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Numerical deduction of a formula for some infinite sums

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1. Introduction

In a recent attempt to evaluate the mean arrival time of the registered pulse number k from a decaying radioactive source, the mean activity of which is decreasing exponentially in time (for a preliminary account see [1]), we came across some infinite sums of inverse products which were all of the type

$$ {}_kS_K \equiv \sum_{n=1}^{\infty} \left[n \prod_{j=k}^{K} (n+j) \right]^{-1} , \tag{1} $$

with $k = 1, 2, 3, \ldots$ and $K \geqslant k$.

The simplest example of this kind corresponds to $k = 1$, where we have

$$ {}_1S_K = \sum_{n=1}^{\infty} \left[\prod_{j=0}^{K} (n+j) \right]^{-1} . $$

For this case, some explicit results like

$$ \sum_{n=1}^{\infty} \frac{1}{n\,(n+1)} = 1 , \qquad \sum_{n=1}^{\infty} \frac{1}{n\,(n+1)\,(n+2)} = \frac{1}{4} $$

$$ \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n\,(n+1)\,(n+2)\,(n+3)} = \frac{1}{18} \tag{2} $$

can be found in [2] which allow us to expect that

$$ {}_1S_K = \frac{1}{K \cdot K!} , \qquad \text{for} \quad K = 1, 2, \ldots , \tag{3} $$

will be the corresponding general formula (see note page 12).

However, analogous expressions for $k > 1$ seem to be unknown. As they were needed, we made an attempt (admittedly in desperation) to find them hopefully by direct numerical evaluation.

2. Results of the numerical evaluation

For determining explicitly some of the simpler sums which are defined in a general way by (1), a very short and elementary computer program was written. For practical reasons, the summation was limited to n = 32 000 terms, but partial sums were printed out for each 2 000 consecutive terms for checking purposes. In fact, in order to minimize rounding problems, the actual summation process was performed in reverse order, i.e. from n = 32 000 to n = 1. In order to get reliable numerical results a special set of subroutines, called "improved extended precision", was used. With these programs, which were written by P. Carré of BIPM for an IBM 1130 computer, one can be sure to get at least 11 significant figures for an operation, since the "mantissa" of a real number is represented by 40 bits.

The numerical results obtained in this way by direct summation are reproduced in Table 1 for k = 1 to 4 and $x \equiv K - k = 0$ to 3.

k	x = 0*	x = 1	x = 2	x = 3
1	0.999 968 751	0.249 999 999 512	0.055 555 555 556	0.010 416 666 667
2	0.749 968 751	0.138 888 888 401	0.024 305 555 556	0.003 750 000 000
3	0.611 079 863	0.090 277 777 290	0.013 055 555 556	0.001 712 962 963
4	0.520 802 086	0.064 166 666 178	0.007 916 666 667	0.000 904 195 011

Table 1 - Results obtained for the sums ${}_kS_K$ by the computer applying (1) up to n = 32 000 terms, with x = K - k.

* given with only 9 decimals, for reasons indicated in the text

The results expected according to (3) for k = 1 are given in Table 2 together with their difference with respect to those given in Table 1. The uncertainties associated with Δ_x are estimated from the partial sums; they explain why the empirical values for x = 0 are stated with 9 decimals only.

K	x	$_1S_K$	Δ_x
1	0	1 = 1.000 000 000	$(31\ 249 \pm 2) \cdot 10^{-9}$
2	1	1/4 = 0.250 000 000 000	$(488 \pm 1) \cdot 10^{-12}$
3	2	1/18 = 0.055 555 555 556	$< 1 \cdot 10^{-12}$
4	3	1/96 = 0.010 416 666 667	

Table 2 - Theoretical results for $_1S_K$ and empirical differences
$\Delta_x = {_1S_K}(\text{theor.}) - {_1S_K}(\text{emp.})$

The observed differences Δ_x for x = 0 and 1 are readily explained by the fact that the summation was stopped too early, as is revealed by the partial sums given in Table 3. For $x \geqslant 2$ a summation over some 10 000 terms was always more than sufficient.

n_1	n_2	k = 1	k = 2	k = 3	k = 4
22 001	24 000	0.000 003 788	0.000 003 787	0.000 003 787	0.000 003 787
24 001	26 000	3 205	3 205	3 205	3 204
26 001	28 000	2 747	2 747	2 747	2 747
28 001	30 000	2 381	2 381	2 381	2 381
30 001	32 000	2 083	2 083	2 083	2 083

Table 3 - Some empirical partial sums $\Delta_k S_k \equiv \sum_{n=n_1}^{n_2} \frac{1}{n\,(n+k)}$ for k = K = 1 to 4

The fact that the partial sums $\Delta_k S_k$ listed in Table 3 are practically independent of k (as was to be expected for $n \gg k$) allows us to use the values Δ_x given in Table 2 for correcting the observed values in the columns x = 0 and 1 of Table 1. In this way the new empirical sums indicated in Table 4 are obtained which correspond to an infinite summation.

k	x = 0	x = 1
1	1.000 000 000 (2)*	0.250 000 000 000 (1)*
2	0.750 000 000	0.138 888 888 889
3	0.611 111 112	0.090 277 777 778
4	0.520 833 335	0.064 166 666 666

Table 4 - Corrected empirical sums ${}_kS_K$ of Table 1 (for infinite summation)

3. Interpretation of the numerical data

With the data given in Tables 4 and 1, we might now be in a position to guess what the formulae for the exact results are likely to be. For this, our first task is an attempt to express the numerical results obtained for the infinite sums in the simpler form of fractions. An example will be sufficient to explain how these are obtained. Consider the sum

$${}_3S_6 \cong 0.001\,712\,962\,963\ (1)\ ,$$

which is interpreted as

$${}_3S_6 = 0.001\,71\overline{2\,96}\ ,$$

where the sequence "296" is supposed to repeat itself to infinity. Hence

$${}_3S_6 = (171 + \frac{296}{999}) \cdot 10^{-5} = \frac{171\,125}{999 \cdot 10^5} = \frac{37}{21\,600}\ .$$

Likewise we can easily find probable exact fractions for all the sums evaluated (except for ${}_4S_7$ where no periodicity is yet visible); they are given in Table 5.

Our next objective is to find formulae from which the ratios listed in Table 5 can be obtained in a general way. This will inevitably imply some amount of "educated guessing". The case k = 1 being settled by (3), we begin with k = 2.

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* estimated uncertainty in units of the last digit for a given value of x

k	x = 0	1	2	3
1	1	$\frac{1}{4}$	$\frac{1}{18}$	$\frac{1}{96}$
2	$\frac{3}{4}$	$\frac{5}{36}$	$\frac{7}{288}$	$\frac{3}{800}$
3	$\frac{11}{18}$	$\frac{13}{144}$	$\frac{47}{3\,600}$	$\frac{37}{21\,600}$
4	$\frac{25}{48}$	$\frac{77}{1\,200}$	$\frac{19}{2\,400}$	?

Table 5 - Probable exact values of ${}_kS_K$

a) The case k = 2

For interpreting the empirical sequence 3/4, 5/36, 7/288, 3/800, ... we first have a look at the denominators. They are

$$\begin{aligned} \text{for } K = 2: &\quad 4 = 2! \cdot 2\,, \\ 3: &\quad 36 = 3! \cdot 6\,, \\ 4: &\quad 288 = 4! \cdot 12\,, \\ 5: &\quad 800 = 5! \cdot 20/3\,. \end{aligned}$$

If 3/800 is replaced by 9/2 400, the sequence of denominators is represented by K! K(K-1). The corresponding numerators are then 3, 5, 7, 9, ... Therefore, the observed results can be described by

$$ {}_2S_K = \frac{2K-1}{(K-1)\,K \cdot K!} \,. \qquad (4)$$

b) The case k = 3

The sequence to be examined is 11/18, 13/144, 47/3 600, 37/21 600, ... For the denominators we try

$$\begin{aligned} \text{for } K = 3: &\quad 18 = 3! \cdot 3\,, \\ 4: &\quad 144 = 4! \cdot 6\,, \\ 5: &\quad 3\,600 = 5! \cdot 30\,, \\ 6: &\quad 21\,600 = 6! \cdot 30\,. \end{aligned}$$

In analogy with (4) we try $(K-2)(K-1)K \cdot K!$ which yields

K	(K - 2) (K - 1) K K!		
3	36	=	18 · 2
4	576	=	144 · 4
5	7 200	=	3 600 · 2
6	86 400	=	21 600 · 4

Use of these new denominators gives for the corresponding numerators the sequence

$11 \cdot 2 = 22, \quad 13 \cdot 4 = 52, \quad 47 \cdot 2 = 94, \quad 37 \cdot 4 = 148, \; \ldots$

For further analysis, the following scheme of successive differences may be useful:

x = K - 3	y_x	Δy_x	$\Delta^2 y_x$
0	<u>22</u>		
		<u>30</u>	
1	52		<u>12</u>
		42	
2	94		12
		54	
3	148		

It shows that the numerators y_x can be expressed by a power series in x of the second order, i.e.

$$y_x = a_o + a_1 x + a_2 x^2 \; .$$

According to Newton's interpolation formula, a power series of order r can also be written as a difference equation of the form (see any good textbook on numerical mathematics, as e.g. [3])

$$y_x = \sum_{i=0}^{r} \binom{x}{i} \Delta^i y_o \, , \qquad \text{with} \quad \Delta^o y_o \equiv y_o \; .$$

In our case (the needed "first" differences $\Delta^i y_o$ are underlined in the preceding table) this gives for the numerators

$$y_x = 22 + \binom{x}{1} 30 + \binom{x}{2} 12$$
$$= 22 + 24\,x + 6\,x^2 ,$$

or in terms of K, since $x = K - 3$,

$$y(K) = 4 - 12\,K + 6\,K^2 .$$

We can therefore expect for the case $k = 3$ the formula

$$_3S_K = \frac{2\,(2 - 6\,K + 3\,K^2)}{(K-2)\,(K-1)\,K \cdot K!} . \tag{5}$$

c) The case $k = 4$

For the empirical sequence 25/48, 77/1 200, 19/2 400, ... we try, in analogy with the previous results, for the denominators the formula $(K-3)\,(K-2)\,(K-1)\,K \cdot K!$. This gives

K	$(K-3)\,(K-2)\,(K-1)\,K \cdot K!$
4	576 = 48 · 12
5	14 400 = 1 200 · 12
6	259 200 = 2 400 · 108
7	4 233 600

In the same way as before, we now obtain for the numerators the series $25 \cdot 12 = 300$, $77 \cdot 12 = 924$, $19 \cdot 108 = 2\,052$, 3 828, ..., where the last value is the result of the multiplication $4\,233\,600 \times 904\,195\,011 \cdot 10^{-12}$.

This sequence is again analyzed by a difference scheme, i.e. we form:

$x = K - 4$	y_x	$\Delta\, y_x$	$\Delta^2 y_x$	$\Delta^3 y_x$
0	300			
		624		
1	924		504	
		1 128		144
2	2 052		648	
		1 776		
3	3 828			

As before, this leads to the power series

$$y_x = 300 + \binom{x}{1} 624 + \binom{x}{2} 504 + \binom{x}{3} 144$$

$$= 300 + 420\,x + 180\,x^2 + 24\,x^3 \;,$$

or in terms of $K = x + 4$, after some elementary rearrangements,

$$y(K) = -36 + 132\,K - 108\,K^2 + 24\,K^3 \;.$$

Hence, we can suggest for the case $k = 4$ the formula

$$_4S_K = \frac{12\,(-3 + 11\,K - 9\,K^2 + 2\,K^3)}{(K-3)\,(K-2)\,(K-1)\,K \cdot K!} \;. \qquad (6)$$

4. <u>The general formula for $_kS_K$</u>

Obviously, our next task is to try to find a general formula for $k = 1, 2, \ldots$ of which the previous results (3) to (6) are special cases.

As for the denominator of the general expression looked for, it is not difficult to guess that it will be given by

$$K! \prod_{j=0}^{k-1} (K - j) \;. \qquad (7a)$$

The case of the numerator is less obvious. Let us, just for trying, write it in the form

$$(k-1)! \sum_{j=1}^{k} c_{k,j} \cdot K^{j-1} \;. \qquad (7b)$$

A comparison of (7b) with the more explicit results (3) to (6) yields for the unknown coefficients $c_{k,j}$ the empirical values given in Table 6.

k	$c_{k,1}$	$c_{k,2}$	$c_{k,3}$	$c_{k,4}$
1	1			
2	-1	2		
3	2	-6	3	
4	-6	22	-18	4

Table 6 - Values of the coefficients $c_{k,j}$ appearing in the suggested expression (7b) for the numerator of a general formula for $_kS_K$

A first look at Table 6 does not reveal a simple way to form the coefficients. Some familiarity with combinatorial problems, however, and in particular the characteristic alternating sequence of signs, might suggest a closer comparison with Stirling numbers. Indeed, this reveals that we apparently have the simple relation

$$c_{k,j} = j \cdot s(k,j) \ , \qquad (8)$$

where $s(k,j)$ are Stirling numbers of the first kind. For an excellent treatment of these numbers see [4]. A more extended tabulation (for $1 \leqslant k \leqslant 25$ and $1 \leqslant j \leqslant k$) can be found in [5]. In Table 7 we give a list of $s(k,j)$ till $k = 6$.

If the results contained in (7) and (8) are combined, the general formula looked for turns out to be given by

$$_k S_K = \frac{(k-1)! \sum_{j=1}^{k} j \cdot s(k,j) \cdot K^{j-1}}{K! \prod_{j=0}^{k-1} (k-j)} \ . \qquad (9)$$

This is the main result of the present study.

k	j = 1	2	3	4	5	6
1	1					
2	-1	1				
3	2	-3	1			
4	-6	11	-6	1		
5	24	-50	35	-10	1	
6	-120	274	-225	85	-15	1

Table 7 - Some Stirling numbers $s(k,j)$ of the first kind

5. Some additions and conclusions

In order to strengthen our confidence in the above heuristic formula (9), it may be worthwhile to compare it with some further results of numerical summations not already used in the derivation of (9). In fact, these new data would have been of little value before since they cannot be easily transformed into a fraction, as will be seen from Table 8.

k	K	numerical sum	expected value
5	7	0.005 204 081 633	$\frac{51}{9\ 800}$
	8	0.000 526 502 268	$\frac{743}{1\ 411\ 200}$
	9	0.000 049 314 479	$\frac{1\ 879}{38\ 102\ 400}$
6	8	0.003 624 574 830	$\frac{341}{94\ 080}$
	9	0.000 329 244 352	$\frac{2\ 509}{7\ 620\ 480}$
	10	0.000 027 964 118	$\frac{2\ 131}{76\ 204\ 800}$

Table 8 - Some further numerical results for the sums ${}_kS_K$ and the corresponding expected values according to (9)

It is easy to verify that the expected values agree in all the six cases listed in Table 8 with those obtained numerically for the 12 decimals given. This perfect agreement substantiates our claim that (9) must be the correct formula. We are confident, therefore, that if some day a general expression for ${}_kS_K$ will be derived on sound mathematical grounds - assuming that this has not yet been done -, it will be identical with our heuristic result.

The method described in this report to derive a general formula for some infinite sums on the basis of a number of sufficiently accurate numerical results is obviously far from being a general approach. In particular, no progress would have been possible if the numerical sums had not shown a periodicity which allowed their transformation into fractions. This is clearly a very special situation. In fact, in our original problem (compare [1]) another possible decomposition looked quite tempting, too. This would have led us to consider sums of the type

$$ {}_kT_n(x) = \sum_{j=0}^{\infty} \frac{x^{k+j}}{n \prod_{s=0}^{j} (n+k+s)} , \qquad (10)$$

with k = 1, 2, ... and n = 1, 2, ...

This is quite different from (1). Apparently, the only known result related to this type is [6]

$$T = \sum_{i=0}^{\infty} \frac{x^i}{\prod_{s=0}^{i} (n+s)} = \frac{(n-1)!}{x^n} \sum_{i=n}^{\infty} \frac{x^i}{i!} \quad . \tag{11}$$

Since

$$T = \frac{1}{n} + \sum_{i=1}^{\infty} \frac{x^i}{\prod_{s=0}^{i} (n+s)}$$

$$= \frac{1}{n} + \sum_{i=0}^{\infty} \frac{x^{i+1}}{n \prod_{s=0}^{i} (n+1+s)} = \frac{1}{n} + {}_1T_n(x) \ , \tag{12}$$

the case $k = 1$ of (10) can be considered as settled, but no solutions are known for $k > 1$. However, already an expression like (11) would no doubt be too complicated to be recognizable from numerical results.

Since the problem described by (1) represents a rather special case with features we can hardly hope to find again, one may wonder why the way in which we obtained formula (9) has been described here in some detail. Perhaps a possible justification can be that the lucky outcome of this search might be an encouragement to look also in more complicated situations for ways to transform the problem into a form where a purely numerical attack is not quite hopeless, at least for obtaining a partial solution. This, in turn, can then serve as a starting point for further trials.

I am very grateful to P. Carré for the kind interest he has shown in the problems treated in this report.

References

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Note added in proof

The surmised equation (3) is in fact correct since it corresponds to the following expression we have just found in [6] as no. 285 in the form

$$\sum_{j=0}^{\infty} \frac{j!}{(n+j)!} = \frac{1}{(n-1)\cdot(n-1)!} \; ,$$

which can be readily transformed into

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)\,(j+2)\,\ldots\,(j+n)} = \sum_{j=1}^{\infty} \left\{ \prod_{s=0}^{n-1} (j+s) \right\} = {}_1S_{n-1} \; .$$

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