Rapport BIPM-85/5

The interval distribution for a generalized dead time*

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Abstract

We evaluate the explicit expression for the interval density of an original Poisson process which has been deformed by the insertion of a generalized dead time of the Albert-Nelson type. The known formulae corresponding to the two traditional types (extended or non-extended) are obtained as special cases.

1. Introduction

For any random process occurring in time, the interval density is clearly a quantity of basic importance. In the field of pulse counting, the time which separates two consecutive arrivals of events can nowadays also readily be measured experimentally, for instance by means of an electronic time-to-amplitude converter. It is therefore essential to have a set of reliable model distributions at hand with which the measurements can be compared and thus interpreted in terms of the parameters to be determined.

For the case of a Poisson process, the interval density has the simple form of a decaying exponential function. If the original process has been distorted by a dead time which belongs to one of the two traditional types, the corresponding modified densities are also well known and of a rather simple form. For a review of these and related features, see e.g. [1].

The recent practical application of dead times of a generalized type makes it desirable to have at one's disposal also an explicit formal description of the corresponding interval density. Its shape is expected to be somehow "intermediate" between those belonging to the two usual types of dead time, where the exact meaning of such a statement is probably best left unspecified.

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* This report is dedicated to Alfred Spernol on the occasion of his recent sixtieth birthday, in recognition of his important contribution to the field of radioactivity measurements and as a tribute to his human qualities.

For the moment, the only relevant information available on the interval density is its Laplace transform, which was first found by Takács [2]. That author has shown that if an original Poisson process (of rate ρ) has passed through a generalized dead time (of the Albert-Nelson type), characterized by the two parameters τ and θ, the transformed interval density is given by the expression

$$ {}_{\theta}\tilde{f}(s;\rho) = \left(\frac{\theta\rho + s}{\rho + s}\right) \left[\frac{\rho\, e^{-(\theta\rho+s)\tau}}{s + \theta\rho\, e^{-(\theta\rho+s)\tau}}\right] . \tag{1} $$

As for most applications we have to know the density as a function of time, the main aim of the present study consists in finding the corresponding interval density ${}_{\theta}f(t;\rho)$ in the time domain.

2. Inversion of the transform

For finding the original of the transform (1), it will be useful to remember that for an extended dead time (i.e. $\theta = 1$) the corresponding Laplace transform is given by [3]

$$ {}_{e}\tilde{f}(s;\rho) = \frac{\rho\, e^{-(\rho+s)\tau}}{s + \rho\, e^{-(\rho+s)\tau}} , \tag{2} $$

a result which can also be readily obtained from (1). As a consequence, it is possible to write (1) in the form

$$ {}_{\theta}\tilde{f}(s;\rho) = \left[\frac{\rho}{\rho + s} + \frac{s}{\theta(\rho + s)}\right] {}_{e}\tilde{f}(s;\theta\rho) . \tag{3} $$

The interpretation of the transforms appearing in the bracket on the right-hand side of (3) is straightforward. For the exponential interval density of an unperturbed Poisson process we have

$$ \phi(t) \equiv \rho\, e^{-\rho t} , \qquad \text{for } t > 0 , \tag{4} $$

thus

$$ \tilde{\phi}(s) = \frac{\rho}{\rho + s} . $$

Application of the well-known rule valid for a derivative, namely

$$ \mathcal{L}\left\{\frac{d\phi}{dt}\right\} = s\,\tilde{\phi}(s) - \phi(+0) = s\,\tilde{\phi}(s) - \rho , \tag{5} $$

shows that the bracket in the transform (3) corresponds to the original function

$$\mathcal{L}^{-1}\{\tilde{\phi}(s) + \frac{1}{\theta s} s \tilde{\phi}(s)\} = \phi(t) + \frac{1}{\theta\rho}\left[\frac{d\phi}{dt} + \rho\,\delta(t)\right]$$

$$= \phi(t) + \frac{1}{\theta\rho}\frac{d\phi}{dt} + \frac{1}{\theta}\delta(t)$$

$$= \frac{1}{\theta}\delta(t) - \left(\frac{1-\theta}{\theta}\right)\phi(t) ,$$

since $\frac{d\phi}{dt} = -\rho\,\phi(t)$.

Therefore, the general rule for finding the original interval density is given by the convolution

$$ {}_{\theta}f(t;\rho) = \frac{1}{\theta}\{\delta(t) - (1-\theta)\,\phi(t)\} * {}_{e}f(t;\theta\rho)$$

$$= \frac{1}{\theta}\,{}_{e}f(t;\theta\rho) - \left(\frac{1-\theta}{\theta}\right)\phi(t) * {}_{e}f(t;\theta\rho) . \quad (6)$$

A simple check is possible here for the limit $\theta = 1$, where we find indeed that

$$ {}_{\theta=1}f(t;\rho) = {}_{e}f(t;\rho) ,$$

as expected. The case $\theta = 0$ is more difficult and will be treated at the end of this report.

For the general case, the only problem which remains to be solved for a complete inversion of (1) is then given by the need to perform a convolution of the type

$$\phi(t) * {}_{e}f(t;\theta\rho) , \quad (7)$$

since this will be needed for evaluating the second contribution appearing in (6).

Let us first recall the explicit form which has been found to hold for the interval density after an extended dead time. It can be written, according to eq. (21) in [3], as

$$ {}_{e}f(t;\theta\rho) = \sum_{j=1}^{J} A'_j(t) ,$$

$$\text{with} \quad A'_j(t) = \frac{\theta\rho}{(j-1)!}\left[-\theta\rho\,(t-j\tau)\right]^{j-1} e^{-j\theta\rho\tau} , \quad (8)$$

for $t > j\tau$ and where J is the largest integer below t/τ.

Our task will therefore be accomplished if we can give a general expression for

$$B_j'(t) = \phi(t) * A_j'(t) ,$$

because then, according to (6), we can simply write

$$_\theta f(t;\rho) = \sum_{j=1}^{J} \{\frac{1}{\theta} A_j'(t) - (\frac{1-\theta}{\theta}) B_j'(t)\} . \qquad (9)$$

Let us first perform the following rearrangements

$$B_j'(t) = \frac{(-1)^{j-1}}{(j-1)!} (\theta\rho)^j e^{-j\theta\rho\tau} (t-j\tau)^{j-1} * \rho e^{-\rho t}$$

$$= \frac{(-1)^{j-1}}{(j-1)!} \rho [\theta\rho e^{-\theta\rho\tau}]^j \delta(t-j\tau) * t^{j-1} * e^{-\rho t} . \qquad (10)$$

The second convolution can be written more explicitly if we have recourse to the definition, i.e.

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\alpha) f_2(t-\alpha) d\alpha = \int_{o}^{t} f_1(\alpha) f_2(t-\alpha) d\alpha,$$

since $f(t) = 0$ for $t < 0$.

We then find

$$C_j(t) \equiv t^{j-1} * e^{-\rho t} = e^{-\rho t} \int_{o}^{t} \alpha^{j-1} e^{\rho\alpha} d\alpha . \qquad (11)$$

This type of integral is simple to solve. A first integration by parts yields

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx ,$$

and repeated application leads to

$$\int x^n e^{ax} dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} [\frac{x^{n-1} e^{ax}}{a} - \frac{n-1}{a} \int x^{n-2} e^{ax} dx]$$

$$= \frac{x^n e^{ax}}{n} - \frac{n x^{n-1} e^{ax}}{a^2} + \frac{n(n-1)}{a^2} [\frac{x^{n-2} e^{ax}}{a} - \frac{n-2}{a} \int x^{n-3} e^{ax} dx]$$

$$= \ldots = e^{ax} \sum_{k=o}^{n} (-1)^k \frac{n!}{(n-k)!} \frac{x^{n-k}}{a^{k+1}} . \qquad (12)$$

We therefore have in our case (since $0^o = 1$) for (11)

$$C_j(t) = e^{-\rho t} \left[e^{\rho t} \sum_{k=o}^{j-1} (-1)^k \frac{(j-1)!}{(j-1-k)!} \frac{t^{j-1-k}}{\rho^{k+1}} \right]_o^t$$

$$= \left\{ \sum_{k=o}^{j-1} (-1)^k \frac{(j-1)!}{(j-1-k)!} \frac{t^{j-1-k}}{\rho^{k+1}} \right\} + (-1)^j \frac{(j-1)!}{\rho^j} e^{-\rho t} . \qquad (13)$$

Hence, for the original coefficient B_j' this then leads with (10) to

$$B_j'(t) = (-1)^{j-1} \rho [\theta\rho\, e^{-\theta\rho\tau}]^j \left\{ \sum_{k=o}^{j-1} \frac{(-1)^k}{(j-1-k)!} \frac{(t-j\tau)^{j-1-k}}{\rho^{k+1}} + \frac{(-1)^j}{\rho^j} e^{-\rho(t-j\tau)} \right\}$$

or also, after some rearrangements, to

$$B_j'(t) = -\rho[\theta\, e^{-\theta\rho\tau}]^j \left\{ e^{-\rho(t-j\tau)} + (-1)^j \sum_{k=o}^{j-1} \frac{(-1)^k}{(j-1-k)!} [\rho(t-j\tau)]^{j-1-k} \right\} . \qquad (14)$$

The formulae (8) and (14) give the explicit expressions for the coefficients appearing in (9).

In order to avoid ambiguous mathematical expressions (of the type "0/0") which may occur for $\theta = 0$, we prefer to write (9) in the equivalent, but more useful form

$$_\theta f(t;\rho) = \sum_{j=1}^{J} \left[_\theta A_j(t) + (1-\theta)\; _\theta B_j(t) \right] , \qquad (15)$$

in which the new coefficients are now given by

$$_\theta A_j(t) = \frac{A_j'(t)}{\theta} = \rho \frac{e^{-j\theta x}}{(j-1)!} (-\theta T_j)^{j-1}$$

and (16)

$$_\theta B_j(t) = \frac{-B_j'(t)}{\theta} = \rho\, e^{-j\theta x}\, \theta^{j-1} \left[e^{-T_j} - (-1)^j \sum_{k=1}^{j} \frac{(-1)^k}{(j-k)!} T_j^{j-k} \right] ,$$

where we have used the abbreviations

$$x = \rho\tau \quad \text{and}$$

$$T_j = \rho(t-j\tau) .$$

It is also possible, of course, to bring the final result into the more compact form

$$_{\theta}f(t;\rho) = {}_{\theta}A_{tot}(t) + (1-\theta)\, {}_{\theta}B_{tot}(t) , \tag{17}$$

with

$$_{\theta}A_{tot}(t) = \sum_{j=1}^{J} {}_{\theta}A_j(t) \quad \text{and} \quad {}_{\theta}B_{tot}(t) = \sum_{j=1}^{J} {}_{\theta}B_j(t) , \tag{18}$$

although this brings little advantage. A simple computer program allows us to evaluate directly $_{\theta}A_{tot}$ and $_{\theta}B_{tot}$.

3. Limiting cases

It is of interest to check the behaviour of (15) for the traditional types of dead times.

For an extended dead time, i.e. for $\theta = 1$, we have

$$_{1}f(t;\rho) = \sum_{j=1}^{J} {}_{1}A_j(t) = \rho \sum_{j=1}^{J} \frac{(-T_j)^{j-1}}{(j-1)!} e^{-jx} = {}_{e}f(t;\rho) , \tag{19}$$

as results from a comparison with (8), for $\theta = 1$.

For a non-extended dead time, i.e. for $\theta = 0$, we obtain first from (17) and (18)

$$_{o}f(t;\rho) = {}_{o}A_{tot}(t) + {}_{o}B_{tot}(t) = {}_{o}A_1(t) + {}_{o}B_1(t) .$$

Since it follows from (16) that

$$_{\theta}A_1(t) = \rho\, e^{-\theta x} \quad \text{and} \quad {}_{\theta}B_1(t) = \rho\, e^{-\theta x}(e^{-\rho t+x} - 1) ,$$

we obtain (still for $\theta = 0$)

$$_{o}f(t;\rho) = \rho + \rho(e^{-\rho t+x} - 1) = \rho\, e^{-\rho(t-\tau)} = {}_{n}f(t;\rho) . \tag{20}$$

Both the results (19) and (20) clearly correspond to our expectation. Obviously, these simple checks cannot be taken as a proof for the correctness of the general formulae (15) and (16); they only guarantee that the limiting cases are correct.

A more detailed discussion of the generalized interval density (including possible approximations) as well as a comparison with experimental results have to be postponed to a later study.

Mme M. Boutillon should be thanked for her careful reading of a draft version.

References

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(September 1985)