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Counting statistics of a decaying source*

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1. Introduction

It is for well over half a century that the simple Poisson formula

$$P_\mu(k) = \frac{\mu^k}{k!} \cdot e^{-\mu} \tag{1}$$

for the probability of observing exactly $k = 0, 1, \ldots$ events within a given fixed time interval, where $\mu > 0$ is the expectation value, has been recognized as the appropriate statistical law for describing the emissions of a radioactive source [1, 2]. The usual derivation of (1) is based on the hypothesis of a constant source activity. In principle, the simultaneous assumptions of decay and of a time-independent (mean) count rate are contradictory, of course. Nevertheless, in practice this is often an excellent approximation to reality and the Poisson law can for most practical situations be taken as a solid basis for the counting statistics. In addition, the consequences of (1) have been repeatedly verified by experiment to a high degree of accuracy (see e.g. [3], also for earlier references).

However, there also exist many short-lived radionuclides where a realistic measuring time becomes inevitably comparable with the half-life, and in this case the influence of decay cannot be neglected. It is the purpose of this report to study in some detail how the simple Poisson law (1) has to be modified in order to account for the finite lifetime of the source.

For an early, but still very interesting attempt to tackle this problem we refer to [4]. As long as the total number of radioactive atoms forming the source is sufficiently large - and it is in fact rather difficult to find an experimental situation where this condition is not met -, the strict connection between disintegration and momentary activity as well as the mathematical complications due to the corresponding binomial distributions can be avoided by assuming that the activity diminishes in time according to an exponential law. Such a simplification is in particular appropriate for a low detection efficiency of the emitted radiation.

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* This report is dedicated to my friend and collaborator Albrecht Rytz on the occasion of his sixtieth birthday.

We therefore assume that the registered count rate $\rho(t)$ is given by

$$\rho(t) = \rho_o \cdot e^{-\lambda t} + \beta , \tag{2}$$

where $1/\lambda$ is the mean lifetime of the source and β the background rate. Possible count-rate dependent experimental effects (as, for example, those due to dead time) which could disturb the assumed strict proportionality between activity and measured count rate are supposed to be negligibly small.

There seem to exist mainly two practical situations where the effect of decay has to be taken into account. In the first case a single measurement of duration T is made, and this measurement is then repeated a large number of times with new sources of nominally the same initial activity. This situation may be realized for very short-lived isotopes in a target which is periodically reactivated by a suitable nuclear reaction and measured immediately afterwards. From the statistical point of view this case poses no problem. In fact, as a simple consequence of the superposition principle, also an inhomogeneous Poisson process still follows the Poisson statistics (1), but with μ replaced by an effective expectation value

$$\mu_{eff} = \int_o^T \rho(t)\, dt . \tag{3}$$

Hence, the distribution of the results of n measurements will be Poissonian, provided that we have statistically equivalent initial conditions for the n runs. It will be obvious that a given activation (of fixed time and intensity) will not result in the production of exactly the same number N of radioactive atoms. However, it can be shown that this is not only unnecessary, but that it is rather a Poisson distribution of N which is the physically required (and experimentally achievable) initial condition for $t = 0$.

The second experimental situation is likely to be of more direct interest. Here a given single, rapidly decaying source is measured for a large number of consecutive short time intervals of equal length t_o. Whereas decay during t_o may be negligible, this will not be true for the total period of observation, which is $T = n \cdot t_o$ if the n measuring intervals follow each other without interruption. For experimental reasons it may be necessary that each measuring interval is followed by an "inactive" time t_o' (used, for instance, for classification of the number of events k observed in the preceding "live" time t_o); we shall then have $T = n(t_o + t_o')$. Provided that t_o' is constant it will be without influence on the statistical distribution of the observed number of events per interval t_o for the total measuring time T. It is for this second type of experimental situation that the effect of decay on the measured "Poisson distribution" will be analyzed in what follows.

2. Derivation of the modified distribution

Assuming $t_o' = 0$ for the sake of simplicity, and also that the total measuring time T is subdivided into a large number n of equal counting intervals $t_o = T/n$, we obtain for the probability of observing exactly k events in t_o, as a consequence of (1) and (2), the expression

$$_\lambda P(k) = \frac{1}{n} \sum_{j=0}^{n-1} \frac{(\rho_j t_o)^k}{k!} \cdot e^{-\rho_j t_o} , \tag{4}$$

with $\rho_j = \beta + \rho_o \cdot e^{-j \cdot \lambda t_o}$.

Hence, with $\rho_o t_o = \mu_o$ and $\beta t_o = g$,

$$_\lambda P(k) = \frac{t_o}{T} \sum_j (g + \mu_o \cdot e^{-j\lambda t_o})^k \cdot \frac{1}{k!} \cdot \exp(-g - \mu_o \cdot e^{-j\lambda t_o})$$

$$= \frac{t_o \cdot e^{-g}}{T \cdot k!} \sum_{j=0}^{n-1} (g + \mu_o \cdot e^{-j\lambda t_o})^k \cdot \exp(-\mu_o \cdot e^{-j\lambda t_o}) .$$

Since $n \gg 1$ and with $j t_o = t$, we find that the probability for observing k events is given in good approximation by

$$_\lambda P(k) = \frac{e^{-g}}{T \cdot k!} \int_o^T (g + \mu_o \cdot e^{-\lambda t})^k \cdot \exp(-\mu_o \cdot e^{-\lambda t}) \, dt$$

$$= \frac{e^{-g}}{T \cdot k!} \sum_{r=0}^{k} \binom{k}{r} g^{k-r} \mu_o^r \int_o^T \exp(-r\lambda t - \mu_o \cdot e^{-\lambda t}) \, dt . \tag{5}$$

In the absence of background, this expression can be simplified considerably since for $g = 0$ we have $g^{k-r} = \delta_{k-r,o}$, hence

$$_\lambda P(k) = \frac{1}{T \cdot k!} \mu_o^k \int_o^T \exp(-k\lambda t - \mu_o \cdot e^{-\lambda t}) \, dt$$

$$= {}_o P(k) \cdot \frac{1}{T} \int_o^T \exp\left[-k\lambda t + \mu_o(1 - e^{-\lambda t})\right] dt . \tag{6}$$

Some approximate forms of (6) involving power-series expansions are discussed in the Appendix. It is easy to verify that quite generally

$$\lim_{\lambda \to 0} {}_\lambda P(k) = {}_o P(k) \text{ , where } {}_o P(k) = P_{\mu_o + g}(k)$$

is the corresponding Poisson probability without decay.

3. Search for a closed form of (5)

a) The case $k > 0$

The integral appearing in (5) can also be written in the form

$$\int_0^T \exp(-r\lambda t - \mu_o \cdot e^{-\lambda t})\, dt = \int_0^\infty \exp(\ldots)\, dt - \int_T^\infty \exp(\ldots)\, dt \equiv I_o(r) - I_1(r) .$$

With the variable $x = \lambda(t - T)$ and the abbreviation $\vartheta = \lambda T$, the second term is

$$I_1(r) = \int_0^\infty \exp\left\{ - r(x+\vartheta) - \mu_o \cdot e^{-(x+\vartheta)} \right\} \frac{dx}{\lambda}$$

$$= \frac{e^{-r\vartheta}}{\lambda} \int_0^\infty \exp(-rx - \mu_1 \cdot e^{-x})\, dx ,$$

where $\mu_1 = \mu_o \cdot e^{-\vartheta}$.

Likewise we get for the first integral, with $\lambda t = x$,

$$I_o(r) = \frac{1}{\lambda} \int_0^\infty \exp(-rx - \mu_o \cdot e^{-x})\, dx .$$

Let us first consider the case where $r > 0$. With $s = 0$ or 1 we then obtain

$$I_s(r) = \frac{e^{-rs\vartheta}}{\lambda} \int_0^\infty \exp(-rx - \mu_s \cdot e^{-x})\, dx .$$

This integral can be evaluated by means of the formula [5]

$$\int_0^\infty \exp(-kx - \mu e^{-x})\, dx = \mu^{-k} \cdot \gamma(k,\mu) , \qquad (7)$$

where $\gamma(k,\mu) = \int_0^\mu e^{-x} \cdot x^{k-1}\, dx$, for $k > 0$,

is the incomplete gamma function.

Hence one can write for $r > 0$

$$I_o(r) - I_1(r) = \frac{1}{\lambda}\left\{\mu_o^{-r}\cdot\gamma(r,\mu_o) - \mu_1^{-r}\cdot e^{-r\vartheta}\cdot\gamma(r,\mu_1)\right\}$$

$$= \frac{1}{\lambda\cdot\mu_o^r}\left\{\gamma(r,\mu_o) - \gamma(r,\mu_1)\right\}, \qquad \text{for } \lambda \neq 0. \tag{8}$$

The case $r = 0$ has to be treated separately. Its contribution to the sum in (5) is

$$\binom{k}{0} g^k \int_o^T \exp(-\mu_o\cdot e^{-\lambda t})\, dt .$$

Putting $\mu_o\cdot e^{-\lambda t} = y$ we find

$$g^k \int_{\mu_o}^{\mu_1} e^{-y}\frac{dy}{(-y)} = \frac{g^k}{\lambda}\left\{\int_{\mu_1}^{\infty}\frac{e^{-y}}{y}\,dy - \int_{\mu_o}^{\infty}\frac{e^{-y}}{y}\,dy\right\} .$$

By means of the exponential integral function, which is defined by

$$E_1(\mu) = \int_{\mu}^{\infty}\frac{e^{-y}}{y}\,dy , \qquad \text{for } \mu > 0 , \tag{9}$$

we can also write for this contribution

$$\frac{g^k}{\lambda}\left\{E_1(\mu_1) - E_1(\mu_o)\right\} .$$

Therefore, for $k > 0$ equation (5) takes on the form

$${}_\lambda P(k) = \frac{e^{-g}}{\vartheta k!}\left\{g^k\left[E_1(\mu_1) - E_1(\mu_o)\right] + \sum_{r=1}^{k}\binom{k}{r} g^{k-r}\left[\gamma(r,\mu_o) - \gamma(r,\mu_1)\right]\right\}. \tag{10}$$

b) <u>The case $k = 0$</u>

This problem can now be readily settled. From (5) we conclude that

$${}_\lambda P(0) = \frac{e^{-g}}{T}\int_o^T \exp(-\mu_o\cdot e^{-\lambda t})\, dt .$$

By a rearrangement which corresponds exactly to the one just applied to the case $r = 0$ we find

$${}_\lambda P(0) = \frac{e^{-g}}{\vartheta}\left\{E_1(\mu_1) - E_1(\mu_o)\right\} . \tag{11}$$

Finally, in the absence of background (g = 0) the general expressions (10) and (11) reduce to

$$ {}_{\lambda}P(k=0) = \frac{1}{\vartheta}\left[E_1(\mu_1) - E_1(\mu_o)\right] \quad \text{and} $$

$$ {}_{\lambda}P(k \geqslant 1) = \frac{1}{\vartheta \cdot k!}\left[\gamma(k,\mu_o) - \gamma(k,\mu_1)\right] , \tag{12a} $$

since $0^{k-r} = \delta_{k-r,0}$, as noted previously.

This result can also be written as a product of the original Poisson distribution (at t = 0) and a correction factor C_k. We then have

$$ {}_{\lambda}P(k) = {}_{o}P(k) \cdot C_k \quad , $$

where

$$ C_k = \frac{e^{\mu_o}}{\vartheta} \cdot \begin{cases} \left[E_1(\mu_1) - E_1(\mu_o)\right] \quad , & \text{for } k = 0 \\ \mu_o^{-k}\left[\gamma(k,\mu_o) - \gamma(k,\mu_1)\right] , & \text{" } k \geqslant 1. \end{cases} \tag{12b} $$

To visualize the effect of the decay on the distribution of the counts k, some graphical representations of ${}_{\lambda}P(k)$ are given in Figs. 1 and 2 for two values of initial expectation μ_o and some reduced decay parameters $\vartheta = \lambda T$.

4. Moments of the modified distribution

Considering the complicated structure of the formulae for the probabilities ${}_{\lambda}P(k)$, one might get the impression that the evaluation of the moments of k, defined for order r by

$$ {}_{\lambda}E(k^r) = \sum_{k=0}^{\infty} k^r \cdot {}_{\lambda}P(k) \quad , $$

will be a most cumbersome, if not impossible matter. It is interesting to note, therefore, that the necessary calculations can be considerably simplified just by interchanging the order of the two averaging processes involved, namely over k and T.

For any given fixed moment t, the count rate $\rho(t)$, given by (2), corresponds to a momentary expectation value E(k) of the number of events k, hence

$$ E_t(k) = \rho(t) \cdot t_o = g + \mu_o \cdot e^{-\lambda t} = \mu(t) \ . \tag{13} $$

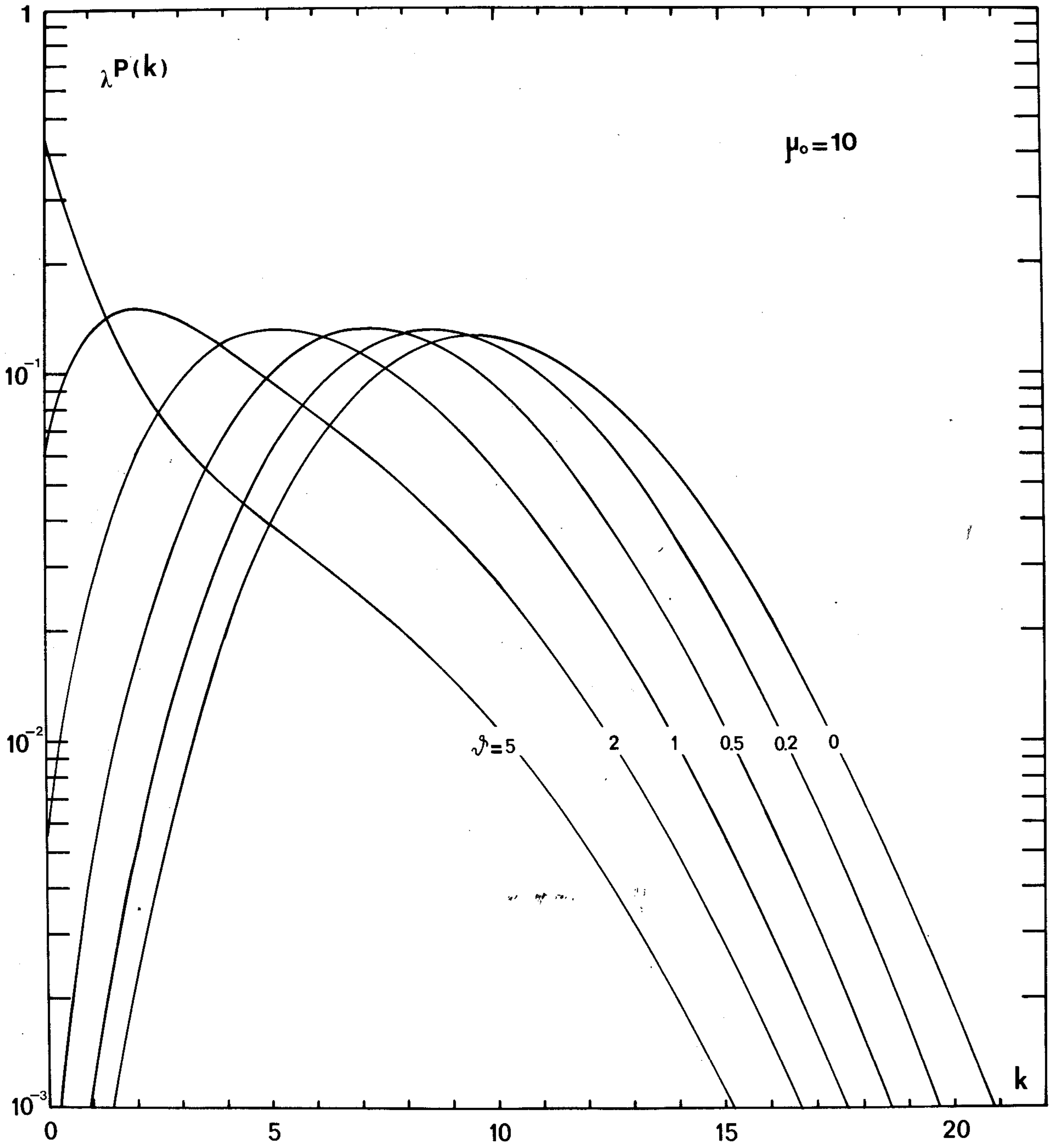

Figure 1 - Decay-modified Poisson probabilities $_\lambda P(k)$, for $\mu_o = 10$ and $\vartheta = \lambda T = 0, 0.2, 0.5, 1, 2$ and 5. Background is assumed to be negligible. The case $\vartheta = 0$ is a pure Poisson distribution.

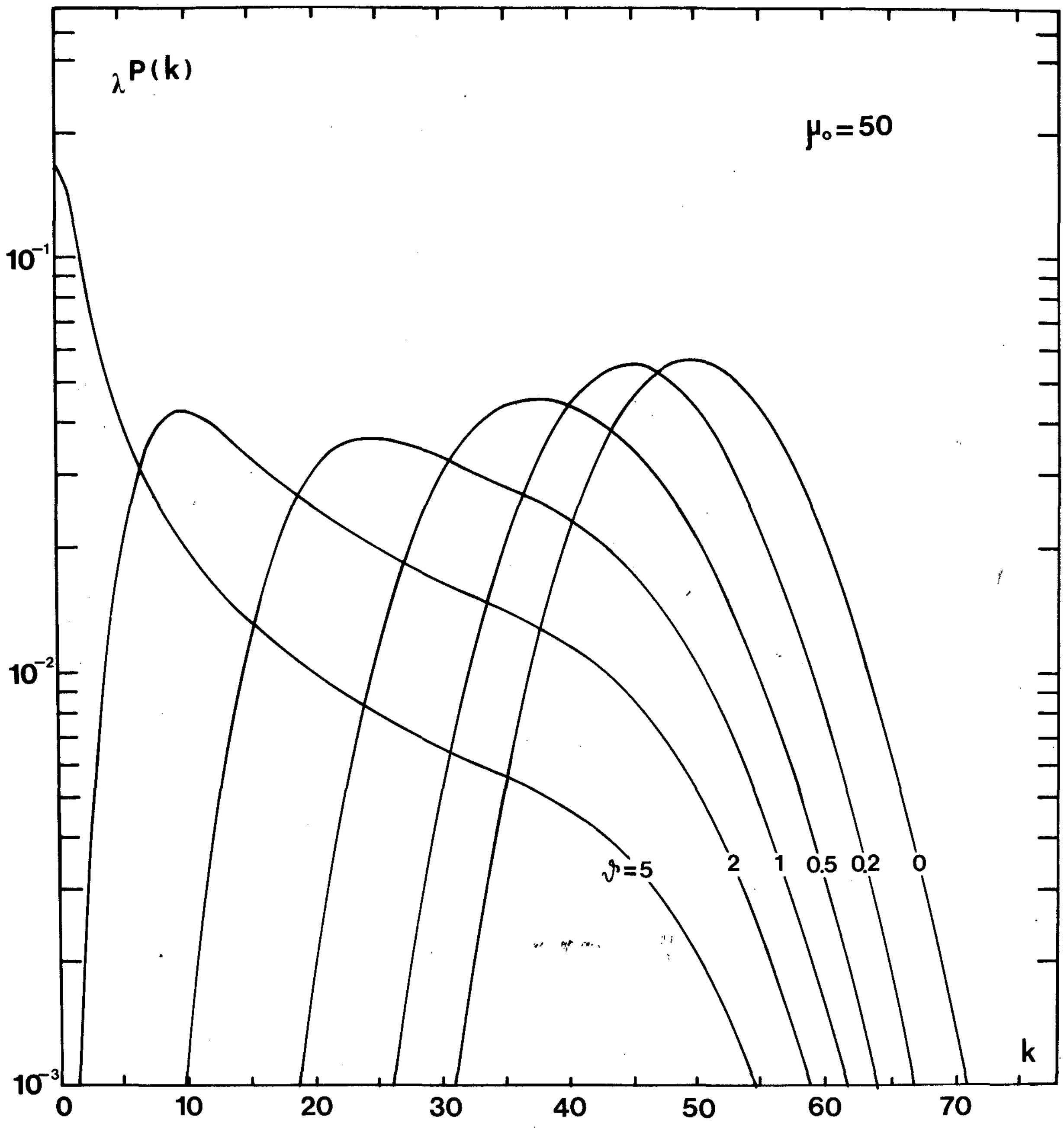

Figure 2 - Same as Fig. 1, but for $\mu_o = 50$.

Since the process, also with the background component included, still is at any moment of the Poisson type, the corresponding probabilities are, according to (1), given by

$$P_{\mu(t)} = \frac{\mu^k(t)}{k!} \cdot e^{-\mu(t)} .$$

Performing first the averaging over the total measuring time T, we get for the first moment

$$_\lambda\{E_t(k)\} \equiv {_\lambda E(k)} = \frac{1}{T}\int_0^T \mu(t)\, dt ,$$

hence by means of (13)

$$_\lambda E(k) = g + \frac{\mu_o}{T}\int_0^T e^{-\lambda t}\, dt = g + \frac{\mu_o}{\vartheta}(1 - e^{-\vartheta}) , \text{ for } \vartheta \neq 0. \quad (14)$$

The variance V(k) can be obtained in a similar simple way which has been indicated earlier by Lewis et al. [6] . As a result of the Poisson nature we have

$$V_t(k) = E_t(k^2) - E_t^2(k) = \mu(t) ,$$

thus also

$$E_t(k^2) = V_t(k) + E_t^2(k) = \mu(t) + \mu^2(t) .$$

Time averaging then yields

$$_\lambda\{V_t(k)\} \equiv {_\lambda V(k)} = {_\lambda\{E_t(k^2)\}} - {_\lambda\{E_t(k)\}^2} , \quad (15a)$$

or more explicitly

$$_\lambda V(k) = \frac{1}{T}\int_0^T \left[\mu^2(t) + \mu(t))\right] dt - \left[\frac{1}{T}\int_0^T \mu(t)\, dt\right]^2 . \quad (15b)$$

Some rearrangements then lead with (13) to

$$_\lambda V(k) = \frac{1}{T}\int_0^T (g + \mu_o e^{-\lambda t})^2\, dt + {_\lambda E(k)} - {_\lambda E^2(k)}$$

$$= g^2 + \frac{2g\mu_o}{\vartheta}(1 - e^{-\vartheta}) + \frac{\mu_o^2}{2\vartheta}(1 - e^{-2\vartheta}) + {_\lambda E(k)} - {_\lambda E^2(k)} .$$

Since $_{\lambda}E^2(k) = g^2 + \frac{2g\,\mu_o}{\vartheta}(1 - e^{-\vartheta}) + \frac{\mu_o^2}{\vartheta^2}(1 - e^{-\vartheta})^2$,

we can also write

$$
\begin{aligned}
_{\lambda}V(k) &= \frac{\mu_o^2}{2\vartheta}(1 - e^{-2\vartheta}) + {}_{\lambda}E(k) - \frac{\mu_o^2}{\vartheta^2}(1 - e^{-\vartheta})^2 \\
&= {}_{\lambda}E(k) + \frac{\mu_o^2}{\vartheta^2}(1 - e^{-\vartheta})\left[\frac{\vartheta}{2}(1 + e^{-\vartheta}) - (1 - e^{-\vartheta})\right] \\
&= {}_{\lambda}E(k) + \frac{\mu_o^2}{\vartheta^2}(1 - e^{-\vartheta})^2\left\{\frac{\vartheta}{2}\cdot\frac{1 + e^{-\vartheta}}{1 - e^{-\vartheta}} - 1\right\} \\
&= {}_{\lambda}E(k) + \left[{}_{\lambda}E(k) - g\right]^2\left\{\frac{\vartheta}{2}\cdot\frac{1 + e^{-\vartheta}}{1 - e^{-\vartheta}} - 1\right\}, \text{ for } \vartheta \neq 0.
\end{aligned}
\tag{16}
$$

For $g = 0$ this corresponds to the result given in [6] as eq. 26. It follows from (16) that the ratio

$$
R \equiv \frac{_{\lambda}V(k) - {}_{\lambda}E(k)}{\left[{}_{\lambda}E(k) - g\right]^2} = \frac{\vartheta}{2}\left[\frac{1 + e^{-\vartheta}}{1 - e^{-\vartheta}}\right] - 1 \tag{17}
$$

is only a function of ϑ. Therefore, if the experimental mean and variance of k are inserted, this relation allows an evaluation of the half-life of the corresponding nuclide, since

$$
T_{1/2} = (1/\lambda)\ln 2 = (T/\vartheta)\ln 2.
$$

For $\vartheta \ll 1$ a power-series development gives the approximation

$$
\begin{aligned}
R &\cong \frac{\vartheta}{2}\cdot\frac{2 - \vartheta + \vartheta^2/2}{\vartheta - \vartheta^2/2 + \vartheta^3/6} - 1 \\
&\cong \vartheta\left(1 - \frac{\vartheta}{2} + \frac{\vartheta^2}{4}\right)\frac{1}{\vartheta}\left(1 + \frac{\vartheta}{2} + \frac{\vartheta^2}{12}\right) - 1 \cong \frac{\vartheta^2}{12}.
\end{aligned}
$$

In general, however, it will be preferable to base such a determination on the entire empirical distribution of the values k and to deduce the half-life from a best fit of the theoretical values $_{\lambda}P(k)$. In this way, a possible systematic deviation in the shape of the curve can be readily seen in a graphical plot of the data while it would remain undetected in an evaluation using only the first two empirical moments.

In view of some practical applications of this method for measuring half-lives, the evaluation of the probabilities ${}_{\lambda}P(k)$ has been implemented in an automatic computer program. This has allowed us to verify the different formulae given for the moments and combinations of them (in particular eq. 17) to a very high degree of accuracy.

At present we are studying various methods to take the influence of a dead time into account. General results have been obtained for the moments and their description is in preparation.

APPENDIX

Some approximations of equation 6

In the exact form given in (6) as

$$ {}_{\lambda}P(k) = {}_{o}P(k) \cdot \frac{1}{T} \int_{o}^{T} \exp\left[- k\lambda t + \mu_o (1 - e^{-\lambda t}) \right] dt \ , \qquad (A1) $$

the exponent can be developed into a power series yielding

$$ \int_{o}^{T} \exp \left\{ -k\lambda t + \mu_o \left[1 - (1 - \lambda t \pm \ldots) \right] \right\} dt \cong \int_{o}^{T} e^{-\lambda (k - \mu_o) t} \, dt = \frac{1 - e^{-\lambda (k - \mu_o) T}}{\lambda (k - \mu_o)} \ . $$

Hence, for $\vartheta = \lambda T \ll 1$ we have the simple approximation

$$ {}_{\lambda}P(k) \cong {}_{o}P(k) \left[\frac{1 - e^{-\vartheta (k - \mu_o)}}{\vartheta (k - \mu_o)} \right] \ , \qquad (A2) $$

which is valid for any $k \geqslant 0$.

a) A linear approximation

For checking the moments, it is more practical to have a power series in ϑ. Development of (A2) up to linear terms gives readily

$$ {}_{\lambda}P(k) \cong {}_{o}P(k) \left[1 - \frac{\vartheta}{2} (k - \mu_o) \right] \ . \qquad (A3) $$

For the expectation value we then find in this approximation

$$_{\lambda}E(k) = \sum_{k=0}^{\infty} k \cdot {}_{\lambda}P(k) \cong \sum k \cdot P - \frac{\vartheta}{2} \sum k(k-\mu_o) \cdot P \quad ,$$

where the Poisson probability $_oP(k)$ is simply written as P and all sums are over k.

The ordinary moments of k for a Poisson distribution with parameter μ_o are known [7] to be given by

$$m_r \equiv \sum k^r \cdot P = \sum_{j=1}^{r} S(r,j) \cdot \mu_o^j \quad , \qquad \text{(A4)}$$

where S(r,j) is a Stirling number of the second kind. Hence, the first moments are

$$m_1 = \mu_o \quad , \qquad m_3 = \mu_o + 3\mu_o^2 + \mu_o^3 \quad ,$$
$$m_2 = \mu_o + \mu_o^2 \quad , \qquad m_4 = \mu_o + 7\mu_o^2 + 6\mu_o^3 + \mu_o^4 \; . \qquad \text{(A5)}$$

This gives for the expectation of the modified distribution (A3)

$$_{\lambda}E(k) \cong m_1 - \frac{\vartheta}{2}(m_2 - m_1\mu_o) = \mu_o\left(1 - \frac{\vartheta}{2}\right) \; . \qquad \text{(A6)}$$

Likewise the second moment is

$$_{\lambda}E(k^2) \cong m_2 - \frac{\vartheta}{2}(m_3 - m_2\mu_o) = \mu_o\left\{1 + \mu_o - \frac{\vartheta}{2}(1 + 2\mu_o)\right\} \; .$$

Hence, the variance – also up to linear terms in ϑ – turns out to be

$$_{\lambda}V(k) = {}_{\lambda}E(k^2) - {}_{\lambda}E^2(k) \cong \mu_o\left(1 - \frac{\vartheta}{2}\right) \; . \qquad \text{(A7)}$$

It follows that in this lowest approximation expectation value and variance are equal.

b) A quadratic approximation

In order to obtain more information, it is obviously necessary to go at least to second order in ϑ. One might feel tempted, for the sake of simplicity, to start again from (A2). However, since this is in part already a first-order approximation, we can hardly hope to get all terms of second order correctly. Nevertheless, the necessary correction could possibly be guessed. So let us make an attempt which gives

$$_{\lambda}P'(k) \cong {}_oP(k)\left\{1 - \frac{1}{2}(k-\mu_o)\vartheta + \frac{1}{6}(k-\mu_o)^2\vartheta^2\right\} \; .$$

A check of the normalization yields

$$\sum_\lambda P'(k) = \sum P - \frac{\vartheta}{2}\left(\sum kP - \mu_o\right) + \frac{\vartheta^2}{6}\left(\sum k^2 P - 2\,\mu_o \sum kP + \mu_o^2\right)$$

$$= 1 - \frac{\vartheta}{2}\left(m_1 - \mu_o\right) + \frac{\vartheta^2}{6}\left(m_2 - 2\,m_1\,\mu_o + \mu_o^2\right)$$

$$= 1 + \frac{1}{6}\,\mu_o\,\vartheta^2 \;.$$

Hence, the normalization can be arranged by adding a correction term, i.e.

$$_\lambda P(k) \cong {}_\lambda P'(k) - {}_o P(k)\left\{\frac{1}{6}\,\mu_o\,\vartheta^2\right\} \;. \qquad \text{(A8)}$$

Obviously, this heuristic approach needs confirmation. For this purpose we start again from (A1). Development up to second order gives

$$\frac{1}{T}\int_o^T \exp\left\{-k\lambda t + \mu_o(1 - e^{-\lambda t})\right\} dt \cong \frac{1}{T}\int_o^T \exp\left\{-k\lambda t + \mu_o \lambda t - \frac{\mu_o}{2}\lambda^2 t^2\right\} dt$$

$$= \frac{1}{T}\int_o^T \exp\left\{-(k - \mu_o)\,\lambda t - \frac{\mu_o}{2}(\lambda t)^2\right\} dt \;. \qquad \text{(A9a)}$$

With the abbreviations

$$\frac{k - \mu_o}{\mu_o \lambda} = \alpha \qquad \text{and} \qquad \frac{1}{2}\,\mu_o \lambda^2 = \beta \,,$$

the exponent takes the form

$$\left\{\ldots\right\} = -2\alpha\beta \cdot t - \beta \cdot t^2 = -\beta\,(\alpha + t)^2 + \alpha^2 \beta \;.$$

With the new variable $x = t + \alpha$ we therefore simply have

$$\frac{1}{T}\int_o^T \exp\left\{\ldots\right\} dx \cong \exp(\alpha^2 \beta) \cdot \frac{1}{T}\int_\alpha^{T+\alpha} \exp(-\beta x^2)\, dx \;. \qquad \text{(A9b)}$$

A series development of the integrand gives

$$\frac{1}{T}\int_{\alpha}^{T+\alpha} \exp(-\beta x^2)\, dx = \frac{1}{T}\int_{\alpha}^{T+\alpha} (1 - \beta x^2 + \frac{1}{2}\beta^2 x^4 \mp \dots)\, dx$$

$$= \frac{1}{T}\left\{T - \frac{\beta}{3}\left[(T+\alpha)^3 - \alpha^3\right] + \frac{\beta^2}{10}\left[(T+\alpha)^5 - \alpha^5\right] \mp \dots\right\}$$

$$= 1 - \frac{\beta}{3}(T^2 + 3\alpha T + 3\alpha^2)$$

$$+ \frac{\beta^2}{10}(T^4 + 5\alpha T^3 + 10\alpha^2 T^2 + 10\alpha^3 T + 5\alpha^4) \mp \dots$$

If we retain only terms up to ϑ^2 and $(k-\mu_o)^2$, (A9) can also be written as

$$e^{\alpha^2\beta} \cdot \frac{1}{T}\int_{\alpha}^{T+\alpha} e^{-\beta x^2}\, dx$$

$$\cong \left[1 + \frac{(k-\mu_o)^2}{2\mu_o}\right]\left\{1 - \frac{(k-\mu_o)^2}{2\mu_o} - (k-\mu_o)\frac{\vartheta}{2} - \frac{1}{6}\mu_o\vartheta^2 + \frac{1}{4}(k-\mu_o)^2\vartheta^2\right\}$$

$$= 1 - \frac{1}{2\mu_o}(k-\mu_o)^2 - \frac{\vartheta}{2}(k-\mu_o) - \frac{\vartheta^2}{6}\mu_o + \frac{\vartheta^2}{4}(k-\mu_o)^2$$

$$+ \frac{1}{2\mu_o}(k-\mu_o)^2 - \frac{\vartheta^2}{6}\mu_o\frac{(k-\mu_o)^2}{2\mu_o}$$

$$= 1 - \frac{\vartheta}{2}(k-\mu_o) - \frac{\vartheta^2}{6}\mu_o + \vartheta^2(k-\mu_o)^2\frac{1}{6} .$$

Our second-order approximation to the modified Poisson probabilities is therefore

$$_{\lambda}P(k) \cong {}_{o}P(k)\left\{1 - \frac{\vartheta}{2}(k-\mu_o) + \frac{\vartheta^2}{6}\left[(k-\mu_o)^2 - \mu_o\right]\right\} . \qquad \text{(A10)}$$

This expression agrees with (A8), confirming thereby the validity of the surmised correction term.

Let us now look at the moments of k which correspond to the approximation (A10). The expectation value is

$$_{\lambda}E(k) \cong \sum kP - \frac{\vartheta}{2}\sum k(k-\mu_o)P + \frac{\vartheta^2}{6}\left[\sum k(k-\mu_o)^2 P - \mu_o\sum kP\right]$$

$$= m_1 - \frac{\vartheta}{2}(m_2 - \mu_o m_1) + \frac{\vartheta^2}{6}(m_3 - 2\mu_o m_2 + \mu_o^2 m_1 - \mu_o m_1) .$$

Use of the explicit expressions (A5) for the Poisson moments m_r leads to

$$_{\lambda}E(k) \cong \mu_o (1 - \frac{\vartheta}{2} + \frac{\vartheta^2}{6}) . \qquad \text{(A11)}$$

This is in agreement with a power-series development of the exact formula (14) since

$$\frac{\mu_o}{\vartheta}(1 - e^{-\vartheta}) = \mu_o (1 - \frac{\vartheta}{2} + \frac{\vartheta^2}{6} \mp \dots) .$$

For the second ordinary moment we obtain likewise from (A10)

$$_{\lambda}E(k^2) \cong m_2 - \frac{\vartheta}{2}(m_3 - \mu_o m_2) + \frac{\vartheta^2}{6}(m_4 - 2\mu_o m_3 + \mu_o^2 m_2 - \mu_o m_2) .$$

Substitution of (A5) then leads after some rearrangements to the approximation

$$_{\lambda}E(k^2) \cong \mu_o \left[1 + \mu_o - \frac{\vartheta}{2}(1 + 2\mu_o) + \frac{\vartheta^2}{6}(1 + 4\mu_o) \right] .$$

For the variance we therefore find, up to terms proportional to ϑ^2,

$$\begin{aligned} _{\lambda}V(k) &= {_{\lambda}E(k^2)} - {_{\lambda}E^2(k)} \\ &\cong \mu_o \left[1 + \mu_o - \frac{\vartheta}{2}(1 + 2\mu_o) + \frac{\vartheta^2}{6}(1 + 4\mu_o) \right] - \mu_o^2 \left[1 - \vartheta + \vartheta^2 \cdot \frac{7}{12} \right] , \end{aligned}$$

which gives finally

$$_{\lambda}V(k) \cong \mu_o \left[1 - \frac{\vartheta}{2} + \frac{\vartheta^2}{6}(1 + \mu_o/2) \right] . \qquad \text{(A12)}$$

For comparison with the previous exact result (17), we form the ratio

$$R = \frac{_{\lambda}V(k) - {_{\lambda}E(k)}}{_{\lambda}E^2(k)} .$$

Substitution of (A11) and (A12) leads to

$$R \cong \frac{\mu_o \left[1 - \frac{\vartheta}{2} + \frac{\vartheta^2}{6}(1 + \mu_o/2) - 1 + \frac{\vartheta}{2} - \frac{\vartheta^2}{6} \right]}{\mu_o^2 (1 - \vartheta + \frac{7}{12}\vartheta^2)} \cong \frac{\vartheta^2}{12} . \qquad \text{(A13)}$$

This result agrees with the first non-vanishing term resulting from the series development of (17). We therefore conclude that (A10) is a valid second-order approximation of (12), provided that $\vartheta \ll 1$.

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