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Counting statistics of a Poisson process with dead time

Part I : General relations

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1. Introduction

In most experiments in nuclear physics, the numerical results are ultimately based on the counting of events which arrive at random. The permanent demand for improved precision and accuracy has given rise to a renewed interest in counting statistics permitting a reliable and efficient interpretation of the measured data.

A number of problems connected with dead time corrections of experimental counting distributions, first treated in the early thirties and fourties, have recently been the subject of new studies [1 to 6]. However, their practical usefulness is often somewhat limited, either by basing the arguments on over-simplified assumptions [1, 2, 4] or by the restricted evaluation of asymptotic expressions for the moments [5, 6]. A comprehensive survey [3], although including the case of extended dead times, is also incomplete and not always reliable in the details. In particular, the equilibrium or stationary process, which corresponds nowadays to a very common experimental situation, is hardly ever mentioned in these papers.

In view of the fact that a rigorous mathematical method for dealing with such problems, based on the operational calculus, has been indicated a long time ago [7], this is somewhat surprising. It seems that the very concise and elegant treatment of Jost [8], as well as the lucid exposition of Feller [9], which rightly emphasizes the narrow connection that exists between counting problems and renewal theory [10], have not always been well considered. In fact, the two papers [8] and [9], still classics in this field, provide us practically with all which is needed for tackling counting problems in a rigorous way. We must conclude, therefore, that R. Jost has been too optimistic in stating at the end of his paper: "It will be evident how this method can be applied to other problems".

Apart from the papers mentioned above (and probably many others of which we may be unaware), there is a certain number of articles that treat problems in counting statistics which arise e.g. in connection with variable dead times, pulsed sources, or dead times in series. However, all these more specialized questions will not be considered in what follows. In the present elementary review, of which this is the first part, emphasis lies primarily on exact results for the probability distributions, the expectations and the variances for the number of registered counts for some simple and well defined experimental situations. They should offer optimum conditions for a reliable comparison with the measurements and for extracting thereby a maximum of information. It is true that these exact expressions are usually more complicated than the corresponding approximations used till now. But this drawback seems to be largely compensated by the advantage -especially when an electronic computer is available - of avoiding possible systematic errors. In deriving the results, which are not always new, the essential steps will be discussed at some length. In particular, due attention will be given to the different possibilities of choosing the beginning of the counting-interval. The corresponding details will be worked out in later reports, whereas in this first part we shall restrict ourselves to the discussion of some general principles. More specific assumptions about the stochastic character of the original process and the effect of the dead time will be made in chapter four only.

2. Some definitions and basic facts

The simplest and probably the most powerful approach to problems in counting statistics consists in looking primarily at the development in time, i.e. at the interval-distributions. At a later stage only, the probability-distributions for the number of events in a given time interval will be determined. The reason for this procedure lies in the fact that the arrivals of pulses form what is called a renewal process [10], the statistical behaviour of which is completely defined by the probability density of successive events and a prescription relative to the choice of the time origin. As Feller [9] has clearly demonstrated, the problems then reduce to special instances of the theory of summation of independent random variables.

Since the time origin does not necessarily coincide with the arrival of an event, the density for the first renewal will in general be different from those for the subsequent intervals. In what follows we shall always use the following notation:

$g(t)$ = density for the arrival of the first event,

$f(t)$ = density for intervals between subsequent pulses.

In general*), $g(t)$ is only identical with $f(t)$ if the time origin is determined by the arrival of a pulse. In this case the sequence is said to form an ordinary renewal process [10].

Since the interval from the beginning to the arrival of event number k can evidently be thought of as composed of the waiting time to the first event and the sum of the $k-1$ independent intervals between succeeding pulses, the density $g_k(t)$ for event number k is given by the convolution

$$g_k(t) = g(t) * f_{k-1}(t) \quad , \qquad \text{for} \quad k \geqslant 1 \quad , \tag{1}$$

where $f_i(t) = \left\{ f(t) \right\} * i$

is a i-fold auto-convolution of the basic interval-density $f(t)$.

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*) The only, but important, exception is given by the Poisson process with its complete lack of memory.

In particular, we define $f_o(t) = \delta(t)$ and $f_1(t) = f(t)$; thus $g_1(t) = g(t)$.

The function $g_k(t)$ represents therefore the density for the effective arrival time of event number k. We may mention that all the densities $f_k(t)$ and $g_k(t)$ are correctly normalized to unity, provided this is the case for the initial densities f(t) and g(t).

It will soon be convenient to have also the cumulative distributions at our disposal. Therefore, we define for $k \geqslant 1$

$$F_k(t) \equiv \int_o^t f_k(x)\, dx \quad \text{and}$$

$$G_k(t) \equiv \int_o^t g_k(x)\, dx \quad . \tag{2}$$

For formal reasons, we may demand for k=0 that

$$F_o(t) = G_o(t) = U(t) ,$$

where U is the unit step function defined by

$$U(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{"} \quad t > 0 \end{cases} \quad .$$

Note, however, that the fictitious event at t = 0 is never counted. The normalization of $f_k(t)$ and $g_k(t)$ implies that

$$\lim_{t \to \infty} F_k(t) = \lim_{t \to \infty} G_k(t) = 1 \quad .$$

By means of the operational calculus, the relations can often be brought in a more convenient form. Therefore, whenever this seems to be advantageous, Laplace transforms will be used according to the definition

$$\tilde{f}(s) \equiv \mathcal{L}\{f(t)\} \equiv \int_o^\infty f(t) \cdot e^{-st}\, dt \quad .$$

As is well known, the main advantage in using operational methods stems from the fact that original convolutions reduce to simple products of the transforms. With this in mind, relation (1), for example, can be immediately transposed into the equivalent form ($k \geqslant 1$)

$$\tilde{g}_k(s) = \tilde{g}(s) \cdot \tilde{f}_{k-1}(s) = \tilde{g}(s) \cdot \left[\tilde{f}(s)\right]^{k-1} \tag{1'}$$

whereas its transformed cumulative distribution is

$$\tilde{G}_k(s) = \frac{1}{s} \cdot \tilde{g}_k(s) = \frac{\tilde{g}(s)}{s} \cdot \left[f(s)\right]^{k-1} \quad . \tag{2'}$$

So far, we have considered exclusively the interval-distributions. Now we have to establish a connection between them and the number of events in a given time interval since it is the probability for a given number of renewals that we are finally interested in.

If we designate by k_t the total number of random events that take place within a given time interval t and by t_k the time up to the k-th event, it is evident [10] that a relation of the form

$$k_t < k$$

implies the simultaneous validity of

$$t_k > t \quad .$$

In terms of the corresponding probabilities, this means

$$\text{Prob}\,(k_t < k) = \text{Prob}\,(t_k > t) = \int_t^\infty g_k(x)\,dx = 1 - G_k(t) \quad . \tag{3}$$

On the other hand, the probability $W_k(t)$ for exactly k renewals within an interval t is determined by the difference

$$W_k(t) = \text{Prob}\,(k_t < k+1) - \text{Prob}\,(k_t < k) \quad ,$$

thus

$$W_k(t) = G_k(t) - G_{k+1}(t) \quad . \tag{4}$$

This is the general relation for the probability that, in a renewal process with a known interval density, exactly k events happen during a time t. The practical evaluation of this expression for some specific cases will be studied later.

3. Expectation and variance for the number of events

A detailed comparison between the experimental and the theoretical distribution functions provides a sensitive check for the validity of the different assumptions made with regard to the underlying process. If this does not seem needed, or if the main interest lies in arriving at corrected mean values only, the evaluation of the first few moments may be appropriate.

The first moment, or expectation of k, is defined by

$$E_k(t) \equiv \hat{k} \equiv \sum_{k=0}^{\infty} k \cdot W_k(t) \quad .$$

Using (4), this leads to

$$E_k(t) = \sum_{k=1}^{\infty} k \cdot \left[G_k(t) - G_{k+1}(t)\right] = \sum_{1}^{\infty} k \cdot G_k(t) - \sum_{1}^{\infty} (k-1) \cdot G_k(t) = \sum_{k=1}^{\infty} G_k(t) \ . \quad (5)$$

The corresponding Laplace transform is

$$\tilde{E}_k(s) = \sum_{k=1}^{\infty} \tilde{G}_k(s) = \frac{1}{s} \sum_{1}^{\infty} \tilde{g}_k(s) \ .$$

But since according to (1)

$$\tilde{g}_k(s) = \tilde{g}(s) \cdot \left[\tilde{f}(s)\right]^{k-1} ,$$

we obtain

$$\tilde{E}_k(s) = \frac{\tilde{g}(s)}{s} \sum_{k=0}^{\infty} \left[\tilde{f}(s)\right]^k = \frac{\tilde{g}(s)}{s \left[1 - \tilde{f}(s)\right]} \ . \quad (6)$$

For the second moment about the origin*), we have likewise

$$E_{k^2}(t) \equiv \sum_{k=1}^{\infty} k^2 \cdot W_k(t) = \sum k^2 \cdot G_k(t) - \sum k^2 \cdot G_{k+1}(t)$$

$$= \sum k^2 \cdot G_k(t) - \sum (k-1)^2 \cdot G_k(t) = \sum (2k - 1) \cdot G_k(t)$$

$$= 2 \cdot \sum_{k=1}^{\infty} k \cdot G_k(t) - E_k(t) \ .$$

For the variance, i.e. the second moment with respect to the expectation, this leads to

$$V_k(t) \equiv \sigma_k^2 \equiv E_{k^2}(t) - E_k^2(t)$$

$$= 2 \cdot \sum_{k=1}^{\infty} k \cdot G_k(t) - E_k(t) - E_k^2(t) \quad (7)$$

The sum in the first term, i.e.

$$J(t) = \sum_{k=1}^{\infty} k \cdot G_k(t) \ ,$$

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*) A method for obtaining moments of higher order is outlined in the Appendix.

can be written, after transformation, as

$$\tilde{J}(s) = \sum_{1}^{\infty} k \cdot \tilde{G}_k(s) = \frac{\tilde{g}(s)}{s} \cdot \sum_{1}^{\infty} k \cdot \left[\tilde{f}(s)\right]^{k-1} = \frac{\tilde{g}(s)}{s \cdot \left[1 - \tilde{f}(s)\right]^2} \quad . \tag{8}$$

These expressions will be needed later for evaluating the asymptotic values, i.e. in the limit $t \gg \tau$.

Whereas for the expectation this is done by means of (6), the corresponding expression for the variance

$$\tilde{V}_k(s) = \frac{2 \cdot \tilde{g}(s)}{s \left[1 - \tilde{f}(s)\right]^2} - \frac{\tilde{g}(s)}{s \left[1 - \tilde{f}(s)\right]} - \mathcal{L}\left\{E_k^2(s)\right\} \tag{9}$$

contains in the last term the transform of the square of the expectation which will have to be evaluated first.

4. Poisson process and non-extended dead time

All the relations given so far are quite general and thus independent of any specific counter model. The only fundamental hypothesis is the recurrent nature of the underlying process.

In order to arrive at formulae for distributions which can be compared directly with the results of measurements, more specific assumptions have to be made. For practical as well as theoretical reasons, we shall assume that the following two simplifications hold strictly:

a) The initial series of events, e.g. the pulses originating from a radioactive source, forms a Poisson process with constant count rate ρ .

b) The dead time inserted into the original series is of the non-extended type, i.e. every registered event is followed by a constant time interval during which eventual other pulses are ignored.

We are aware, of course, that weaker assumptions could have been chosen and that a good part of the corresponding formalism has actually been worked out, in particular for a more general type of dead time (see e.g. [11]). However, instead of discussing the corresponding problems here, we rather prefer to restrict ourselves to a more detailed description of the simple situation outlined above.

As a matter of fact, this limitation does not narrow too much the practical usefulness of the results since reliable electronic circuits imposing a constant dead time of the non-extended type are readily available nowadays and the strictly Poissonian character of the radiation emitted in a radioactive decay can hardly be seriously questioned.*)

Our first hypothesis, namely that the original sequence of events forms a Poisson process with density ρ , implies that, for a fixed time interval of length $t > 0$, the probability for exactly k events is given by

$$P(k) = \frac{(\rho t)^k}{k!} \cdot e^{-\rho t} \quad . \tag{10}$$

This corresponds to an exponential interval-density**) ${}_o f(t)$ for consecutive pulses for

$$_o f(t) = P(o) \cdot \rho = U(t) \cdot \rho \cdot e^{-\rho t} \quad , \tag{11}$$

where U is the unit step function defined earlier.

The Laplace transform of (11) is simply

$$_o \tilde{f}(s) = \frac{\rho}{\rho + s} \quad . \tag{12}$$

Since the densities of multiple intervals correspond to repeated self-convolutions, i.e.

$$_o f_k(t) = \left\{ {}_o f(t) \right\}^{*k} \quad ,$$

its transform is given by

$$_o \tilde{f}_k(s) = \left(\frac{\rho}{\rho + s} \right)^k \quad .$$

By the use of tables, for instance, the corresponding original is found to be a gamma density of the form

$$_o f_k(t) = U(t) \cdot \frac{\rho \cdot (\rho t)^{k-1} \cdot e^{-\rho t}}{(k-1)!} \quad . \tag{13}$$

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*) As for the recent observations of Berkson [12], we think that no conclusion can be drawn. The fact that the experimental values for variance and expectation are not always equal, as would be expected for a pure Poisson process, is not too surprising. Whereas a reduction of the variance is a simple consequence of the presence of a dead time, as will be discussed later in more detail, any instability of the source or of the electronic counting device would result in an increase of the relative variance. Therefore, conclusive evidence for possible deviations from a Poisson process can only be expected from a more critical and elaborate analysis of experimental data.

**) The subscript "o" reminds that all this is without dead time. At a later stage, it will therefore automatically disappear.

Now, the effect of the dead time has to be taken into account, which is supposed to be of the non-extended type. In general, the change in the interval-density, caused by the insertion of such a dead time, can be quite involved [13]. However, this happens to be very simple just for our case of a Poisson process where the exponential function (11) is simply shifted in time by the amount of the dead time, thus

$$f(t) = {}_{o}f(t) * \delta(t - \tau) = U(t-\tau) \cdot \rho \cdot e^{-(t-\tau)} \quad . \tag{14}$$

The Laplace transform of this new density is therefore

$$\tilde{f}(s) = {}_{o}\tilde{f}(s) \cdot e^{-s\tau} = \frac{\rho \cdot e^{-s\tau}}{\rho + s} \quad . \tag{15}$$

Likewise, the density for multiple intervals is now given by

$$f_k(t) = \left\{ f(t) \right\}^{*k}$$

with the transform

$$\tilde{f}_k(s) = \left[\tilde{f}(s) \right]^k = \left(\frac{\rho}{\rho + s} \right)^k \cdot e^{-ks\tau} \quad . \tag{16}$$

The corresponding original is therefore of the type (13), where all the times are retarded by $k \cdot \tau$, thus

$$f_k(t) = U(t - k\tau) \cdot \frac{\rho \left[\rho \, (t-k\tau) \right]^{k-1} \cdot e^{-\rho(t-k\tau)}}{(k-1)!} \quad . \tag{17}$$

We may call this a shifted gamma density (for $k \geqslant 1$).

The corresponding cumulative distribution is

$$F_k(t) = \int_0^t f_k(x) \, dx$$

$$= \int_0^t U(x-k\tau) \cdot \frac{\rho \left[\rho \, (x-k\tau) \right]^{k-1}}{(k-1)!} \cdot e^{-\rho(x-k\tau)} \, dx$$

$$= \int_0^{\rho(t-k\tau)} U(z) \cdot \frac{z^{k-1}}{(k-1)!} \cdot e^{-z} \, dz \quad .$$

Putting

$$\rho(t - k\tau) \equiv T_k \quad , \tag{18}$$

we obtain

$$F_k(t) = \frac{U(T_k)}{\Gamma(k)} \cdot \int_o^{T_k} z^{k-1} \cdot e^{-z}\, dz \quad , \tag{19}$$

which is an incomplete gamma function. Although this function has been extensively tabulated [14], it is much more convenient for our purpose to take advantage of the relation which connects it with the cumulative Poisson distribution, namely [15]

$$\int_o^x e^{-t} \cdot t^{n-1}\, dt = \Gamma(n) \cdot \sum_{i=n}^{\infty} \frac{x^i}{i!} \cdot e^{-x} \quad . \tag{20}$$

Again, tables for the cumulative Poisson probabilities would be readily available (e.g. [16]), but the use of numerical values at this early stage of the development cannot be recommended. If (20) is applied to (19), instead, F can be expressed by a sum of Poisson-like terms, namely

$$F_k(t) = U(T_k) \cdot \left\{ 1 - \sum_{i=1}^{k-1} P_k(i) \right\} \quad , \tag{21}$$

where $P_k(i)$ stands for the shifted Poisson probability

$$P_k(i) \equiv \frac{T_k^i}{i\,!} \cdot e^{-T_k} \tag{22}$$

with T_k defined in (18).

Thus, $F_o(t) = U(t)$ and $F_k(t)$ vanishes for $k\tau \geqslant t$.

The latter statement follows from the fact that, according to the definitions of U and T_k ,

$$U(T_k) = \begin{cases} 1 & \text{for } 0 \leqslant k \leqslant K \\ 0 & \text{" } \quad k > K \; , \end{cases}$$

where

$$K \equiv \left[\left[\frac{t}{\tau} \right]\right] \tag{23}$$

denotes the largest integer below t/τ .

5. On the choice of the time origin

The general formula (4) shows how the probabilities $W_k(t)$ are determined by the functions $G_k(t)$ which, in turn, depend on g(t), i.e. the density for the first interval. On the other hand, this first interval is not uniquely determined by the nature of the underlying renewal process, which is completely specified by f(t). On the contrary, g(t) can also be strongly influenced by the way the time origin, i.e. the beginning of the counting-interval, is chosen. Therefore, since the resulting distributions depend on the choice of the time origin, our description, too, must take into account the specific experimental conditions if we want to arrive at formulae which can be really applied to the results of measurements.

From now onwards, therefore, some definite assumption has to be made about the selection principle according to which the beginning of the time interval for the counting period is determined. This will then permit to specify the function g(t) for the first interval-density.

For this purpose, essentially two different models will be proposed. In the first case, the time origin is chosen completely at random. This situation and the corresponding statistics will be studied in some detail in part II. The other case, where the beginning is related to a counted event, gives rise to some different developments, to be described in part III from a general point of view. From this it will be easy to obtain results for the special situations where the counting-period is initiated by a registered event or where the beginning never falls within the dead time of a previous pulse.

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APPENDIX

Evaluation of higher moments for the number of renewals

The direct way for determining moments, as used in deriving formulae (6) to (8), cannot be followed in general since the corresponding summations would become too much involved. However, there is a method of circumventing this difficulty. It consists in determining first the so-called factorial moments and then using their relation to the ordinary moments for obtaining these.

The factorial moment of order n of a positive integral random variable x is defined by the expectation

$$m_{(n)}(x) \equiv E\left\{x_{(n)}\right\} , \qquad \text{(A1)}$$

where $x_{(n)} \equiv x \cdot (x-1) \cdot (x-2) \cdot \ldots \cdot (x-n+1) = \binom{x}{n} \cdot n!$
is a "falling n-factorial" [17] .

In our case, this yields as a factorial moment of order n the expression

$$E_{k_{(n)}}(t) = \sum_k k_{(n)} \cdot W_k(t) = \sum_k k_{(n)} \cdot G_k(t) - \sum_k k_{(n)} \cdot G_{k+1}(t) .$$

Since for $k+1 = k'$

$$k_{(n)} = \frac{k'-n}{k'} \cdot k'_{(n)} ,$$

we can also write

$$E_{k_{(n)}}(t) = \sum_k k_{(n)} \cdot G_k(t) - \sum_{k'} \left(1 - \frac{n}{k'}\right) k'_{(n)} \cdot G_{k'}(t)$$

$$= n \sum_{k'} \frac{k'_{(n)}}{k'} \cdot G_{k'}(t) = n \sum_k k_{(n-1)} \cdot G_{k+1}(t) .$$

But according to (1)

$$\widetilde{G}_k(s) = \frac{\widetilde{g}(s)}{s} \cdot \left[\widetilde{f}(s)\right]^{k-1} ,$$

which gives

$$\widetilde{E}_{k_{(n)}}(s) = n \cdot \frac{\widetilde{g}(s)}{s} \cdot \sum k_{(n-1)} \cdot \left[\widetilde{f}(s)\right]^k . \qquad \text{(A2)}$$

Among the formulae which look promising for evaluating this sum the best we found was [18]

$$\sum_{n=0}^{\infty} z^n \cdot (n+1) \cdot (n+2) \cdot (n+3) \cdot \ldots \cdot (n+m-1) = \frac{(m-1)!}{(1-z)^m} ,$$

where n and m are positive integers and $|z| < 1$.

Putting n+m-1 = x and m-1 = r , we obtain in fact

$$\sum_{x=0}^{\infty} x_{(r)} \cdot z^x = \frac{z^r \cdot r\,!}{(1-z)^{r+1}} \quad , \qquad \text{for} \quad r > 0 \ . \tag{A3}$$

This allows us to evaluate the Laplace transform of the factorial of order n as

$$\tilde{E}_{k_{(n)}}(s) = n\,! \cdot \frac{\tilde{g}(s)}{s} \cdot \frac{\left[\tilde{f}(s)\right]^{n-1}}{\left[1-\tilde{f}(s)\right]^{n}} \ . \tag{A4}$$

The corresponding relation for the ordinary moments is now easily obtained by applying a general relation which connects powers with factorials in a general way, i.e. the decomposition [17]

$$x^n = \sum_{k=1}^{n} S(n,k) \cdot x_{(k)} \quad , \qquad \text{for} \ n > 0 \quad , \tag{A5}$$

where S are the Stirling numbers of the second kind for which tables are available [15, 19] . In view of the definitions

$$m_n(x) = E\left\{x^n\right\} \quad \text{and}$$

$$m_{(n)}(x) = E\left\{x_{(n)}\right\}$$

for the ordinary and the factorial moments, respectively, (A5) is equivalent to

$$m_n(x) = \sum_k S(n,k) \cdot m_{(k)}(x) \quad .$$

Applying this relation to (A4) gives

$$\tilde{E}_{k^n}(s) = \sum_j S(n,j) \cdot \tilde{E}_{k_{(j)}}(s) = \frac{\tilde{g}(s)}{s} \sum_{j=1}^{n} j\,!\, S(n,j) \cdot \frac{\left[\tilde{f}(s)\right]^{j-1}}{\left[1-\tilde{f}(s)\right]^{j}} \ . \tag{A6}$$

This is the general formula for the Laplace transform of the ordinary moment of order n for the number of renewals in t. A similar result is given without proof by Takács [20] .

The numerical values for the coefficientsare listed below for $n \leq 5$.

	j = 1	2	3	4	5
n = 1	1	-	-	-	-
2	1	2	-	-	-
3	1	6	6	-	-
4	1	14	36	24	-
5	1	30	150	240	120

Table for the coefficients $j\,! \cdot S(n,j)$ in equation (A6).

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