B.I.P.M.
F-92 SEVRES

Report BIPM-106

On the influence of two consecutive dead times

Jörg W. Müller

Introduction

It is evident that in general the insertion of a dead time in an electronic circuit will change the stochastic features of a sequence of transmitted pulses. In an earlier communication [7], the deformation of an original interval-distribution $I(t)$ to the new probability density function $f_1(t)$, caused by a dead time τ_1, has been treated.

However, it often happens that for experimental reasons, dead times have to be imposed at different points in a circuit. It may be of interest therefore to know the combined effect of such dead times "in series" (Fig. 4)*) on the final interval-density $W(t)$, a function, from which most of the physically interesting quantities of the process (e.g. count rates) can be easily obtained.

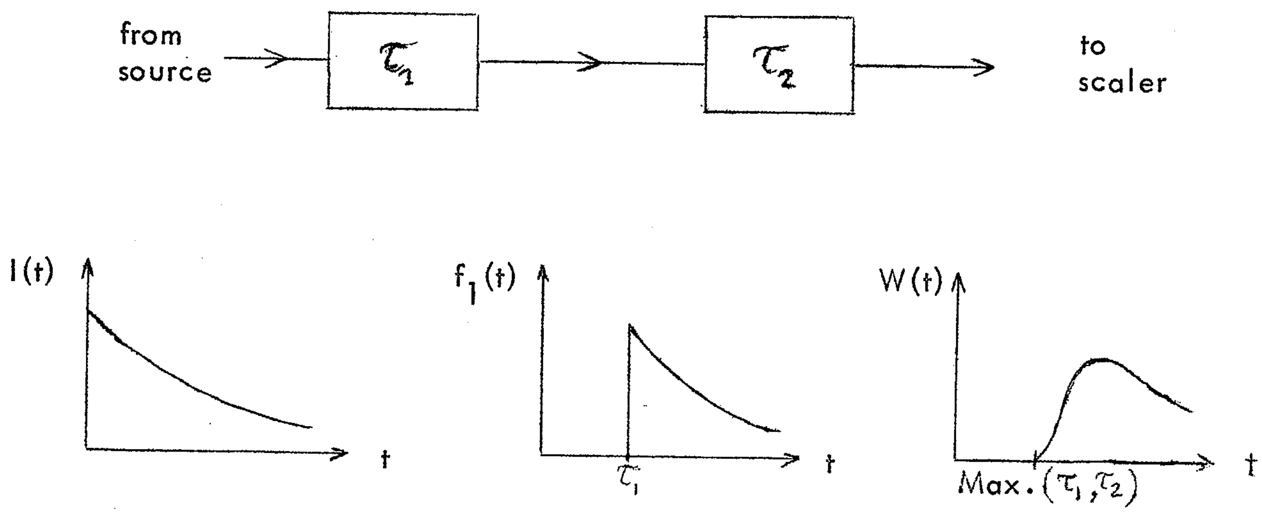

Fig. 4: Block diagram and corresponding interval-densities (schematically drawn)

*) Since this report is a direct continuation of [7], we simply proceed with the consecutive enumeration of the formulae, figures, and references.

For the sake of simplicity, let us confine ourselves in the following discussion to two consecutive non-extended dead times and to the case of most evident practical interest, i.e. to an original Poisson distribution

$$I(t) = U(t) \cdot \varrho \cdot e^{-\varrho t} \quad , \tag{13}$$

U being the unit function defined in (3) and ϱ the "source" rate. In this case (cf. (7)), the interval-density after the insertion of τ_1 is easily shown to be

$$f_1(t) = U(t - \tau_1) \cdot \varrho \cdot e^{-\varrho(t - \tau_1)} \quad . \tag{14}$$

As a result of the complete independence of different events in a renewal process, the density for a k-fold interval is simply given by the convolution

$$f_k(t) = \left\{ f_1(t) \right\}^{*k} \quad .$$

Applying integral tranforms, e.g., this may be readily shown to correspond to

$$f_k(t) = U(t-k\cdot\tau_1) \cdot \frac{\varrho^k}{(k-1)!} \cdot (t-k\cdot\tau_1)^{k-1} \cdot e^{-\varrho(t-k\cdot\tau_1)} \quad , \tag{15}$$

which is a displaced gamma distribution.

Determination of the interval-distribution

When we calculate the final density $W(t)$, our procedure is analogous to that of [7] . Therefore, we must first determine the probability density function for the interval between two registrations, in which exactly k intermediate pulses (in the original sequence) have been lost because they arrived during the dead time τ_1. Inserting (14) and (15) into the general formula (3), straightforward manipulation leads to

$$W_k(t) = U(t-\tau_2)\cdot\int_0^{\tau_2} f_k(x)\cdot f_1(t-x)\,dx$$

$$= U(t-\tau_2)\cdot\frac{\rho^{k+1}}{(k-1)!}\cdot\int_{k\cdot\tau_1}^{M}(x-k\cdot\tau_1)^{k-1}\cdot e^{-\rho\{t-(k+1)\tau_1\}}\,dx$$

$$= U(t-\tau_2)\cdot\frac{\rho^{k+1}}{k!}\cdot(M-k\cdot\tau_1)^k\cdot e^{-\rho\{t-(k+1)\tau_1\}}, \qquad (16)$$

where $M = \text{Min.}\left\{\tau_2\,,\ \text{Max.}(t-\tau_1\,,\ k\cdot\tau_1)\right\}$,

since the unit functions cause the integrand to vanish for $x < k\tau_1$ and $x > t-\tau_1$.

We can assume without any loss of generality that $\tau_1 \leqslant \tau_2$. Then, the following relations hold for the different domains of t:

$$W_k(t) = \begin{cases} 0 & \text{for } t < M' \\ \dfrac{\rho^{k+1}}{k!}\cdot e^{\rho(k+1)\tau_1}\cdot\left\{t-(k+1)\tau_1\right\}^k\cdot e^{-\rho t} & \text{" } M' < t < \tau_1+\tau_2 \\ \dfrac{\rho^{k+1}}{k!}\cdot e^{\rho(k+1)\tau_1}\cdot(\tau_2-k\cdot\tau_1)^k\cdot e^{-\rho t} & \text{" } t > \tau_1+\tau_2 \end{cases} \qquad (17)$$

with $M' = \text{Max.}\left\{\tau_2\,,\ (k+1)\tau_1\right\}$.

Fig. 5 is a graphical representation analogous to Fig. 1 and 2. It shows immediately that for $\tau_1 \gg \tau_2$ the "truncated" density $f_1''(t)$ always coincides with the original $f_1(t)$. In this case, therefore, the second dead time τ_2 has no influence whatsoever.

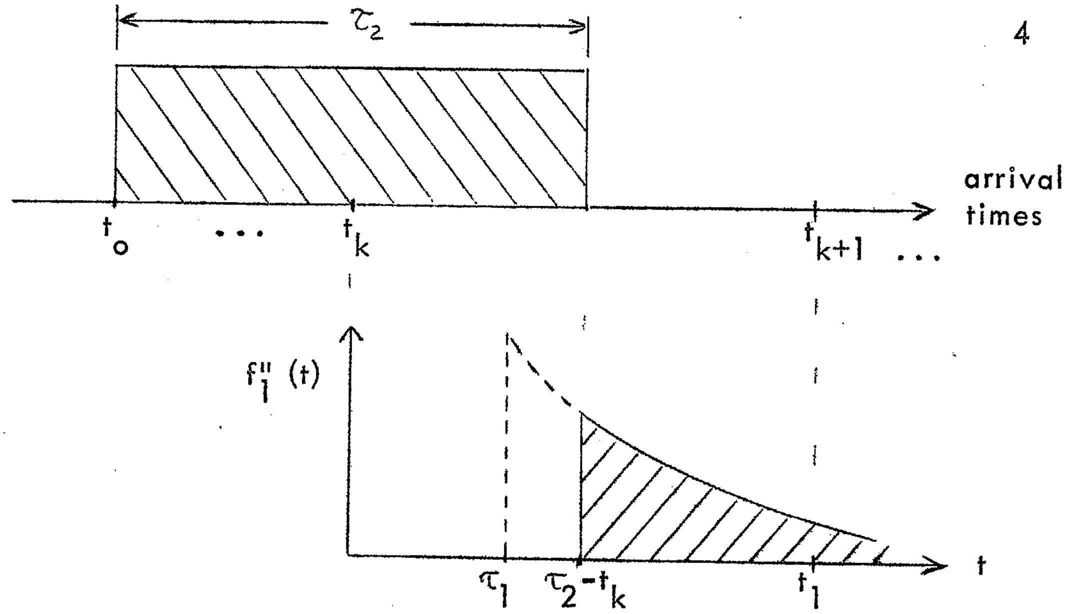

Fig. 5

Since there is a minimum distance $\tau_1 \geqslant 0$ between the pulses with the interval-density $f_1(t)$, it is evident that there exists an upper limit K for the number k of possible events which may happen during the second dead time τ_2. If a bracket like $[x]$ is used to signify the largest integer below x, then this limit is simply given by

$$K = \left[\tau_2 / \tau_1 \right] \tag{18}$$

and according to (4), the observable interval-density after both dead times is determined by the sum

$$W(t) = \sum_{k=0}^{K} W_k(t) \quad , \tag{19}$$

which is in general finite (except for $\tau_1 = 0$).

For purposes of control, the interval-densities for the two special cases where $\tau_1 = \tau_2$ and $\tau_1 = 0$ may be conveniently checked. For convenience, we write τ for τ_2.

1) For $\tau_1 = \tau_2$, the summation in (19) reduces to one term, since $K = [1] = 0$. In this case, (17) leads easily to

$$W(t) = W_o(t) = U(t-\tau) \cdot \rho \cdot e^{-\rho(t-\tau)} \quad ,$$

from which we conclude that the effect of two consecutive dead times of equal duration is identical to that of one alone.

2) For $\tau_1 = 0$, it follows from (17) that

$$W_k(t) = U(t-\tau)\ \frac{\rho\,(\rho\tau)^k}{k!}\cdot e^{-\rho t} \quad .$$

As in (7), this then leads directly to

$$W\ (t) = U(t-\tau)\cdot\rho\cdot e^{-\rho(t-\tau)} .$$

Although it is possible to measure directly the frequency distribution of the intervals [6] and to check by this means $W(t)$ or even $W_k(t)$, it is evidently much simpler to determine the corresponding count rate. This quantity is of great practical importance and therefore currently measured with high precision. Several fields, such as absolute activity measurements, also call for the utmost in accuracy. However, since we have just realized that the interval-densities $W_k(t)$ depend, in general, on both dead times, it is clearly of interest to know how, when taken together, they will affect the experimental count rate R. This quantity, in turn, is given by the average time interval $\bar{t}$ between successive pulses, i.e.

$$R = 1/\bar{t} \; . \tag{20}$$

Therefore, our next problem is to calculate

$$\bar{t} = \int_0^\infty \sum_{k=0}^{K} t\cdot W_k(t)\ dt \equiv \sum_{k=0}^{K} t_k \quad , \tag{21}$$

where t_k is an abbreviation for

$$t_k = \int_0^\infty t\cdot W_k(t)\ dt \quad ,$$

which may be interpreted as the average interval between two output-pulses, when k intermediate events in the original sequence have been eliminated by the first dead time τ_1 .

By means of (17), this yields

$$t_k = \frac{\rho^{k+1}}{k!} \cdot e^{\rho(k+1)\tau_1} \cdot \left[\int_{M'}^{\tau_1+\tau_2} t \left\{ t-(k+1)\tau_1 \right\}^k \cdot e^{-\rho t} \, dt + \int_{\tau_1+\tau_2}^{\infty} t \left\{ \tau_2 - k\tau_1 \right\}^k \cdot e^{-\rho t} \, dt \right] .$$

After a number of simple, but somewhat lengthy rearrangements, this can also be written as

$$t_k = \frac{\rho^k}{k!} \cdot e^{-\rho T_k} \cdot \left[T_k^k \left(\frac{1}{\rho} + \tau_1 + \tau_2 - T_k \right) + e^{\rho \tau_1} \cdot T_{k+1}^{k+1} \right] + (k+1) \cdot \left(\frac{1}{\rho} + \tau_1 \right) \cdot \left[P(k+1, \rho T_k) - P(k+1, \rho T_{k+1}) \right] , \qquad (22)$$

where $T_k = \mathrm{Max.} \left\{ \tau_2 - k\tau_1 , \; 0 \right\}$

and $$P(n, \lambda) = \sum_{j=n}^{\infty} \frac{e^{-\lambda} \cdot \lambda^j}{j!}$$

is the cumulative Poisson distribution, the numerical values of which are well tabulated in [8].

<u>Special cases</u>

Before treating the general case and displaying some rather surprising results of the numerical calculations based on (21) and (22) - a task which has to be deferred to another report - we may first evaluate the average interval $\bar{t}$ for two limiting cases, where the results are considerably simpler and where they can be readily checked. Again, τ means τ_2.

1) Let $\tau_1 = \tau_2$.

Therefore, we have $K = 0$ and there remain only the terms $T_o = \tau$ and $T_1 = 0$.

Since $P(1, \rho\tau) = 1 - e^{-\rho\tau}$ and $P(1,0) = 0$, we immediately obtain as expected

$$\bar{t} = \sum_{k=o}^{K} t_k = t_o = \frac{1}{\rho} + \tau \; .$$

2) Let $\tau_1 = 0$.

In this case, we have $K = \infty$ and $T_k = \tau$ holds for any k.

Since now the differences $P(k+1, \rho T_k) - P(k+1, \rho T_{k+1})$ all vanish, there only remains

$$t_k = \frac{\rho^k}{k!} \cdot e^{-\rho\tau} \cdot \left(\frac{\tau^k}{\rho} + \tau^{k+1}\right) = \left(\frac{1}{\rho} + \tau\right) \cdot e^{-\rho\tau} \cdot \frac{(\rho\tau)^k}{k!} \; .$$

Thus, we get

$$\bar{t} = \sum_{k=o}^{\infty} t_k = \left(\frac{1}{\rho} + \tau\right) \cdot e^{-\rho\tau} \sum_k \frac{(\rho\tau)^k}{k!} = \frac{1}{\rho} + \tau \; ,$$

which is the well-known result for the average interval between pulses after they have passed a single non-extended dead time.

References

[7] J.W. Müller: "On the interval-distribution for recurrent events with a non-extended dead time", Report BIPM-105 (1967)

[8] E.C. Molina: "Poisson's Exponential Binomial Limit", (Van Nostrand, Princeton; 1942)

(March, 1968)