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Inversion of the Takács formula*

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Abstract

A generalized model for a dead time has been proposed long ago by Albert and Nelson, which contains the usual two types as limiting cases. There exists a simple formula for the output count rate, due to Takács, if the input pulses form a Poisson process. In most practical applications, however, only the output can be measured and one would like to know the corresponding original input rate. We indicate two equivalent general formulae which allow us to do this analytically. They both involve Stirling numbers of the second kind and are generalizations of the expressions known to hold for the traditional types.

1. Introduction

The effect of a dead time on a series of events in time consists in retaining from the original sequence a certain subset. Those events which are rejected give rise to counting losses. The statistical behaviour of a dead time can therefore be specified by a selection rule. Traditionally, two different selection rules are discussed which correspond to the well-known dead times of the non-extended type and the extended type, respectively.

A long time ago Albert and Nelson [1] proposed a more general mathematical model for the selection of events out of a random sequence. The complete characterization of such a generalized dead time then requires, in addition to the usual time interval τ, a further parameter θ which corresponds to a probability. The generalization consists in supposing that the arrival of an event during the action of a previously imposed dead time prolongs the latter (in a way known from the action of an extended dead time), but only with probability θ, and that the random choice between extension or non-extension is made for any arrival independently of previous decisions. This mechanism assures that for

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* Dedicated to Albrecht Rytz on the occasion of his sixty-fifth birthday

$\theta = 0$ we have the usual non-extended dead time, whereas $\theta = 1$ corresponds to the extended type. Obviously, such a model, chosen essentially for its mathematical simplicity, in no way guarantees that the effect produced by an experimental dead time follows this scheme exactly, but it certainly provides more flexibility in the quantitative description of the behaviour of "naturally" occurring dead times.

Among the various relations which can be derived for such a generalized dead time, its effect on the count rate of an original Poisson process is clearly of particular interest. As Takács [2] has been the first to indicate such a relation, we shall call the corresponding equation Takács' formula. It says that the count rate r, observed at the output of a dead time characterized by the parameters τ and θ, is given by

$$r = \frac{\theta\rho}{e^{\theta\rho\tau} + \theta - 1}, \qquad (1)$$

provided that the sequence of events at the input of the device forms a Poisson process with count rate ρ. No attempt will be made here to derive (1); we restrict ourselves to showing that for the limiting cases $\theta = 0$ and $\theta = 1$ the well-known traditional formulae are recovered.

For $\theta = 1$ we readily find from (1)

$$r_e = \rho\, e^{-\rho\tau}, \qquad (2)$$

as expected for an extended (e) dead time.

The case $\theta = 0$ requires a power expansion, for instance in the form

$$r_n = \lim_{\theta\to 0}\left\{\frac{\theta\rho}{1 + \theta\rho\tau + \frac{1}{2}\,\theta^2\rho^2\tau^2 + \ldots + \theta - 1}\right\}$$

$$= \lim_{\theta\to 0}\left\{\frac{\theta\rho}{\theta(1 + \rho\tau + \frac{1}{2}\,\theta\rho^2\tau^2 + \ldots)}\right\} = \frac{\rho}{1 + \rho\tau}, \qquad (3)$$

which is indeed the formula valid for a non-extended (n) dead time.

It happens, however, that in most practical applications the experimentally measured quantity is r and that one would like to know (for given parameters τ and θ) the corresponding original count rate ρ, as this will normally be the quantity which is of physical interest. This raises the problem of inverting (1), in other words of finding a mathematical expression which gives ρ as an explicit function of r (for known parameters τ and θ)*.

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* If a numerical value of ρ is all we wish to obtain, then a purely numerical solution by some iterative process may be more appropriate; any programmable pocket computer will suffice for this.

At first sight the problem looks rather hopeless, in particular if we realize that (1) is clearly a generalization of (2) and that already finding an inversion for (2) was an enterprise [3] which could only be brought to a good end by means of some lucky guessing (see also the new information contained in Appendix 1).

2. An attempt at an iterated inversion

Let us first briefly consider a possible way of inverting (1). With the abbreviations $r\tau = z$, $\rho\tau = x$, $\theta x = x'$ and $\theta - 1 = \eta$, we have

$$z = \frac{x'}{e^{x'} + \eta}$$

and thus also

$$x' = z(e^{x'} + \eta) \ .$$

A power-series development gives

$$\begin{aligned} x' &= z\left[1 + x' + \frac{1}{2}x'^2 + \ldots + \eta\right] \\ &\cong z\left[1 + z(1 + x' + \eta) + \frac{1}{2}z^2(1+\eta)^2 + \eta\right] \\ &= z\left[1 + z(1 + z[1+\eta] + \eta) + \frac{1}{2}z^2(1+\eta)^2 + \eta\right] \\ &= z\left[1 + z + z^2\theta + z\eta + \frac{1}{2}z^2\theta^2 + \eta\right] \\ &= z\left[\theta + z\theta + z^2(\theta + \frac{1}{2}\theta^2)\right] \\ &= z\theta\left[1 + z + z^2(1 + \frac{\theta}{2})\right] , \end{aligned}$$

hence

$$x = z\left[1 + z + z^2(1 + \frac{\theta}{2}) + \ldots\right] . \tag{4}$$

This shows that the method of iterated inversion works in principle; however, it is too cumbersome for deriving approximations of higher order. It is interesting to note in (4) that up to second order in z the quantity x is independent of the parameter θ. This behaviour is well known from the two traditional types of dead time.

3. A more systematic procedure

The approach used this time is essentially the same as the one applied previously in [3], although the detailed elaboration is somewhat more cumbersome. By putting $\rho\tau = x$ and $r\tau = z$, the Takács formula (1) can also be written as

$$z = \frac{\theta x}{e^{\theta x} + \theta - 1} . \tag{5}$$

A graphical representation of (5) is given in Fig. 1 for some values of θ.

The solution x we try to find here lies in the domain $0 < x < x_{max}$, i.e. in the region on the left-hand side of Fig. 1, before z reaches its maximum value (for a given θ). The exact position of this limiting value x_{max} can be readily found from (5) by requiring that $\partial z/\partial x = 0$. This leads to the equation

$$e^{\theta x} = \frac{1 - \theta}{1 - \theta x} , \tag{6}$$

the solution $x = x_{max}$ of which is represented graphically in Fig. 2.

No attempt will be made in this report to obtain the "second" solution $x' > x_{max}$, which corresponds to the same output z (for $\theta > 0$). As x' implies very large counting losses, it will normally be of little practical interest.

It will also be noted in Fig. 1 that curves belonging to different values of θ never cross each other. This is formally assured by the fact that, for $\theta x > 0$, the partial derivative $\partial z/\partial\theta$ exists everywhere and does not vanish. Hence, for a given value of τ at a known original count rate ρ, a single measurement of the output rate r (or z) is (in principle) sufficient for obtaining the parameter θ.

After these preliminary remarks let us now tackle the inversion problem. A development of the exponential function in (5) gives

$$z = \frac{x}{1 + x + \frac{1}{2}\theta x^2 + \frac{1}{6}\theta^2 x^3 + \frac{1}{24}\theta^3 x^4 + \frac{1}{120}\theta^4 x^5 + \frac{1}{720}\theta^5 x^6 + \ldots} . \tag{7}$$

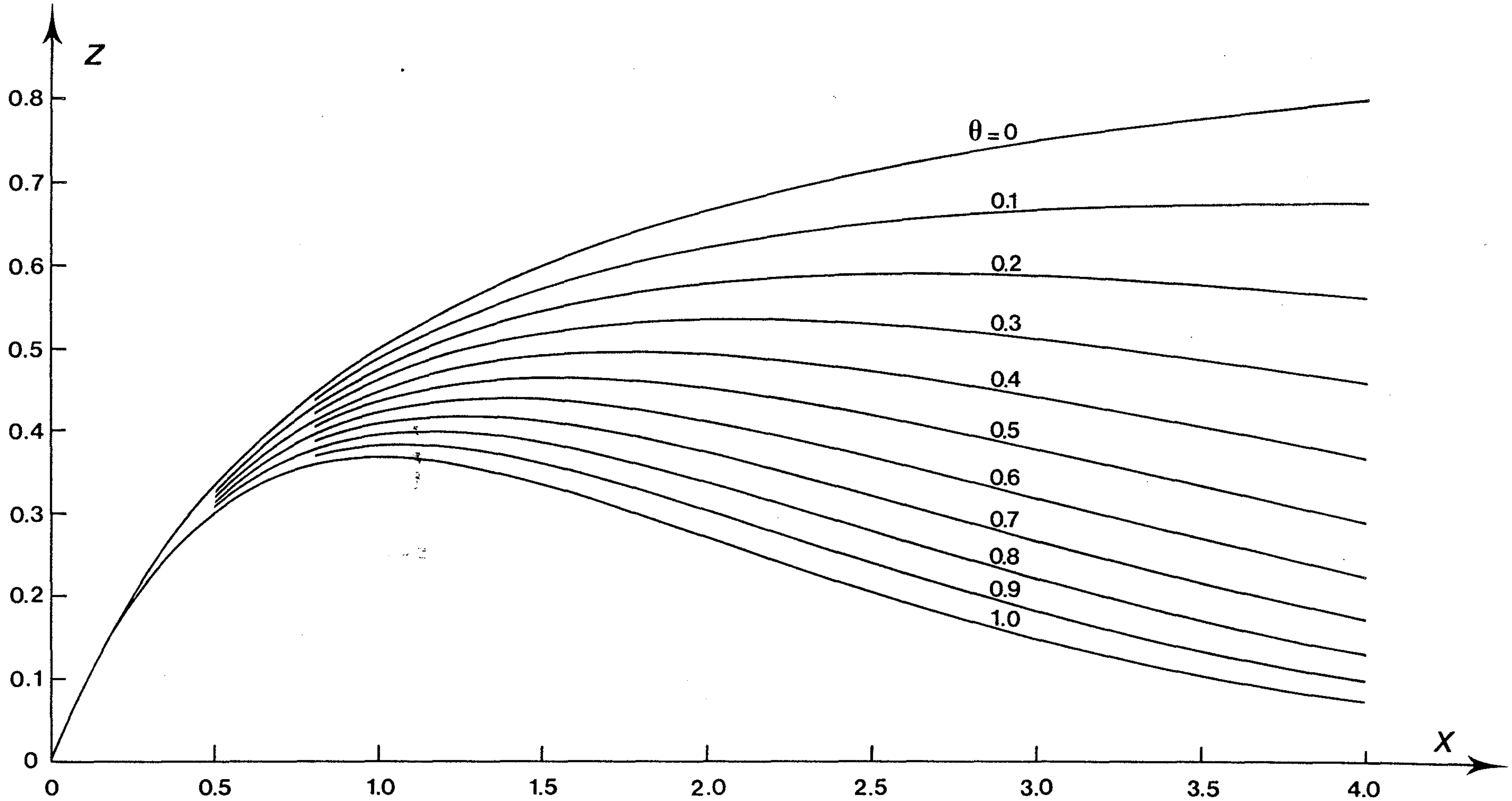

Fig. 1. Graphical representation of the Takács formula (5). For $\theta > 0$ the output count rate z passes through a maximum value when the input count rate x is increased. The notation used is $x = \rho\tau$ and $z = r\tau$.

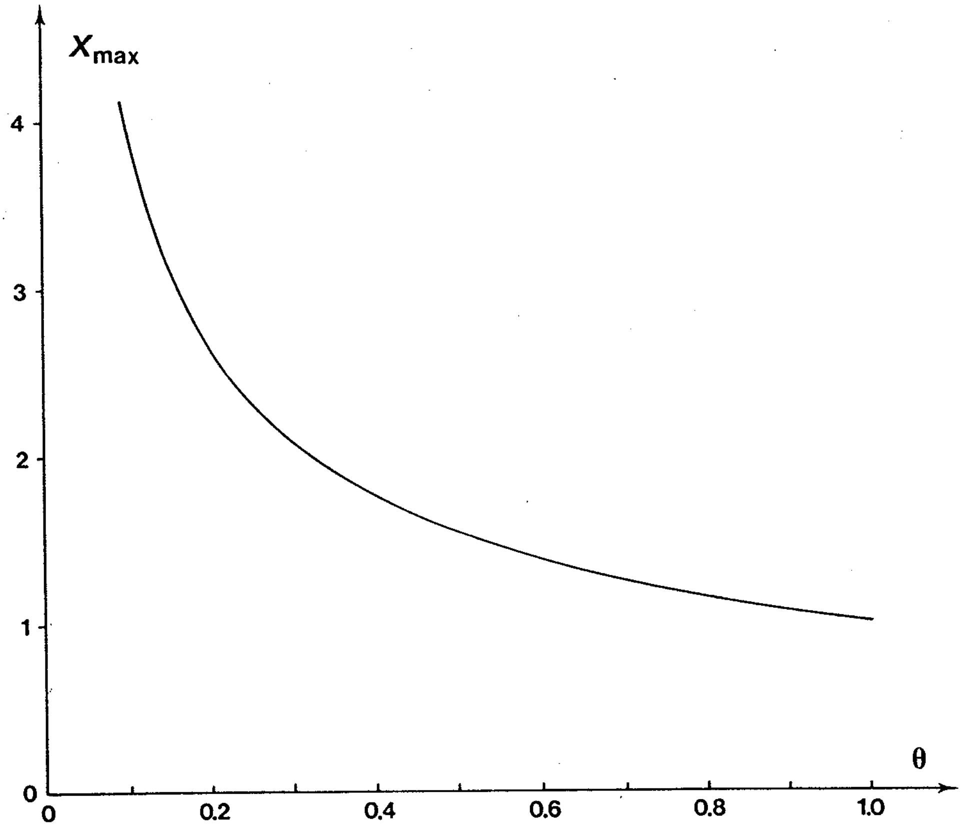

Fig. 2. Input rate for which the output rate is at a maximum, for a given value of θ, with $x_{max} = \tau\rho_{max}$.

Inversion of the denominator

$$D = 1 + \sum_{j=1}^{\infty} a_j x^j$$

to the reciprocal

$$1/D = 1 + \sum_{j=1}^{\infty} b_j x^j$$

can easily be performed by means of the explicit expressions indicated recently in [4]**. Since $a_1 = 1$, $a_2 = \theta/2$, $a_3 = \theta^2/6$, etc., we find for the new coefficients $b_1 = -1$, $b_2 = 1 - \theta/2$, $b_3 = -1 + \theta - \theta^2/6$, etc.

** We are pleased to note that, in the meantime, the general formulae given in [4] have been proved to be correct by P. Carré. His simple and very elegant method can also be applied to other operations with power series.

Thus (7) can be expressed in the equivalent form (up to seventh order in x)

$$\begin{aligned} z &= x(1 + \sum_{j=1}^{\infty} b_j x^j) \\ &= x - x^2 + (1 - \tfrac{1}{2}\theta)x^3 - (1 - \theta + \tfrac{1}{6}\theta^2)x^4 \\ &\quad + (1 - \tfrac{3}{2}\theta + \tfrac{7}{12}\theta^2 - \tfrac{1}{24}\theta^3)x^5 \\ &\quad - (1 - 2\theta + \tfrac{5}{4}\theta^2 - \tfrac{1}{4}\theta^3 + \tfrac{1}{120}\theta^4)x^6 \\ &\quad + (1 - \tfrac{5}{2}\theta + \tfrac{13}{6}\theta^2 - \tfrac{3}{4}\theta^3 + \tfrac{31}{360}\theta^4 - \tfrac{1}{720}\theta^5)x^7 \mp \ldots . \end{aligned} \tag{8}$$

Our next step consists in reverting this series, which means expressing x in the form of a power series of z. As the formulae given for this purpose in the popular tables of Dwight [5] also just go up to seventh order, their application is immediate and leads to

$$\begin{aligned} x &= z + z^2 + (1 + \tfrac{1}{2}\theta)z^3 + (1 + \tfrac{3}{2}\theta + \tfrac{1}{6}\theta^2)z^4 \\ &\quad + (1 + 3\theta + \tfrac{7}{6}\theta^2 + \tfrac{1}{24}\theta^3)z^5 \\ &\quad + (1 + 5\theta + \tfrac{25}{6}\theta^2 + \tfrac{5}{8}\theta^3 + \tfrac{1}{120}\theta^4)z^6 \\ &\quad + (1 + \tfrac{15}{2}\theta + \tfrac{65}{6}\theta^2 + \tfrac{15}{4}\theta^3 + \tfrac{31}{120}\theta^4 + \tfrac{1}{720}\theta^5)z^7 + \ldots . \end{aligned} \tag{9}$$

We now come to the "artistic" part of this study which consists in guessing from the explicit terms available in (9) what the general series development might be. This attempt could very well lead to no clear result. However, it looks as if luck were once more with us in this attempt. Indeed, a careful examination of the terms appearing in (9) reveals a close relation of the coefficients with the Stirling numbers of the second kind - although the reason for this connection is quite mysterious for the moment. Thus we note, for instance, that the coefficient of z^6, i.e.

$$1 + 5\theta + \frac{25}{6}\theta^2 + \frac{5}{8}\theta^3 + \frac{1}{120}\theta^4 ,$$

can be interpreted as

$$S(5,5) + \frac{S(5,4)}{2!}\theta + \frac{S(5,3)}{3!}\theta^2 + \frac{S(5,2)}{4!}\theta^3 + \frac{S(5,1)}{5!}\theta^4 ,$$

and likewise for the other powers of z. This suggests that the general inversion formula for (5) is given by

$$x = z + \sum_{k=1}^{\infty} z^{k+1} \sum_{j=1}^{k} \frac{S(k,k+1-j)}{j!} \theta^{j-1} . \tag{10}$$

A tabulation of the Stirling numbers S(n,k) for n up to 25 (and $1 \leq k \leq n$) can be found in [6]. By means of the convention $S(k,0) = \delta_{0,k}$ the inversion may also be written in the more condensed form

$$x = \sum_{k=1}^{\infty} a_k z^k ,$$

with (10')

$$a_k = \sum_{j=1}^{k} \frac{S(k-1,k-j)}{j!} \theta^{j-1} .$$

Obviously the special cases $\theta = 0$ and $\theta = 1$ are of particular interest, also for checking purposes.

The case $\theta = 0$ is immediate, as the sum over j in (10) now reduces to the single term $j = 1$ (since $0^0 = 1$). Thus

$$x_n = z + \sum_{k=1}^{\infty} z^{k+1} S(k,k) = z \sum_{k=0}^{\infty} z^k = \frac{z}{1 - z} , \tag{11}$$

as expected for a non-extended (n) dead time.

The case $\theta = 1$ (extended dead time) has been dealt with previously [3] and it was found that

$$x_e = \sum_{k=0}^{\infty} \frac{(k+1)^{k-1}}{k!} z^{k+1} . \tag{12}$$

As (10) now gives for the same situation

$$x_e = z + \sum_{k=1}^{\infty} z^{k+1} \sum_{j=1}^{k} \frac{S(k,k+1-j)}{j!} ,$$

we are led to the identity (for $k \geq 1$)

$$\sum_{j=1}^{k} \frac{S(k,k+1-j)}{j!} = \frac{(k+1)^{k-1}}{k!} , \tag{13a}$$

or also (for $k \geq 0$)

$$\sum_{j=1}^{k+1} \frac{S(k,k+1-j)}{j!} = \frac{(k+1)^{k-1}}{k!} , \tag{13b}$$

remembering that $S(k,0) = \delta_{0,k}$. We have not come across such a relation in the literature available to us. However, a way has now been found to prove its correctness; this will be presented in Appendix 2.

4. An alternative formula

Guided by the analogy with the simple form (11) which is valid for a non-extended dead time, one may be tempted to look for an inversion of the Takacs formula which is of similar structure. This amounts to another inversion of (9), which is of the type

$$x = z(1 + \sum_{k=1}^{\infty} a_k z^k) , \qquad (10")$$

with $a_1 = 1$, $a_2 = 1 + \theta/2$, $a_3 = 1 + 3\theta/2 + \theta^2/6$, etc.

For the inverted series

$$x = \frac{z}{1 - \sum_{k=1}^{\infty} b_k z^k} \qquad (14)$$

the new coefficients b_k can again be obtained by applying the explicit formulae given in [4]. After some numerical work we arrive at the expressions

$$\begin{aligned}
b_1 &= 1 , \\
b_2 &= \frac{\theta}{2} , \\
b_3 &= \frac{\theta}{2} (1 + \frac{1}{3}\theta) , \\
b_4 &= \frac{\theta}{2} (1 + \frac{7}{6}\theta + \frac{1}{12}\theta^2) , \\
b_5 &= \frac{\theta}{2} (1 + \frac{5}{2}\theta + \frac{3}{4}\theta^2 + \frac{1}{60} \theta^3) , \\
b_6 &= \frac{\theta}{2} (1 + \frac{13}{3}\theta + 3\theta^2 + \frac{31}{90}\theta^3 + \frac{1}{360}\theta^4) .
\end{aligned} \qquad (15)$$

It is not very difficult to conjecture for the general form of the coefficients b_k, in which we expect to find again Stirling numbers, the expression

$$b_k = \frac{1}{k(k-1)} \sum_{j=1}^{k-1} \frac{S(k,k-j)}{(j-1)!} \theta^j , \qquad (16)$$

for $k \geqslant 2$, whereas $b_1 = 1$.

Let us again compare this general result with the expressions known to hold for the two traditional types:

For $\theta = 0$ we see from (16) that $b_k = 0$, for $k \geqslant 2$, so that we arrive at (11).

For the case $\theta = 1$, already treated in [3], we know that

$$x_e = \frac{z}{1 - \sum_{k=1}^{\infty} \frac{(k-1)^{k-1}}{k!} z^k} . \tag{17}$$

A comparison with (16) therefore leads to the conjectured identity (for $k \geqslant 2$)

$$\sum_{j=1}^{k-1} \frac{S(k,k-j)}{(j-1)!} = \frac{(k-1)^k}{(k-1)!} , \tag{18a}$$

or also (for $k \geqslant 1$)

$$\sum_{j=1}^{k} \frac{S(k,k-j)}{(j-1)!} = \frac{(k-1)^k}{(k-1)!} , \tag{18b}$$

again since $S(k,0) = \delta_{0,k}$. This relation can indeed be shown to be correct (see Appendix 2 for a proof).

In summing up our findings, we can thus make the following statements:

If we assume that an original Poisson process (with count rate ρ) has been distorted by a "generalized" dead time (characterized by the parameters τ and θ), and that the count rate measured at the output is r, then the initial count rate ρ can be evaluated by two equivalent formulae. With the abbreviations $x = \rho\tau$ and $z = r\tau$ we have

$$x = z + z \sum_{k=1}^{\infty} \alpha_k z^k , \quad \text{with } \alpha_k = \sum_{j=0}^{k-1} \frac{S(k,k-j)}{(j+1)!} \theta^j ,$$

or also (19)

$$x = \frac{z}{1 - z - \sum_{k=2}^{\infty} \beta_k z^k} , \quad \text{with } \beta_k = \sum_{j=1}^{k-1} \frac{S(k,k-j)}{k(k-1)\,(j-1)!} \theta^j .$$

The traditional types of dead times are included as special cases, namely

- for $\theta = 0$, i.e. τ non-extended, with

$$\alpha_k = 1 , \qquad \beta_k = 0 ;$$

(20)

- for $\theta = 1$, i.e. τ extended, with

$$\alpha_k = \frac{(k+1)^{k-1}}{k!} , \qquad \beta_k = \frac{(k-1)^{k-1}}{k!} .$$

It should be remembered that for $\theta > 0$ there are, in fact, two possible solutions x for a given value of z (cf. Fig. 1) and that the above formulae only lead to the lower one. At present we can see no way for obtaining analytically the second solution.

We shall be pleased to hear that someone has succeeded in finding a formal proof of the relations (19).

Acknowledgments

My sincere thanks go once more to Mme M. Boutillon. Appendix 1 is the result of her mathematical competence and kind help.

The dedication of this report to Albrecht Rytz is meant as an expression of my deep gratitude for eighteen years of friendly collaboration, a time during which he has always shown a keen and encouraging interest in my work.

APPENDICES

1. Proof of previous conjectures

The reversion formulae applicable to the case of an extended dead time as given in [3] have essentially been obtained by guessing. Although we have never really doubted their reliability, it is gratifying to note that, in the meantime, the correctness of the two expressions then advanced has been established. They correspond to the formulae given here as (12) and (17).

As far as (12) is concerned, this seems to be a result known for a long time by the specialists. So L. Comtet [7], while discussing the inversion formula of Lagrange (on page 162), treats this case as a mere example and calls it "le plus classique sans doute". Hence, the formula (12) may be taken as well established, and it was only our ignorance which has created a problem.

However, it still remained to prove (17). This has been recently achieved by M. Boutillon of our laboratory and she has generously handed over to me her notes from which I have extracted what follows.

The basic idea is to check the compatibility of (12) and (17), i.e. of

$$x_e(12) = z\left[1 + \sum_{j=1}^{\infty} \frac{(j+1)^{j-1}}{j!} z^j\right] \tag{A1}$$

and

$$x_e(17) = \frac{z}{1 - \sum_{j=1}^{\infty} \frac{(j-1)^{j-1}}{j!} z^j} . \tag{A2}$$

This would be established if we could show that their ratio is unity. We therefore form

$$\frac{x_e(12)}{x_e(17)} = \left[1 + \sum_{j=1}^{\infty} \frac{(j+1)^{j-1}}{j!} z^j\right] \left[1 - \sum_{j=1}^{\infty} \frac{(j-1)^{j-1}}{j!} z^j\right]$$

$$= 1 + \sum_{j=1}^{\infty} \frac{(j+1)^{j-1}}{j!} z^j - \sum_{j=1}^{\infty} \frac{(j-1)^{j-1}}{j!} z^j - (\textstyle\sum \ldots)(\textstyle\sum \ldots)$$

$$\equiv 1 + \sum_{k=1}^{\infty} C_k z^k . \tag{A3}$$

We now try to prove that $C_k = 0$. For this we first write

$$C_k = \frac{(k+1)^{k-1}}{k!} - \frac{(k-1)^{k-1}}{k!} - \sum_{j=1}^{k-1} \frac{(j-1)^{j-1}}{j!} \frac{(k-j+1)^{k-j-1}}{(k-j)!}$$

$$\equiv \frac{(k+1)^{k-1}}{k!} - B , \tag{A4}$$

with

$$B = \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (j-1)^{j-1} (k-j+1)^{k-j-1} . \tag{A5}$$

For complicated binomial sums one will invariably have recourse to J. Riordan's book [8]. Indeed, we find there, on page 18, Abel sums of the type (his eq. 14)

$$A_n(x,y;\ p,q) = \sum_{k=0}^{n} \binom{n}{k} (x+k)^{k+p} (y+n-k)^{n-k+q} .$$

In our case we thus have

$$\sum_{j=0}^{k} \binom{k}{j} (j-1)^{j-1} (k-j+1)^{k-j-1} = A_k(-1,1;\ -1,-1) \ . \qquad (A6)$$

As Riordan's eq. (20) says that

$$A_n(x,y;\ -1,-1) = \left(\frac{1}{x} + \frac{1}{y}\right) (x + y + n)^{n-1} \ ,$$

we find for our application that

$$A_k(-1,1;\ -1,-1) = (-1 + 1)\ k^{k-1} = 0 \ . \qquad (A7)$$

This now allows us to write for (A4)

$$C_k = \frac{(k+1)^{k-1}}{k!} - \frac{1}{k!} \left[A_k(\ldots) - \binom{k}{0} (-1)^{-1} (k+1)^{k-1}\right]$$

$$= \frac{(k+1)^{k-1}}{k!} - \frac{1}{k!} \left[0 + (k+1)^{k-1}\right] = 0 \ , \qquad (A8)$$

which establishes the equivalence of (A1) and (A2). Since (12) is known to be true, the above reasoning proves that (17) is also a valid formula.

2. The transition to an extended dead time

When we tried to pass from the suggested general formula (10) to the special case $\theta = 1$, agreement with the result already known to hold for an extended dead time could only be established by accepting the identity (13), i.e.

$$\sum_{j=1}^{k+1} \frac{S(k,k+1-j)}{j!} = \frac{(k+1)^{k-1}}{k!} \ , \quad \text{for } k \geqslant 0 \ .$$

In order to prove this relation, we first write it in the equivalent form

$$\sum_{j=0}^{k} \frac{S(k,j)}{(k+1-j)!} = \frac{1}{(k+1)!} \sum_{j=0}^{k} j!\ S(k,j) \binom{k+1}{j} \ , \qquad (A9)$$

where we have used the identity

$$\frac{1}{(k+1-j)!} = \frac{j!}{(k+1)!} \binom{k+1}{j} \ . \qquad (A10)$$

With the reccurrence relation for binomial coefficients

$$\binom{k+1}{j} = \binom{k+2}{j+1} - \binom{k+1}{j+1} , \tag{A11}$$

we have also for (A9)

$$\frac{1}{(k+1)!} \left[\sum_{j=0}^{k} j!\ S(k,j) \binom{k+2}{j+1} - \sum_{j=0}^{k} j!\ S(k,j) \binom{k+1}{j+1} \right] . \tag{A12}$$

By means of the formula (taken from [6], p. 825)

$$\sum_{j=0}^{k} j!\ S(k,j) \binom{n}{j+1} = \sum_{j=0}^{n-1} j^k , \tag{A13}$$

we finally obtain

$$\sum_{j=0}^{k} \frac{S(k,j)}{(k+1-j)!} = \frac{1}{(k+1)!} \left[\sum_{j=0}^{k+1} j^k - \sum_{j=0}^{k} j^k \right] = \frac{(k+1)^{k-1}}{k!} ,$$

which proves (13).

A similar problem occurred when we attempted to apply (16) to the limiting case $\theta = 1$. This time we were led to conjecture the identity (18), i.e.

$$\sum_{j=1}^{k} \frac{S(k,k-j)}{(j-1)!} = \frac{(k-1)^k}{(k-1)!} , \qquad \text{for } k \geqslant 1 ,$$

or the equivalent form

$$\sum_{j=0}^{k-1} \frac{S(k,j)}{(k-1-j)!} = \frac{(k-1)^k}{(k-1)!} . \tag{A14}$$

Using relations similar to (A10) and (A11), namely

$$\frac{1}{(k-1-j)!} = \frac{j!}{(k-1)!} \binom{k-1}{j}$$

and (A15)

$$\binom{k-1}{j} = \binom{k}{j+1} - \binom{k-1}{j+1} ,$$

we find for (A14)

$$\sum_{j=0}^{k-1} \frac{S(k,j)}{(k-1-j)!} = \frac{1}{(k-1)!} \sum_{j=0}^{k-1} j!\ S(k,j) \binom{k-1}{j}$$

$$= \frac{1}{(k-1)!} \left[\sum_{j=0}^{k} j!\ S(k,j) \binom{k}{j+1} - \sum_{j=0}^{k} j!\ S(k,j) \binom{k-1}{j+1} \right] .$$

Note that in the last expression the sums over j now also include the term with j = k. This is possible as the two new contributions vanish.

Application of (13) then gives

$$\frac{1}{(k-1)!} \left[\sum_{j=0}^{k-1} j^k - \sum_{j=0}^{k-2} j^k \right] = \frac{1}{(k-1)!} (k-1)^k , \qquad \text{(A16)}$$

which confirms (A14) and hence also (18).

It will be obvious that all this does not prove either (10) or (16), but it actually leaves little room for real doubt.

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