B.I.P.M.
F-92 SEVRES

Report BIPM-105

On the interval-distribution for recurrent events with a non-extended dead time

Jörg W. Müller

General case

We consider the arrival of impulses at a counter and assume that this sequence of events forms a recurrent (or renewal) process [1,2]. In this case the time differences between subsequent pulses are equidistributed and independent random variables. Therefore the stochastic behaviour of this process is fully described by the density function of the intervals $t = \{t_{i+1} - t_i\}$, which we may denote by $f_1(t)$.

Let us suppose now that by a non-paralyzable counter or by a circuit specifically designed for this purpose a fixed resolving (or dead) time τ is inserted in this sequence. As a result, an event is registered if, and only if, no other registration has taken place during the preceding time interval of length τ. Any impulse arriving during the dead time is neglected and has no influence whatsoever.

We want now to determine the interval-distribution for such a sequence. The result will show that in general (perhaps contrary to what might be expected) the dead time does not simply produce a cut-off or a shift of the original interval-distribution, but may change its shape completely.

To determine this distribution we start with a registered pulse at $t_o = 0$ (cf. Fig. 1) and seek the probability of getting the next (registered) event at a time t. Since we cannot know how many impulses have arrived during the dead time τ, we shall have to sum all possibilities.

In the case when t_k was the last event within τ, the first pulse to be counted will be t_{k+1} (Fig. 1). The total time t between the two subsequent registrations may thus be subdivided into two independent intervals, namely

$$t = t_{k+1} = t_k + t_1 \quad . \tag{1}$$

Therefore, the corresponding probability density is easily determined formally by means of the convolution

$$W_k(t) = f'_k(t) * f'_1(t) , \qquad t > \tau \tag{2}$$

where $f'_k(t) = \begin{cases} \{f_1(t)\}^{*k} & \text{for } t < \tau \\ 0 & \text{" } t > \tau \end{cases}$

and $f'_1(t) = \begin{cases} 0 & \text{for } t < \tau - t_k \\ f_1(t) & \text{" } t > \tau - t_k \end{cases}$

are two "truncated" density functions for the events k and l, respectively (Fig. 2).

Fig. 1

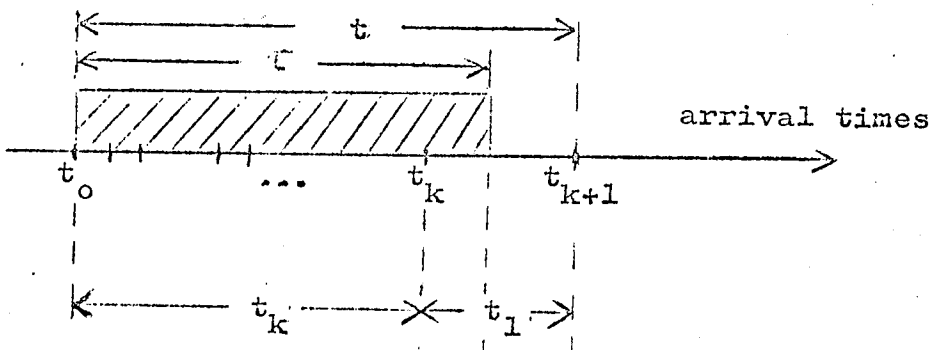
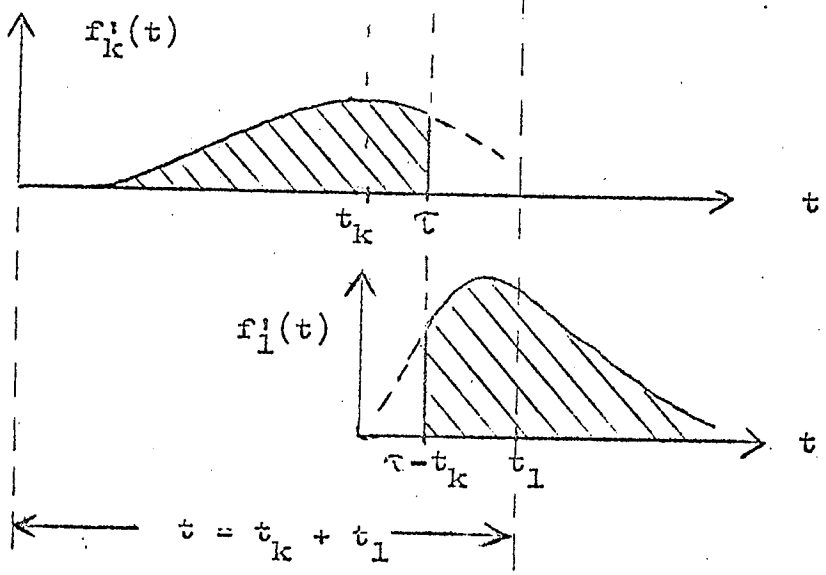


Fig. 2



In order to see how the convolution (2) can be practically evaluated, we write it in the more conventional way as

$$W_k(t) = U(t-\tau) \cdot \int_{-\infty}^{\infty} f'_k(x) \cdot f'_1(t-x)\, dx$$

$$= U(t-\tau) \cdot \int_{0}^{\tau} f_k(x) \cdot f_1(t-x)\, dx \quad , \qquad (3)$$

where $U(y) = \begin{cases} 1 & \text{for } y > 0 \\ 0 & " \quad y < 0 \end{cases}$

is the unit step function.

Thus, the function f'_k can be replaced by f_k if we stop with the integration in (3) at the end of the dead time, whereas the truncation of f_1 has no influence whatsoever, since for $t > \tau$ the values of f_1 for arguments below $\tau - x$ are never used for the integration (3). This can also be seen from Fig. 3 which gives a graphical representation of the convolution (2) in such a way that the corresponding values of the functions f'_k and f'_1 which have to be multiplied are vertically one above the other.

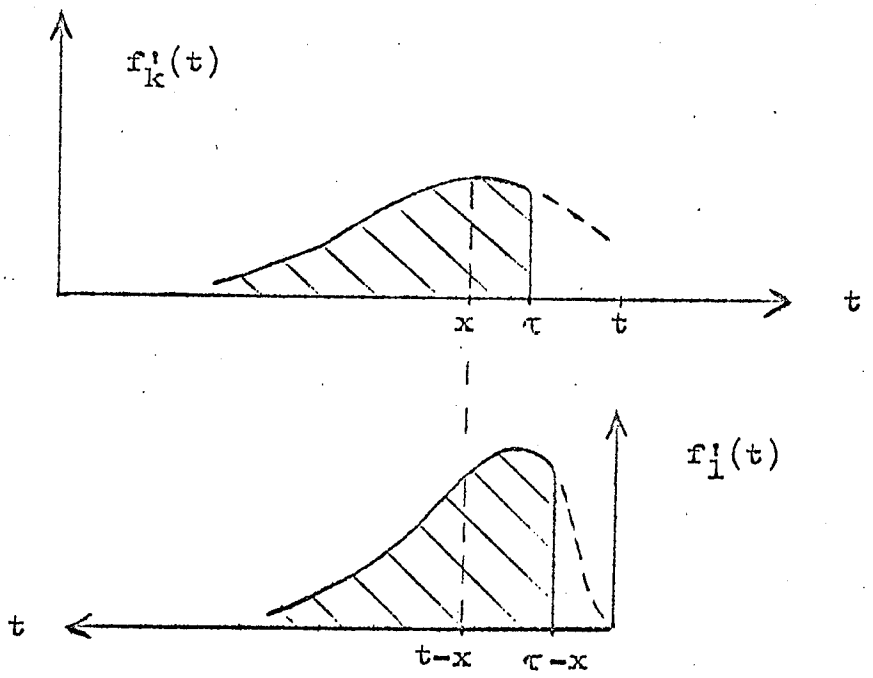


Fig. 3

It is now easy to get the total density by summing all possible combinations. Therefore, the general formula for the interval-distribution as modified by a non-extended dead time is

$$W(t) = U(t-\tau) \cdot \sum_{k=0}^{\infty} W_k(t) = U(t-\tau) \left\{ f_1(t) + \sum_{k=1}^{\infty} \int_0^{\tau} f_k(x) \cdot f_1(t-x)\, dx \right\}, \quad (4)$$

since $f_o(x) = \delta(x)$ and with $f_k \equiv \{f_1\}^{*k}$.

It may be noted that as a result of the definition (2) no special normalization is needed in (4) when the conditional probabilities W_k are added.

Applications

1) As a simple but particularly important example we first determine the interval-distribution for a Poisson process with density ϱ . Since here, as is well known (cf. e.g. [2]), the distribution for a k-fold interval is of the form

$$f_k^{(P)}(t) = \frac{\varrho\,(\varrho t)^{k-1}}{(k-1)!} \cdot e^{-\varrho t} , \quad (5)$$

we get easily from (3)

$$W_k^{(P)}(t) = U(t-\tau) \cdot \frac{\varrho^{k+1}}{(k-1)!} \cdot e^{-\varrho t} \int_0^{\tau} x^{k-1}\, dx = U(t-\tau) \cdot \frac{\varrho\,(\varrho\tau)^k}{k!} \cdot e^{-\varrho t}. \quad (6)$$

Therefore, for a Poisson process, (4) gives the well-known result

$$W^{(P)}(t) = U(t-\tau) \cdot e^{-\varrho t} \cdot \varrho \cdot \sum_{k=0}^{\infty} \frac{(\varrho\tau)^k}{k!} = \varrho \cdot e^{-\varrho(t-\tau)} \quad (7)$$

for $t > \tau$,

i.e. the exponential shape of the density distribution is not changed, but the whole curve is shifted by the amount τ to the right. It should be noted, however, that this simple behaviour is only true for a pure Poisson process, as can easily be shown.

2) Another example which does not lead to very involved numerical computation is given by the interval-distribution at the output of an electronic scaler where a dead time of the non-extended type has been inserted. Besides, in this case the results can easily be checked experimentally by a direct simulation of the process.

It can readily be shown that for a Poisson process as an input, the intervals at the output of a scaler are distributed according to a gamma function. Therefore, the density of the event k after a scaling factor s is given by

$$f_k^{(s)}(t) = \left\{ f_s^{(P)}(t) \right\}^{*k} = \frac{\rho(\rho t)^{ks-1}}{(ks-1)!} \cdot e^{-\rho t} \quad . \tag{8}$$

This leads directly to

$$f_k^{(s)}(x) \cdot f_1^{(s)}(t-x) = \frac{\rho^{(k+1)s} \cdot x^{ks-1} \cdot (t-x)^{s-1}}{(ks-1)! \cdot (s-1)!} \cdot e^{-\rho t} \quad ,$$

i.e.

$$W_k^{(s)}(t) = U(t-\tau) \cdot \int_0^{\tau} f_k^{(s)}(x) \cdot f_1^{(s)}(t-x)\, dx$$

$$= U(t-\tau) \cdot \frac{(\rho t)^{(k+1)s} \cdot e^{-\rho t}}{t\,(ks-1)! \cdot (s-1)!} \cdot B_{\tau/t}(ks, s) \quad , \tag{9}$$

where $B_p(a, b) = \int_c^p y^{a-1} \cdot (1-y)^{b-1}\, dy$

is the incomplete beta function. There exists an extensive tabulation [3]

for the ratios

$$I_p(a, b) = B_p(a, b) / B(a, b) ,$$

where $B(a, b) = \frac{(a-1)!\,(b-1)!}{(a+b-1)!}$

is the usual complete beta function.

On the other hand, we may also take advantage of a relation which exists between the incomplete beta function and the binomial expansion ([4] ,p. 263), i.e.

$$B_p(a, n-a+1) = B(a, n-a+1) \cdot \sum_{i=a}^{n} \binom{n}{i} p^i \cdot (1-p)^{n-i} \quad . \tag{10}$$

This permits a convenient use of the tables giving the cumulative binomial probabilities (e.g. [5]), which may be easier to find. With such a table at hand giving the values for

$$A_p(n, a) = \sum_{i=a}^{n} \binom{n}{i} p^i \cdot (1-p)^{n-i} \quad , \tag{11}$$

we can conveniently evaluate numerically the distribution of the intervals in a Poisson process of density ρ , which is scaled down by a factor s and where a dead time of length τ has been inserted, as

$$W^{(s)}(t) = U(t-\tau) \cdot \sum_{k=0}^{\infty} W_k^{(s)}(t)$$

$$= \frac{\rho\,(\rho t)^{s-1}}{(s-1)!} \cdot e^{-\rho t} \left\{ 1 + \sum_{k=1}^{\infty} \frac{(\rho t)^{ks}}{(ks-1)!} \cdot B(a, b) \cdot A_p(n, a) \right\}$$

for $t > \tau$ (12)

with $p = \tau / t$ and
$n = a + b - 1$.

In the limit $p = 1$, i.e. immediately after the end of the dead time, (12) may be much simplified, since $A_1(n, a) = 1$.

Numerical calculations (for $s = 8$ and different values of p) have shown that an expansion of (12) up to $k=2$ is usually quite sufficient, the remaining terms being negligible. These calculated curves show excellent agreement with the distributions obtained experimentally by means of an automatic electronic interval displayer [6].

References

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[6] P. Bréonce and J.W. Müller: "Construction of an Electronic Interval Displayer", Report BIPM-102 (1966)

(November, 1967)