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On an intricate transition to the Poisson limit

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1. Introduction

The distortion of an original Poisson process produced by insertion of a dead time is a subject which has been studied extensively and for a long time. It may be surprising, therefore, that we are still faced with unsolved problems in this field, some of which are of great practical importance.

The question we try to answer in this report is a minor one and concerns a technical detail. The unexpected mathematical obstacle occurs when one tries to verify a limiting case which, for obvious physical reasons, must lead to a simple Poisson process.

The problem turned up recently when we were looking for approximate expressions for the modified distributions, and its solution was essential for proceeding further.

If $W_k(t)$ denotes the probability of observing k events within a time interval t for an original Poisson process with count rate ρ which is now distorted by a dead time τ, we must obviously expect that

$$\lim_{\tau \to 0} W_k(t) = \frac{\mu^k}{k!} \cdot e^{-\mu} \equiv P_\mu(k) , \tag{1}$$

where $\mu = \rho \cdot t$.

As one usually distinguishes between three counting processes (depending on the choice of the time origin) and two types of dead time, there are six different expressions for $W_k(t)$ to be considered, and (1) should be valid for any of them. In most cases it is very easy to verify, starting from the exact expression of $W_k(t)$ corresponding to a given experimental situation, that relation (1) indeed holds. However, there is one case where the derivation of this limit is not obvious, and it is this problem which we shall treat in what follows.

Since the case of a non-extended dead time has been dealt with earlier and raises no problem for the verification of (1), we restrict ourselves to the situation with an extended dead time. As for the classification of the three counting processes considered, the necessary information can be found in earlier publications ([1] to [4]).

It is rather surprising to see that an exact* formula for $W_k(t)$ was published as long ago as 1943 by Kosten [5] for the case of an extended dead time. In our usual notation it reads, for $t > 0$ and $0 \leqslant k \leqslant K+1$,

$$W_k(t) = \sum_{j=k}^{K+1} \frac{(-1)^{j-k}}{k!\,(j-k)!} (T_{j-1} \cdot e^{-x})^j \quad , \tag{2}$$

where K is the largest integer below t/τ, $x = \rho\tau$ and $T_s = \rho(t - s\tau) = \mu - sx$.

Three years later a new derivation of the same formula was given by Hole [6] .

In 1951 a slightly different expression was published by Ramakrishnan [7] apparently for the same problem which reads, for $0 \leqslant k \leqslant K$,

$$W_k(t) = \sum_{j=k}^{K} \frac{(-1)^{j-k}}{k!\,(j-k)!} (T_j \cdot e^{-x})^j \ . \tag{3}$$

This result has also been obtained in quite a different context by Brockwell and Moyal [8] .

The difference between the two formulae, which was first rather mysterious, is now well understood and stems from alternative initial conditions (at $t = 0$) to which the authors then (and others much later) paid little attention. In our present terminology (2) describes an equilibrium process, whereas (3) corresponds to an ordinary process, and they are related here by

$$_{eq}W_k(t) = {}_{or}W_k(t + \tau) \ .$$

This is a simple consequence of a surprising relation which exists between the corresponding interval densities first noted in 1974 (cf. [9] , eq. 33). For a detailed discussion of the above two expressions for $W_k(t)$ see now also Libert ([2] or [10]).

For the expression given in (2) and (3) the Poisson limit is easy to achieve. Since for $\tau \longrightarrow 0$

$$K \longrightarrow \infty \ , \qquad x \longrightarrow 0 \qquad \text{and } T_j \longrightarrow \mu \quad ,$$

we have obviously for both cases

$$\lim_{\tau \to 0} W_k(t) = \frac{\mu^k}{k!} \sum_{j=k}^{\infty} \frac{(-\mu)^{j-k}}{(j-k)!} = P_\mu(k) \ .$$

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* Minor slips or misprints in this and other early formulae are here tacitly corrected.

2. The case of a "free counter"

There remains the third type of process, called "free counter", to be described. It happens that for an extended dead time this situation leads to a much more complicated expression for $W_k(t)$. Whereas the first two moments of k have been known for a long time, the probability distribution was first indicated in 1969 by an Italian group [11]. It reads in our notation

$$W_0(t) = e^{-\mu} ,$$

$$W_k(t) = (-1)^k \sum_{\ell=k}^{K+1} \binom{\ell-1}{k-1} \cdot R_\ell , \qquad \text{for } 1 \leqslant k \leqslant K+1 , \tag{4a}$$

with $$R_\ell = e^{-\mu} - e^{-(\ell-1)x} \sum_{j=0}^{\ell-1} \frac{(-T_{\ell-1})^j}{j!} .$$

An equivalent, but slightly simpler form is

$$W_0(t) = e^{-\mu} ,$$

$$W_{k+1}(t) = (-1)^k \sum_{\ell=k}^{K} \binom{\ell}{k} \cdot Q_\ell , \qquad \text{for } 0 \leqslant k \leqslant K , \tag{4b}$$

with $$Q_\ell = - R_{\ell+1} = e^{-\ell x} \sum_{j=0}^{\ell} \frac{(-T_\ell)^j}{j!} - e^{-\mu} .$$

The form of the probability distribution has been confirmed in a careful study by Libert (cf. [12], or [13] for more details).

For later application it will also be useful to have available the following explicit expressions for Q_ℓ :

$$\begin{aligned}
Q_0 &= 1 - e^{-\mu} , \\
Q_1 &= e^{-x}\left[1 - (\mu - x)\right] - e^{-\mu} , \\
Q_2 &= e^{-2x}\left[1 - (\mu-2x) + \tfrac{1}{2}(\mu-2x)^2\right] - e^{-\mu} , \\
Q_3 &= e^{-3x}\left[1 - (\mu-3x) + \tfrac{1}{2}(\mu-3x)^2 - \tfrac{1}{6}(\mu-3x)^3\right] - e^{-\mu} .
\end{aligned} \tag{4c}$$

Our problem now is to understand how (4) actually approaches a Poisson distribution for $\tau \longrightarrow 0$, as it should do according to (1).

In the limiting case $\tau = 0$ we have

$$Q_\ell = \sum_{j=0}^{\ell} \frac{(-\mu)^j}{j!} - e^{-\mu} = -\sum_{j=\ell+1}^{\infty} \frac{(-\mu)^j}{j!}$$

which leads to the double sum

$$W_{k+1}(t) = \frac{(-1)^{k+1}}{k!} \sum_{\ell=k}^{\infty} \frac{\ell!}{(\ell-k)!} \sum_{j=\ell+1}^{\infty} \frac{(-\mu)^j}{j!} \; . \tag{5}$$

It is still not easy to see from this form how the Poisson limit will eventually be reached.

Before treating the general case, it may be useful to look at a specific example and see how the necessary transformation can be accomplished. Let us choose for instance the case $k = 2$. Then (5) gives

$$W_3(t) = \frac{(-1)^3}{2!} \sum_{\ell=2}^{\infty} \frac{\ell!}{(\ell-2)!} \sum_{j=\ell+1}^{\infty} \frac{(-\mu)^j}{j!} \; ,$$

which we write in the form of a single sum over j

$$W_3(t) = -\frac{1}{2} \sum_{j=3}^{\infty} {}_2\Lambda_j \frac{(-\mu)^j}{j!} \; .$$

Explicitly, the first terms of the new sum are

- for $j = 3$: $2 \cdot 1 \frac{(-\mu)^3}{3!} = {}_2\Lambda_3 \frac{(-\mu)^3}{3!}$,
- " $j = 4$: $2 \cdot 1 \frac{(-\mu)^4}{4!} + 3 \cdot 2 \frac{(-\mu)^4}{4!} = {}_2\Lambda_4 \frac{(-\mu)^4}{4!}$,
- " $j = 5$: $2 \cdot 1 \frac{(-\mu)^5}{5!} + 3 \cdot 2 \frac{(-\mu)^5}{5!} + 4 \cdot 3 \frac{(-\mu)^5}{5!} = {}_2\Lambda_5 \frac{(-\mu)^5}{5!}$,

...

Hence, the "multiplicity" factor ${}_2\Lambda_j$ is seen to be given by (for $j \geq 3$)

$$\begin{aligned} {}_2\Lambda_j &= \sum_{\ell=2}^{j-1} \frac{\ell!}{(\ell-2)!} = \sum_{\ell=1}^{j-1} \ell^2 - \sum_{\ell=1}^{j-1} \ell \\ &= \frac{(j-1)}{6} j(2j-1) - \frac{(j-1)}{2} j = \frac{1}{3} j(j-1)(j-2) \; , \end{aligned} \tag{6}$$

since $\sum_{\ell=1}^{n} \ell = \frac{n}{2}(n+1)$ and $\sum_{\ell=1}^{n} \ell^2 = \frac{n}{6}(n+1)(2n+1)$.

Therefore, we have for $k = 2$ the probability

$$W_3(t) = \frac{(-1)^3}{2!} \sum_{j=3}^{\infty} \frac{1}{3}\, j(j-1)\,(j-2)\, \frac{(-\mu)^j}{j!}$$

$$= \frac{(-1)^3}{6} \sum_{j=3}^{\infty} \frac{(-\mu)^j}{(j-3)!} = \frac{\mu^3}{6} \sum_{j=3}^{\infty} \frac{(-\mu)^{j-3}}{(j-3)!}$$

$$= \frac{\mu^3}{3!} \cdot e^{-\mu} = P_\mu(3) \ ,$$

as expected.

The generalization for an arbitrary value k is straightforward.

The multiplicity is now (for $k \geqslant 0$ and $j \geqslant k+1$)

$${}_k\Lambda_j = \sum_{\ell=k}^{j-1} \frac{\ell\,!}{(\ell-k)!} = \sum_{\ell=k}^{j-1} \prod_{s=0}^{k-1} (\ell-s) = \sum_{\ell=1}^{j-k} \prod_{s=0}^{k-1} (\ell+s) \ .$$

According to Mangulis [14] there exists a general relation which can be put in the form

$$\sum_{\ell=1}^{n} \prod_{s=0}^{k-1} (\ell+s) = \frac{1}{k+1} \prod_{s=0}^{k} (n+s) \ .$$

The general expression for the multiplicity is therefore

$${}_k\Lambda_j = \frac{1}{k+1} \prod_{s=0}^{k} (j-k+s) = \frac{1}{k+1} \prod_{s=0}^{k} (j-s) \ . \tag{7}$$

We are now in a position to proceed further with (5) which can be written as (for $k \geqslant 0$)

$$W_{k+1}(t) = \frac{(-1)^{k+1}}{k!} \sum_{j=k+1}^{\infty} {}_k\Lambda_j \frac{(-\mu)^j}{j!}$$

$$= \frac{(-1)^{k+1}}{k!} \cdot \frac{1}{k+1} \sum_{j=k+1}^{\infty} \prod_{s=0}^{k} (j-s) \frac{(-\mu)^j}{j!}$$

$$= \frac{(-1)^{k+1}}{(k+1)!} (-\mu)^{k+1} \sum_{j=k+1}^{\infty} \frac{(-\mu)^{j-k-1}}{(j-k-1)!}$$

$$= \frac{\mu^{k+1}}{(k+1)!} \sum_{j=0}^{\infty} \frac{(-\mu)^j}{j!} = \frac{\mu^{k+1}}{(k+1)} \cdot e^{-\mu} = P_\mu(k+1) \ . \tag{8}$$

Since $W_0(t) = e^{-\mu} = P_\mu(0)$, this result shows that there is indeed

$$\lim_{\tau \to 0} W_k(t) = P_\mu(k) , \qquad \text{for any } k \geqslant 0 ,$$

when $W_k(t)$ is given by (4).

3. The ordinary moments

The first two moments of the probability distribution (4) have been first derived by Feller [15] by means of the now classical method of using Laplace transforms for renewal processes. The results for expectation and variance of k are

$$\begin{aligned} E_k(t) &= (\mu - x)\, e^{-x} + 1 - e^{-x} \\ &= 1 + (\mu - x - 1)\, e^{-x} , \qquad \text{for} \quad t \geqslant \tau , \qquad (9) \\ \sigma_k^2(t) &= e^{-x} (\mu - x) \cdot (1 - 2x e^{-x}) - e^{-x} + (1+x)^2 e^{-2x} \\ &= e^{-x} (\mu - 1 - x) + e^{-2x} \left[1 - 2x(\mu - 1) + 3x^2 \right] , \qquad \text{for } t \geqslant 2\tau . \end{aligned}$$

These formulae have since been rederived several times by various methods and were confirmed. Expressions are also known for the region $0 < t < 2\tau$.

Foglio Para and Mandelli Bettoni [11] indicate an interesting form for the first moments which involves the quantities R_ℓ (or Q_ℓ in our notation), which were introduced above in (4). For the first few ordinary moments of k, defined by $M_r(t) = E\{k^r\}$, they give

$$\begin{aligned} M_0(t) &= 1 , \\ M_1(t) &= R_2 - R_1 = Q_0 - Q_1 , \qquad &\text{for } t \geqslant \tau, \qquad (10) \\ M_2(t) &= 3R_2 - R_1 - 2R_3 = Q_0 - 3Q_1 + 2Q_2 , \qquad &\text{" } \ t \geqslant 2\tau . \end{aligned}$$

It is easy to verify that these expressions are compatible with (9). Unfortunately, no hint to the derivation of (10) is given other than the indication that they are obtained from (4) "after straightforward calculations". It was tempting to see how this could actually be achieved.

Let us first check the normalization. From (4) we have

$$M_0(t) = \sum_{k=0}^{K+1} W_k(t) = e^{-\mu} + \sum_{k=0}^{K} (-1)^k \sum_{\ell=k}^{K} \binom{\ell}{k} Q_\ell . \qquad (11)$$

The first contributions to the sum are

$$- \text{ for } k=0: \ +\sum_{\ell=0}^{K} \binom{\ell}{0} Q_\ell \ = \binom{0}{0} Q_0 + \binom{1}{0} Q_1 + \binom{2}{0} Q_2 + \ldots,$$

$$- \text{ " } k=1: \ -\sum_{\ell=1}^{K} \binom{\ell}{1} Q_\ell \ = \qquad - \binom{1}{1} Q_1 - \binom{2}{1} Q_2 - \ldots,$$

$$- \text{ " } k=2: \ +\sum_{\ell=2}^{K} \binom{\ell}{2} Q_\ell \ = \qquad\qquad + \binom{2}{2} Q_2 + \ldots,$$

$$\ldots$$

For $K \geqslant 1$ we therefore have

$$M_0(t) = e^{-\mu} + Q_0 + \sum_{\ell=1}^{K} Q_\ell \sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} .$$

Since $\sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} = (1-1)^\ell = 0,$ for $\ell \geqslant 1$, there only remains

$$M_0(t) = e^{-\mu} + Q_0 = 1, \tag{12}$$

which confirms the correct normalization of (4).

For the ordinary moments of order $r \geqslant 1$ use of (4) yields

$$M_r(t) = \sum_{k=0}^{K+1} k^r \cdot W_k(t) = \sum_{k=1}^{K+1} k^r (-1)^{k+1} \sum_{\ell=k-1}^{K} \binom{\ell}{k-1} Q_\ell . \tag{13}$$

The evaluation of this expression can be performed in the following way. First let us look again at the various terms of the sum over k, the first few of which are

$$- \text{ for } k=1: \ +1^r \sum_{\ell=0}^{K} \binom{\ell}{0} Q_\ell = 1^r \binom{0}{0} Q_0 + 1^r \binom{1}{0} Q_1 + 1^r \binom{2}{0} Q_2 + \ldots,$$

$$- \text{ " } k=2: \ -2^r \sum_{\ell=1}^{K} \binom{\ell}{1} Q_\ell = \qquad - 2^r \binom{1}{1} Q_1 - 2^r \binom{2}{1} Q_2 - \ldots,$$

$$- \text{ " } k=3: \ +3^r \sum_{\ell=2}^{K} \binom{\ell}{2} Q_\ell = \qquad\qquad + 3^r \binom{2}{2} Q_2 + \ldots,$$

$$\ldots$$

Hence we get for the moments

$$M_r(t) = \sum_{\ell=0}^{K} Q_\ell \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (j+1)^r .$$

In order to proceed further with M_r we now consider the sum

$$\begin{aligned} Z(r,\ell) &\equiv \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} (j+1)^r \\ &= \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} \sum_{m=0}^{r} \binom{r}{m} j^m . \end{aligned} \tag{14}$$

By using this quantity, the moments can be written as

$$M_r(t) = \sum_{\ell=0}^{K} Q_\ell \cdot Z(r,\ell) . \tag{15}$$

According to [16], Stirling numbers of the second kind may be defined by the following expression (in our present notation)

$$S(m,\ell) = \frac{1}{\ell!} \sum_{j=0}^{\ell} (-1)^{\ell-j} \binom{\ell}{j} \cdot j^m , \tag{16}$$

which is equivalent to

$$\sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} j^m = (-1)^\ell \ell! \cdot S(m,\ell) .$$

It is therefore possible to write (14) in the form

$$Z(r,\ell) = (-1)^\ell \ell! \sum_{m=0}^{r} \binom{r}{m} \cdot S(m,\ell) . \tag{17}$$

However, since Stirling numbers have the characteristic that $S(m,\ell) = 0$ whenever $\ell > m$, we see also that

$$Z(r,\ell) = 0 \quad \text{for} \quad \ell > m . \tag{18}$$

This allows us to reduce the expression (15) for the moments to the form

$$M_r(t) = \sum_{\ell=0}^{r} Q_\ell \cdot Z(r,\ell) . \tag{19}$$

By means of (18) we now find with (14), always for $r \geqslant 1$,

$$M_r(t) = \sum_{\ell=0}^{r} Q_\ell \, (-1)^\ell \, \ell! \sum_{m=\ell}^{r} \binom{r}{m} \cdot S(m,\ell) \; . \tag{20}$$

This expression can be greatly simplified. For this purpose we use a recurrence relation for Stirling numbers [16] which can be written as (d is a dummy integer)

$$\binom{\ell}{d} S(r,\ell) = \sum_{m=\ell-d}^{r-d} \binom{r}{m} S(r-m,d) \cdot S(m,\ell-d) \; . \tag{21}$$

For $d = 1$ we get, since $S(m,1) = 1$,

$$\binom{\ell}{1} S(r,\ell) = \sum_{m=\ell-1}^{r-1} \binom{r}{m} \cdot S(m,\ell-1) \; ,$$

or likewise, with $\ell+1$ instead of ℓ,

$$\binom{\ell+1}{1} \cdot S(r,\ell+1) = \sum_{m=\ell}^{r-1} \binom{r}{m} \cdot S(m,\ell) \; .$$

From this follows, with the help of a well-known recurrence formula, the interesting relation

$$\begin{aligned} \sum_{m=\ell}^{r} \binom{r}{m} \cdot S(m,\ell) &= \binom{r}{r} \cdot S(r,\ell) + \binom{\ell+1}{1} \cdot S(r,\ell+1) \\ &= S(r+1,\ell+1) \; . \end{aligned} \tag{22}$$

Hence, we can now write (20) in the simple form

$$M_r(t) = \sum_{\ell=0}^{r} (-1)^\ell \, \ell! \cdot S(r+1,\ell+1) \cdot Q_\ell \; , \tag{23}$$

which is generally valid for $r \geqslant 1$ and $t \geqslant r\tau$.

The simplest practical examples lead readily to

$$\begin{aligned} M_1(t) &= Q_0 - Q_1 \; , && \text{for } t \geqslant \tau \; , \\ M_2(t) &= Q_0 - 3Q_1 + 2Q_2 \; , && t \geqslant 2\tau \; , \\ M_3(t) &= Q_0 - 7Q_1 + 12Q_2 - 6Q_3 \; , && t \geqslant 3\tau \; , \\ &\dots \end{aligned} \tag{24}$$

These results fully confirm and slightly generalize the formula (10) given by Foglio Para and Mandelli Bettoni in the excellent section 5 of their paper [11], which, however, at other places was found to be somewhat less reliable for some details.

4. Some central moments

The central moments of k, defined by $\mu_r(t) = E\{(k-M_1)^r\}$, are of obvious practical importance. In addition, explicit expressions up to third order would be most welcome for providing independent checks on the reliability of previously-established asymptotic formulae. Since $\mu_0(t) = M_0(t) = 1$ and $\mu_1(t) = 0$, the cases of real interest are $r = 2$ and $r = 3$.

For the variance we find with the help of (24)

$$
\begin{aligned}
\mu_2(t) &\equiv \sigma_k^2(t) = M_2(t) - M_1^2(t) \\
&= Q_0 - Q_0^2 - 3Q_1 - Q_1^2 + 2Q_0Q_1 + 2Q_2 \ .
\end{aligned}
\tag{25}
$$

Use of the explicit expressions given in (4c) leads, after some rearrangement, to

$$
\mu_2(t) = e^{-x}(\mu-1-x) + e^{-2x}\left[1 - 2x\,(\mu-1) + 3x^2\right], \qquad \text{for} \quad t \geq 2\tau ,
$$

which is in agreement with (9). With the help of the abbreviation $y = e^x$ we can also arrive at the form

$$
\mu_2(t) = \frac{1}{y^2}\left[\mu\,(y-2x) + 1 - y + x(2-y+3x)\right],
$$

which coincides with the asymptotic expression given previously in [17].

For the third central moment we have similarly

$$
\begin{aligned}
\mu_3(t) &= M_3(t) - 3\,M_2(t)\cdot M_1(t) + 2\,M_1^3(t) \\
&= Q_0 - 3Q_0^2 + 2Q_0^3 - 7Q_1 - 9Q_1^2 - 2Q_1^3 + 12Q_2 - 6Q_3 \\
&\quad + 6Q_0\,(2Q_1 + Q_1^2 - Q_2) - 6Q_0^2\,Q_1 + 6Q_1\,Q_2 \ .
\end{aligned}
\tag{26}
$$

Substitution of (4c) into this equation leads to a very long and unwieldy expression which we do not want to reproduce here. The many rearrangements needed to bring it into a more manageable form are completely elementary and of no interest. One can finally arrive at an expression of the form

$$
\mu_3(t) = e^{-x}(\mu-1-x) + 3e^{-2x}\left[1 - 2x\,(\mu-1) + 3x^2\right] - e^{-3x}\left[2 + 6x - 9x^2\,(\mu-1) + 17x^3\right].
$$

By grouping together all the contributions which are proportional to $\mu = \rho t$ we obtain

$$\mu_3(t) = \mu(e^{-x} - 6x\,e^{-2x} + 9x^2\,e^{-3x}) - e^{-x}(1+x) + 3e^{-2x}(1 + 2x + 3x^2)$$

$$- e^{-3x}(2 + 6x + 9x^2 + 17x^3)$$

$$= e^{-3x}(e^{2x} - 6x\,e^{x} + 9x^2) \cdot \mu$$

$$+ e^{-3x}\left\{-2 - 6x - 9x^2 - 17x^3 + 3e^{x}(1 + 2x + 3x^2) - e^{2x}(1 + x)\right\} .$$

By means of the abbreviation $y = e^x$ this can finally be put in the form (always for $t \geqslant 3\tau$)

$$\mu_3(t) = \frac{1}{y^3}\left\{\mu(y - 3x)^2 + y\left[3+6x+9x^2 - y(1+x)\right] - 2 - 6x - 9x^2 - 17x^3\right\} .$$

It is not without relief that we note that this formula is in exact agreement with the corresponding asymptotic expression given previously [18] , which was of a conspicuously complicated form.

It seems to be a general feature of extended dead times that the asymptotic moments of order r for the corresponding counting distributions are in fact rigorous provided that $t \geqslant r\tau$.

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