apart from their obvious practical usefulness, these transforms may also give rise to some interesting problems that occur in their search.

A normalizing transform is constructed for the purpose of transforming a given, usually asymetric distribution of a random quantity into another one in such a way that the distribution for the new variable approximates as closely as possible to a normal (or Gaussian) distribution. Thus, for a given X we try to find a function g(X) so that

$$Y = g(X) \cong \text{normal} , \qquad (1)$$

for instance asymptotically for $X \gg 1$.

The practical advantage then obviously is that the well-known testing "machinery" developed for the normal distribution can be used for Y, at least in an approximate way, since it turns out that exact normalizing transformations only exist for some simple special cases.

In a recent article M.C. Teich [1] has described in a clear and detailed way how one can obtain normalizing transformations for a dead-time-distorted Poisson process, and much of what follows is taken from or inspired by his paper.

Let us consider a random variable X, with expectation value μ and variance σ^2. If σ can be expressed as a function of μ, such that $\sigma = f(\mu)$ is known explicitly, then it is possible to derive an approximate normalizing transformation by evaluating the integral

$$Y = g(X) \cong c \int_o^{\mu} \frac{d\mu}{f(\mu)} \Bigg|_{\mu=X} , \qquad (2) = (T\ 2)^*$$

where c is a constant.

In the following section we first summarize the results obtained by Teich in [1], as they will be of interest in the discussion of the more general result that we shall derive in section 3.

2. Transforms for the traditional types of dead time

a) For the case of a Poisson process modified by a non-extended dead time (n), the first few moments of the counting distribution are well known (see [1] for references). By neglecting the contributions which depend only on the initial conditions (choice of time origin), we have the relations (with $x = \rho\tau$)

$$\mu_n = \frac{\rho t}{1 + x} \quad \text{and} \qquad \sigma_n^2 = \frac{\rho t}{(1 + x)^3} . \qquad (3)$$

From this one can obtain for the required function f the linear relation

$$\sigma_n = f(\mu_n) \cong \sqrt{\mu_n}\,(1 - \mu_n\, \tau/t) . \qquad (4) = (T\ 5)$$

Substitution into (2) leads (in the notation of [1]) to

$$Y = g_n(X) \cong 2ca^2 \int_o^X \frac{dw}{a^2 - w^2}$$

$$= 2ca \operatorname{arctgh} \sqrt{X}/a . \qquad (5) = (T\ 6,7)$$

where $w = \sqrt{\mu_n}$ and $a = \sqrt{t/\tau}$.

If (5) is taken at its face value and developed into a power series, we arrive at

$$g_n(X) = 2c \sqrt{X} \sum_{k=o}^{\infty} \frac{1}{2k+1} (\eta X)^k , \qquad (6)$$

where $\eta = \tau/t$.

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* The label T refers to the corresponding reference in [1].

b) For an extended dead time (e), the first two moments of the counting distribution are (again with $x = \rho\tau$)

$$\mu_e = \rho t\, e^{-x} \quad \text{and}$$
$$\sigma_e^2 = \left(\frac{e^x - 2x}{e^{2x}}\right) \rho t \, . \tag{7}$$

In this case, the linear relation given in [1] is

$$\sigma_e = f(\mu_e) \cong \sqrt{\mu_e}\,\sqrt{1 - 2\mu_e\, \tau/t} \, . \qquad (8) = (\text{T } 12)$$

Substitution into (2) then leads to (in the notation of [1])

$$Y = g_e(X) \cong 2cb \int_o^X \frac{du}{\sqrt{b^2 - u^2}}$$
$$= 2cb \arcsin \sqrt{X}/b \, , \qquad (9) = (\text{T } 13,\ 14)$$

where $u = \sqrt{\mu_e}$ and $b = \sqrt{t/2\tau}$.

By performing again a series development we arrive at

$$g_e(X) = 2c\, \sqrt{X} \sum_{k=o}^{\infty} \frac{(2k-1)!!}{(2k+1)\ k!} (\eta X)^k \, . \tag{10}$$

3. <u>The transform for a generalized dead time</u>

As it has already been noted at the end of [1], it might be of interest to examine the normalizing transformation applicable to a generalized dead time. This is known to be characterized by a parameter $0 \leqslant \theta \leqslant 1$, with the limiting cases corresponding to the traditional types.

As the necessary basic formulae for the moments have been available to us for quite some time (the relevant expressions for the variance, dated March 84 in my files, are still unpublished), it seemed worth while to try to make use of them for the present calculations.

The asymptotic first and second moments for the counting distribution of a Poisson process which has been distorted by a generalized dead time (τ, θ) can be shown to be given by the expressions

$$\mu = \frac{\theta\rho t}{e^{\theta x} + \theta - 1} \quad \text{and}$$
$$\sigma^2 = \frac{\left[e^{\theta x}(e^{\theta x} - 2\theta x) + \theta^2 - 1\right] \theta\rho t}{(e^{\theta x} + \theta - 1)^3} \, . \tag{11}$$

It is easy to verify that for $\theta = 1$ this leads to (7), whereas for $\theta = 0$, after an elementary development, one finds indeed (3).

In order to express σ in terms of μ, as required by (2), we have to go into some lengthy series developments. The starting point is (11), written in the form

$$\frac{\sigma}{\sqrt{\mu}} = \frac{\left[e^{2\theta x} - 2\theta x\, e^{\theta x} + \theta^2 - 1\right]^{1/2}}{e^{\theta x} + \theta - 1} \equiv \frac{B^{1/2}}{A} .$$

Let us perform the power-series development up to order x^5, for example. This then first leads to

$$A = 1 + x + \frac{1}{2}\,\theta x^2 + \frac{1}{6}\,\theta^2 x^3 + \frac{1}{24}\,\theta^3 x^4 + \frac{1}{120}\,\theta^4 x^5$$

and

$$B = 1 + \frac{1}{3}\,\theta x^3 + \frac{1}{3}\,\theta^2 x^4 + \frac{11}{60}\,\theta^3 x^5 .$$

In view of the form of (2), we aim at a series developments which is easy to integrate. For this purpose we evaluate $B^{-1/2}$. Since the currently available formulae for doing this directly do not go to the required order, the rearrangement will be performed in two steps by means of the expressions given in [2]. This first leads to

$$B^{1/2} \cong 1 + \frac{1}{6}\,\theta x^3 + \frac{1}{6}\,\theta^2 x^4 + \frac{11}{120}\,\theta^3 x^5 ,$$

and then to

$$B^{-1/2} \cong 1 - \frac{1}{6}\,\theta x^3 - \frac{1}{6}\,\theta^2 x^4 - \frac{11}{120}\,\theta^3 x^5 .$$

Hence, we arrive at the intermediate result

$$\begin{aligned} \frac{\sqrt{\mu}}{\sigma} &= A\, B^{-1/2} \\ &\cong 1 + x + \frac{1}{2}\,\theta x^2 - \frac{1}{6}\,\theta(1-\theta)x^3 \\ &\quad - \frac{1}{24}\,\theta(4 + 4\theta - \theta^2)x^4 - \frac{1}{120}\,\theta^2\,(30 + 11\theta - \theta^2)x^5. \end{aligned} \tag{12}$$

We now have to express x in terms of μ. In order to do this, we start from the expectation value given in (11) and write it in the form

$$\mu \cong \frac{t}{\tau}\,\frac{x}{1 + x + \frac{1}{2}\,\theta x^2 + \frac{1}{6}\,\theta^2 x^3 + \frac{1}{24}\,\theta^3 x^4}$$

or also, with the abbreviation $\eta = \tau/t$, as

$$\mu \cong \frac{x}{\eta}\,[1 - x + (1 - \frac{1}{2}\,\theta)\,x^2 - (1 - \theta + \frac{1}{6}\,\theta^2)\,x^3 + (1 - \frac{3}{2}\,\theta + \frac{7}{12}\,\theta^2 - \frac{1}{24}\,\theta^3)\,x^4]\,. \tag{13}$$

For the reversion of this series we try the ansatz

$$x = \eta\mu[1 + a_1(\eta\mu) + a_2(\eta\mu)^2 + a_3(\eta\mu)^3 + a_4(\eta\mu)^4]\,. \tag{14}$$

The unknown coefficients can be determined successively - the procedure is very similar to the one we recently applied in [3] - and they are found to be

$$a_1 = 1, \quad a_2 = 1 + \frac{1}{2}\,\theta, \quad a_3 = 1 + \frac{3}{2}\,\theta + \frac{1}{6}\,\theta^2,$$
$$a_4 = 1 + 3\theta + \frac{7}{6}\,\theta^2 + \frac{1}{24}\,\theta^3\,.$$

A closer look at these coefficients suggests the general form

$$a_k = \sum_{j=0}^{k-1} \frac{S(k,\,k-j)}{(j+1)!}\,\theta^j\,, \qquad \text{for } k \geqslant 1\,, \tag{15}$$

where $S(k,\,k-j)$ are Stirling numbers of the second kind.

Hence, for x and its required powers this results in (by putting $\eta\mu = z$), always up to fifth order,

$$\begin{aligned}
x &\cong z[1 + z + (1 + \frac{1}{2}\,\theta)z^2 + (1 + \frac{3}{2}\,\theta + \frac{1}{6}\,\theta^2)z^3 \\
&\qquad + (1 + 3\theta + \frac{7}{6}\,\theta^2 + \frac{1}{24}\,\theta^3)z^4]\,, \\
x^2 &\cong z^2\,[1 + 2z + (3 + \theta)z^2 + (4 + 4\theta + \frac{1}{3}\,\theta^2)z^3]\,, \\
x^3 &\cong z^3\,[1 + 3z + (6 + \frac{3}{2}\,\theta)z^2]\,, \\
x^4 &\cong z^4\,[1 + 4z]\,, \\
x^5 &\cong z^5\,.
\end{aligned} \tag{16}$$

Substitution of these approximations into (12) leads to a lengthy expression which, however, can be rearranged into the form

$$\frac{\sqrt{\mu}}{\sigma} \cong 1 + z + (1 + \frac{1}{2}\theta)z^2 + (1 + \frac{4}{3}\theta + \frac{1}{6}\theta^2)z^3$$
$$+ (1 + \frac{7}{3}\theta + \theta^2 + \frac{1}{24}\theta^3)z^4 \qquad (17)$$
$$+ (1 + \frac{10}{3}\theta + 3\theta^2 + \frac{8}{15}\theta^3 + \frac{1}{120}\theta^4)z^5 .$$

Since $\frac{1}{\sigma} = \frac{1}{f(\mu)}$ is the form we need for the integration required in (2), the evaluation of the normalizing transform looked for is now easy to perform. We thus find (with $\eta = \tau/t$)

$$g(X) \cong 2c\sqrt{X}\,[1 + \frac{1}{3}\eta X + \frac{1}{5}(1 + \frac{1}{2}\theta)(\eta X)^2$$
$$+ \frac{1}{7}(1 + \frac{4}{3}\theta + \frac{1}{6}\theta^2)(\eta X)^3$$
$$+ \frac{1}{9}(1 + \frac{7}{3}\theta + \theta^2 + \frac{1}{24}\theta^3)(\eta X)^4 \qquad (18)$$
$$+ \frac{1}{11}(1 + \frac{10}{3}\theta + 3\theta^2 + \frac{8}{15}\theta^3 + \frac{1}{120}\theta^4)(\eta X)^5] .$$

This is clearly the main result of the present study. It is interesting to note that the type parameter θ does not come into play before second order, as it might have been expected.

4. Discussion

The limiting cases of (18) are of special interest as they correspond to the types usually considered. They give rise to the following expressions

- for $\theta = 0$, i.e. a non-extended dead time:

$$g_n(X) \cong 2c\sqrt{X}\,[1 + \frac{1}{3}\eta X + \frac{1}{5}(\eta X)^2 + \frac{1}{7}(\eta X)^3 + \frac{1}{9}(\eta X)^4 + \frac{1}{11}(\eta X)^5], \qquad (19)$$

- for $\theta = 1$, i.e. an extended dead time:

$$g_e(X) \cong 2c\sqrt{X}\,[1 + \frac{1}{3}\eta X + \frac{3}{10}(\eta X)^2 + \frac{5}{14}(\eta X)^3 + \frac{35}{72}(\eta X)^4 + \frac{63}{88}(\eta X)^5]. \qquad (20)$$

A comparison of these results with the relations (6) and (10), which are based on the formulae given in [1], show that they are in complete agreement for all the orders considered. This is quite surprising since the formulae (4) and (8), on which the determination of the normalizing transformations is based in [1], were supposed to be linear approximations only. Our observation therefore strongly suggests them to be rigorously valid, and this can be verified in the following way.

a) for equation (4):
From (3) we have the relation

$$\sigma_n^2 = \frac{\mu_n}{(1+x)^2} = \mu_n(1 - \frac{x}{1+x})^2 = \mu_n(1 - \frac{\rho t \ \tau/t}{1+x})^2 ,$$

thus

$$\sigma_n^2 = \mu_n(1 - \mu_n \ \tau/t)^2 ; \tag{21}$$

b) for equation (8):
From (7) it follows that

$$\sigma_e^2 = \frac{1}{e^x} (e^x - 2x) \ \mu_e = (1 - 2\rho t \ e^{-x} \ \tau/t) \ \mu_e ,$$

thus

$$\sigma_e^2 = \mu_e \ (1 - 2\mu_e \ \tau/t) . \tag{22}$$

The results (21) and (22) prove that the equations (4) and (8) are not just linear approximations, but indeed exact. This also explains why Teich's solutions (5) and (9), or the equivalent series expansions (6) and (10), are rigorous.

Obviously, the new generalized transform (18) is only an approximate solution to the problem, but for the time being this seems to be more than adequate for the possible practical applications.

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References

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