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The moments of self-convolutions

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Abstract

A general formula which makes use of the Bell polynomials is given for the ordinary moments of self-convolutions.

1. Introduction

The need to dispose of a general expression for the moments of a variable, the probability density of which is given by a multiple self-convolution, arises in various fields. Thus, we have been led to this problem in an attempt to describe the counting losses due to energy selection [1], and recently the same problem has reappeared in the description of doubly-stochastic Poisson processes [2]. It is the purpose of this report to derive a general expression which can be useful for such applications.

It is well known that a sum s of k independent random quantities x, i.e.

$$s = x_1 + x_2 + x_3 + \ldots + x_k , \tag{1}$$

where all contributions have the same probability density f(x), is described by a density $f_k(x)$ which is the k-fold self-convolution of f(x), hence

$$f_k(x) = \{f(x)\}^{*k} , \qquad k = 1, 2, \ldots \tag{2}$$

This obviously assumes x to be continuous; the formal changes needed for dealing with a discrete variable are well known and need not be repeated here. Besides, this distinction is irrelevant for the moments.

If we note the ordinary moments of x_j or s (of order r) by

$$E(x_j^r) \equiv m_r , \qquad E(s^r) \equiv {}_k m_r , \tag{3}$$

the question arises as to how these two series are interrelated.

2. A naîve approach

Let us first consider the simple case of two variables x_1 and x_2, which are supposed to be independent, but not necessarily identical. For $s = x_1 + x_2$ the probability density is

$$f_2(x) = f^{(1)}(x) * f^{(2)}(x)$$

and the first three moments are readily found to be given by (see e.g. [3])

$$\begin{aligned} {}_2m_1 &= m_1^{(1)} + m_1^{(2)} , \\ {}_2m_2 &= m_1^{(1)} + m_2^{(2)} + 2\, m_1^{(1)}\, m_1^{(2)} , \qquad (4) \\ {}_2m_3 &= m_3^{(1)} + m_3^{(2)} + 3\, [m_1^{(1)}\, m_2^{(2)} + m_1^{(2)}\, m_2^{(1)}] . \end{aligned}$$

Which are the corresponding expressions for a k-fold self-convolution? Let us begin with a very naîve empirical approach by evaluating the first three moments for successive values of k.

a) First moment

Here it will be obvious from (4) that the general formula is simply

$$_km_1 = k\, m_1 . \qquad (5)$$

b) Second moment

By repeated application of (4) one finds for identical variables

- for k = 2: $\quad {}_2m_2 = 2\, m_2 + 2\, m_1^2 ,$

- for k = 3: $\quad {}_3m_2 = m_2 + (2\, m_2 + 2\, m_1^2) + 2\, m_1\, 2\, m_1 = 3\, (m_2 + 2\, m_1^2),$

- for k = 4: $\quad {}_4m_2 = 2\, (2\, m_2 + 2\, m_1^2) + 2\, (2\, m_1)^2 = 4\, (m_2 + 3\, m_1^2) .$

This might suggest for the second moment a general formula of the type

$$_km_2 = k\, [m_2 + (k-1)\, m_1^2] . \qquad (6)$$

This can be shown to be correct by forming the variance

$$\begin{aligned} {}_k\sigma^2 &= {}_km_2 - {}_km_1^2 \\ &= k\, [m_2 + (k-1)\, m_1^2] - (k\, m_1)^2 \\ &= k\, (m_2 - m_1^2) = k\, \sigma^2 , \qquad (7) \end{aligned}$$

a relation which is known to hold generally for self-convolutions.

c) Third moment

As before we can conclude from (4)

- for k = 2: $\quad {}_2m_3 = 2\ (m_3 + 3\ m_2\ m_1)$,

- for k = 3: $\quad {}_3m_3 = 3\ (m_3 + 6\ m_2\ m_1 + 2\ m_1^3)$,

- for k = 4: $\quad {}_4m_3 = 4\ (m_3 + 9\ m_2\ m_1 + 6\ m_1^3)$.

This suggests as a possible general formula for the third moment

$$ {}_km_3 = k\ [m_3 + 3\ (k-1)\ m_2\ m_1 + (k-1)(k-2)\ m_1^3]\ . \tag{8}$$

A proof of the validity of (8) can again be obtained by forming the third central moment, namely

$$\begin{aligned} {}_k\mu_3 &= {}_km_3 - 3\ {}_km_2\ {}_km_1 + 2\ {}_km_1^3 \\ &= k\ [m_3 + 3\ (k-1)\ m_2\ m_1 + (k-1)(k-2)\ m_1^3] \\ &\quad - 3\ k[m_2 + (k-1)\ m_1^2]\ k\ m_1 + 2\ (k\ m_1)^3\ . \end{aligned}$$

After some elementary rearrangements we find indeed

$$ {}_k\mu_3 = k\ (m_3 - 3\ m_2\ m_1 + 2\ m_1^3) = k\ \mu_3\ . \tag{9}$$

Since the third central moments are known to be additive for convolutions, this proves (8).

3. A better approach

The way in which the results (6) and (8) of the previous section have been obtained cannot be pursued much further; we should make an attempt to replace guessing by a more systematic approach. This is indeed possible if we remember the fact that cumulants have the feature of being additive for convolutions, as is explained in any good textbook (see e.g. [4]).

Let the moment-generating function for a random variable x be defined by (t real)

$$\phi_x(t) \equiv E(e^{tx}) = \sum_{r=0}^{\infty} m_r \frac{t^r}{r!}\ . \tag{10}$$

If its logarithm can be developed into a power series

$$\Psi_x(t) \equiv \ln\ \phi_x(t) = \sum_{r=1}^{\infty} {}_x\kappa_r \frac{t^r}{r!}\ , \tag{11}$$

the coefficients ${}_x\kappa_r$ are called the cumulants of the random variable x.

For a sum of k independent random variables x_j (which may have different distributions)

$$s = x_1 + x_2 + \ldots + x_k \ , \tag{1}$$

it then follows from (10) that

$$\phi_s(t) = \prod_{j=1}^{k} \phi_{x_j}(t) \ , \tag{12}$$

and likewise

$$\Psi_s(t) = \sum_{j=1}^{k} \Psi_{x_j}(t) \ . \tag{13}$$

A look at (11) and (13) now reveals that the cumulant (of order r) of a sum is equal to the sum of the corresponding cumulants of the individual random variables, i.e.

$$_s\kappa_r = \sum_{j=1}^{k} {}_{x_j}\kappa_r \ , \qquad \text{for any order } r \geqslant 1 \ . \tag{14}$$

If it is possible to find a way to pass from the cumulants back to the moments, our problem is solved. This decisive last step is possible, but not quite simple: it involves the use of the so-called Bell polynomials [5]. Instead of tabulating them, it may be more appropriate here to give a short list of the main correspondences among the moments they imply. These are

- for the ordinary moments:

$$\begin{aligned}
m_1 &= \kappa_1 \ , \\
m_2 &= \kappa_2 + \kappa_1^2 \ , \\
m_3 &= \kappa_3 + 3\ \kappa_2\ \kappa_1 + \kappa_1^3 \ , \\
m_4 &= \kappa_4 + 4\ \kappa_3\ \kappa_1 + 3\ \kappa_2^2 + 6\ \kappa_2\ \kappa_1^2 + \kappa_1^4 \ , \\
m_5 &= \kappa_5 + 5\ \kappa_4\ \kappa_1 + 10\ \kappa_3\ \kappa_2 + 10\ \kappa_3\ \kappa_1^2 + 15\ \kappa_2^2\ \kappa_1 + 10\ \kappa_2\ \kappa_1^3 + \kappa_1^5 \ ;
\end{aligned} \tag{15}$$

- for the central moments:

$$\mu_2 = \kappa_2 \ , \qquad \mu_4 = \kappa_4 + 3\ \kappa_2^2 \ , \tag{16}$$

$$\mu_3 = \kappa_3 \ , \qquad \mu_5 = \kappa_5 + 10\ \kappa_3\ \kappa_2 \ .$$

Since the relation (14) can only be applied to (1) if the cumulants of the original variable x are known, we also need the inverse relations

$$\kappa_1 = m_1 \ ,$$

$$\kappa_2 = m_2 - m_1^2 \ ,$$

$$\kappa_3 = m_3 - 3\ m_2\ m_1 + 2\ m_1^3 \ , \tag{17}$$

$$\kappa_4 = m_4 - 4\ m_3\ m_1 - 3\ m_2^2 + 12\ m_2\ m_1^2 - 6\ m_1^4 \ ,$$

$$\kappa_5 = m_5 - 5\ m_4\ m_1 - 10\ m_3\ m_2 + 20\ m_3\ m_1^2 = 30\ m_2^2\ m_1 - 60\ m_2\ m_1^3 + 24\ m_1^5 \ .$$

Additional explicit relations (up to order 10) can be found in [6].

4. Application to self-convolutions

For the case of k identically distributed variables, the basic relation (14) becomes

$$_k\kappa_r = k\ \kappa_r \ . \tag{18}$$

The applications are now straightforward. Thus, for the central moments one first finds with (16)

$$\begin{aligned} _k\mu_2 &= {_k\kappa_2} = k\ \kappa_2 = k\ \mu_2 \ , \\ _k\mu_3 &= {_k\kappa_3} = k\ \kappa_3 = k\ \mu_3 \ , \end{aligned} \tag{19}$$

confirming thereby the known relations (7) and (9). For fourth and fifth orders we obtain by means of (15) and (18)

$$\begin{aligned} _k\mu_4 &= {_k\kappa_4} + 3\ {_k\kappa_2^2} = k\ \kappa_4 + 3\ (k\ \kappa_2)^2 \\ &= k(\mu_4 - 3\ \mu_2^2) + 3\ k^2\ \mu_2^2 = k\left[\mu_4 + 3\ k(k-1)\ \mu_2^2\right] \ , \\ _k\mu_5 &= {_k\kappa_5} + 10\ {_k\kappa_3}\ {_k\kappa_2} = k\ \kappa_5 + 10\ k^2\ \kappa_3\ \kappa_2 \\ &= k(\mu_5 - 10\ \mu_3\ \mu_2) + 10\ k^2\ \mu_3\ \mu_2 \\ &= k\left[\mu_5 + 10\ k(k-1)\ \mu_3\ \mu_2\right] \ . \end{aligned} \tag{20}$$

These two new formulae show that the simple additivity expressed by (19) is no longer valid for central moments of higher order.

Let us now come back to our original problem, the evaluation of the ordinary moments. For the case r = 1, the result ${}_km_1 = k\ m_1$ is obvious. For r = 2 and 3 one obtains by means of the previously established relations

$$
\begin{aligned}
{}_km_2 &= {}_k\kappa_2 + {}_k\kappa_1^2 = k\ \kappa_2 + k^2\ \kappa_1^2 = k(m_2 - m_1^2) + k^2\ m_1^2 \\
&= k\left[m_2 + (k-1)\ m_1^2\right] , \\
{}_km_3 &= {}_k\kappa_3 + 3\ {}_k\kappa_2\ {}_k\kappa_1 + {}_k\kappa_1^3 = k\ \kappa_3 + 3\ k^2\ \kappa_2\ \kappa_1 + k^3\ \kappa_1^3 \\
&= k(m_3 - 3\ m_2\ m_1 + 2\ m_1^3) + 3\ k^2(m_2 - m_1^2)\ m_1 + k^3\ m_1^3 \\
&= k\left[m_3 + 3(k-1)\ m_2\ m_1 + (k-1)(k-2)\ m_1^3\right] .
\end{aligned}
\tag{21}
$$

These two results are in agreement with the relations (6) and (8) obtained before. Likewise one can find, after a number of similar elementary rearrangements (omitted here), the following expressions for r = 4 and 5

$$
\begin{aligned}
{}_km_4 &= k\{m_4 + (k-1)\left[4\ m_3\ m_1 + 3\ m_2^2\right] + 6(k-1)(k-2)\ m_2\ m_1^2 \\
&\quad + (k-1)(k-2)(k-3)\ m_1^4\} , \\
{}_km_5 &= k\{m_5 + (k-1)\left[5\ m_4\ m_1 + 10\ m_3\ m_2\right] + (k-1)(k-2)\left[10\ m_3\ m_1^2\right. \\
&\quad \left. + 15\ m_2^2\ m_1\right] + 10(k-1)(k-2)(k-3)\ m_2\ m_1^3 \\
&\quad + (k-1)(k-2)(k-3)(k-4)\ m_1^5\} .
\end{aligned}
\tag{22}
$$

It is not difficult now to see what the general expression for a moment of order n will look like. Indeed, a comparison with the Bell polynomials listed by Riordan ([5], table 3) reveals that ${}_km_n$ can be identified with the polynomial Y_n if we put (in order to change from his notation to ours)

$$
\begin{aligned}
f_j &= k_{(j)} \equiv k(k-1)(k-2)\ \dots\ (k-j+1) \\
\text{and} \qquad g_j &= m_j .
\end{aligned}
\tag{23}
$$

If we want to express this fact in a more explicit way, we can write

$$
{}_km_n = n! \sum_{j=1}^{n} k_{(j)} \sum_{\pi(n,j)} \prod_{r=1}^{n} \frac{1}{j_r!} \left(\frac{m_r}{r!}\right)^{j_r} , \tag{24}
$$

where the second sum extends over all partitions $\pi(n,j)$ of m_r such that

$$\sum_{r=1}^{n} j_r = j \qquad \text{and} \qquad \sum_{r=1}^{n} r\, j_r = n ,$$

with $0 \leqslant j_r \leqslant j$. If the combination $n!/\prod_{r=1}^{n} j_r!\,(r!)^{j_r}$ is called a Bell coefficient ${}_nB_j(n+1-j,\, n-j,\, \ldots,\, 2,\, 1)$, as has been done previously in [7], the general formula can also be stated as

$$_k m_n = \sum_{j=1}^{n} k_{(j)} \sum_{\pi(n,j)} {}_nB_j(\ldots) \prod_{r=1}^{n} m_r^{j_r} . \tag{25}$$

Although equation (25) has not been derived here in a formal way, its general validity is more than likely. The appearance of the Bell polynomials in this context is not a surprise as they provide the link between the cumulants and the ordinary moments.

References

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