Evaluation of the third asymptotic moment for counting distributions

by Jörg W. Müller

Bureau International des Poids et Mesures, F-92310 Sèvres

1. Introduction

In a previous report [1] formulae have been derived for the first two asymptotic moments of renewal processes. If these results are applied to Poisson distributions distorted by the presence of a dead time, they permit to determine the numerical value of the dead time involved. For dead times inserted on purpose with special electronic circuits, there usually exist simpler and more accurate methods of measurement (see e.g. [2]). However, there are cases where they cannot be applied, and this is in particular true for the perturbations which are due to the detector itself or to the first stages of the associated electronic chain. If it is supposed that their combined effect can be described by a "first" dead time τ_1 , then the pulses can be said to pass through a series arrangement of two dead times where the second (τ_2) is inserted artificially and will be assumed to be known.

The influence of such a chain of two dead times on the count rate is well understood for those cases where both τ_1 and τ_2 belong to one of the usual "types", i.e. if they are strictly extended or non-extended. There are then four different combinations of types. Since only the case $\alpha \equiv \tau_1/\tau_2 < 1$ is of real interest (otherwise τ_2 has no effect at all), it is always possible (cf. Fig. 1) to write the ratio of output to input rate (or the total transmission) in the form

$$T_{tot} \equiv \frac{R}{\rho} = \frac{R_o}{\rho} \cdot \frac{R}{R_o} \equiv T_2 \cdot T_1 \ , \tag{1}$$

where $R_o = R$ for $\tau_1 = 0$.

Whereas T_2 is the usual dead-time correction taking only τ_2 into account, T_1 describes the additional influence of the first dead time on the output rate R.

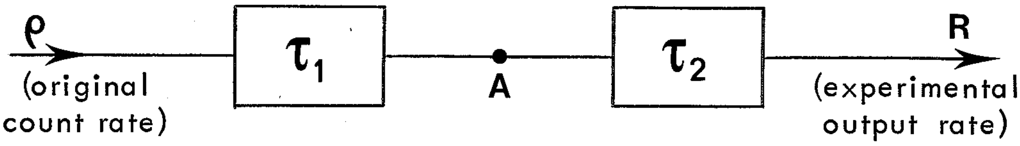

Fig. 1 - The notation used for the count rates in a series arrangement of two dead times. For measuring τ_1 , the statistical properties of the process are studied at A.

Formulae and graphs for T_1 have been given previously (see in particular [3] ; for additional information also [4]). For most practical applications τ_2 will be of the non-extended type and then only two arrangements of the dead times are possible. For the high count rates we are now particularly interested in, the graphical plot given previously (Fig. 3 in [3]) is insufficient. It has therefore been extended to values of $x \equiv \rho\tau_2 = 1.5$. This graph, together with an analogous plot for τ_2 extended, will be reproduced later.

Since τ_2 can be measured with sufficient accuracy by the methods available (see e.g. [2] or [3]), the major remaining problem consists in determining experimentally both value and "type" of the first dead time. We suggest to do this by a careful analysis of the distortion produced on an original Poisson process.

At first sight, one might feel tempted to study directly with a time-to-amplitude converter the distribution of the time intervals between successive events. This seems to be a simple and promising way since the probability densities corresponding to the traditional two types of dead times are well known. We have in fact performed such measurements, but the result is rather disappointing; the observed density shows no clear resemblance to any of the two theoretical shapes. In fact, there is no real basis to expect that an "intermediate" type (- whatever this may mean) should also have some "intermediate" shape for the density. All we can possibly hope for is that some simple and well-defined characteristic quantity (with which a definite numerical value can be associated) will be close to the ones corresponding to a certain type of dead time.

For a simple characterization of the amount of distortion produced on a Poisson process, it is practical to use for instance the variance-to-mean ratio

$$V \equiv \sigma_k^2(t) / \hat{k}(t) \tag{2}$$

of the counting process consisting of events which have passed the first dead time (i.e. at point A in Fig. 1). The behaviour of V is quite well known for the different values and types of dead times (see [5] and [1]).

Whereas for small distortions the type has little influence, the numerical value should be known for larger distortions before information about the type can be obtained. Obviously, what we need is some additional experimental characteristic. Since we cannot be sure that the numerical value of τ_1 (as well as its type) will remain constant for different count rates, measuring for instance V for sources of different activity does not solve the problem. However, additional information on the perturbation can be obtained by determining for example also the third moment of the empirical counting distribution. In fact, this data can only be used if we are in a position to compare it with some theoretical expectation. It is for this reason that we shall make an attempt to calculate the third central moment in what follows. For a preliminary account of the main results see [6].

2. Some general relations

Since the present study is a complement to the results obtained in [1] for the first two moments, we assume the reader to be familiar with the basic concepts used in treating asymptotic properties of renewal processes. The relations derived in the first two sections of [1] will be given here without proof, except for those cases where the derivation of a more general expression seems appropriate. The notation used is largely the same.

The main aim of the present report is to arrive at general expressions for the third central moment, abbreviated by $\mu_3(t)$, for the number of events k occurring in a time interval t. The counting process considered is of the general renewal type where f(t) is the density for the time interval between arrivals, except for the first event observed after the time origin t = 0, which is described by the density ${}_0f(t)$. Again it is practical to evaluate first the factorial moments which are defined for order r by*

$$\psi_r(t) \equiv E\left\{k_{(r)}\right\}, \qquad (3 = \text{I}/3)$$

with $k_{(r)} \equiv k(k-1)(k-2)\ldots(k-r+1)$.

The third central moment can also be expressed in terms of factorial moments (see e.g. [7]) as

$$\mu_3(t) = \psi_3(t) + 3\,\psi_2(t) - 3\,\psi_2(t)\cdot\psi_1(t) + 2\,\psi_1^3(t) - 3\,\psi_1^2(t) + \psi_1(t)\,. \qquad (4)$$

The corresponding equations for $\hat{k}(t) = m_1(t)$ and $\sigma_k^2(t) = \mu_2(t)$ have been given before (I/4).

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* Equations numbered (I/n) refer to (n) in [1]; the correspondances are given to simplify comparison.

In terms of the distributions

$$\Phi_k(t) \equiv \int_0^t \varphi_k(x)\, dx\,, \qquad k = 1, 2, \ldots, \qquad (5 = 1/2)$$

where $\varphi_k(t)$ is the interval density for the arrival of event number k, the factorial moments can be written as

$$\Psi_r(t) = \sum_{k=r}^{\infty} k_{(r)} \left[\Phi_k(t) - \Phi_{k+1}(t) \right] . \qquad (6 = 1/8)$$

This expression may be simplified to

$$\Psi_r(t) = \sum_{k=r}^{\infty} k_{(r)} \cdot \Phi_k(t) - \sum_{k=r}^{\infty} (k-1)_{(r)} \cdot \Phi_k(t)$$

$$= \sum (k-1)\,(k-2) \ldots (k-r+1) \left[k - (k-r) \right] \Phi_k(t)$$

$$= r \sum_{k=r}^{\infty} (k-1)_{(r-1)} \cdot \Phi_k(t) \,. \qquad (7)$$

While for $r = 1$ and 2 this result confirms the earlier relations (I/9) and (I/10), the case $r = 3$ gives the new result

$$\Psi_3(t) = 3 \sum_{k=3}^{\infty} (k-1)\,(k-2)\; \Phi_k(t) \,. \qquad (8)$$

For a modified renewal process (m), the Laplace transform corresponding to (7) is therefore

$${}_m\tilde{\Psi}_r(s) = \frac{r}{s} \sum_{k=r}^{\infty} (k-1)_{(r-1)} \cdot {}_o\tilde{f}(s) \left[\tilde{f}(s)\right]^{k-1}$$

$$= \frac{r}{s} \cdot {}_o\tilde{f}(s) \sum_{k=r-1}^{\infty} k_{(r-1)} \cdot \tilde{f}^{k}(s)$$

$$= \frac{r}{s} \cdot {}_o\tilde{f}(s) \cdot \frac{(r-1)!\, \tilde{f}^{r-1}(s)}{\left[1 - \tilde{f}(s)\right]^{r}}$$

$$= \frac{r!}{s} \cdot \frac{{}_o\tilde{f}(s) \cdot \tilde{f}^{r-1}(s)}{\left[1 - \tilde{f}(s)\right]^{r}} \,, \qquad (9)$$

which is in agreement with a result given some time ago (namely eq. A4 in [8]) and where use has been made of the relation

$$\sum_{n=0}^{\infty} n_{(r)} \cdot x^n = \frac{x^r \cdot r!}{(1-x)^{r+1}} \quad . \tag{10}$$

Again the results for the special cases r = 1 and 2 agree with (I/11) and (I/12), whereas the transformed third factorial moment turns out to be

$$ {}_m\tilde{\Psi}_3(s) = \frac{6}{s} \cdot \frac{{}_o\tilde{f}(s) \cdot \tilde{f}^2(s)}{[1 - \tilde{f}(s)]^3} \quad . \tag{11}$$

3. Asymptotic results

The derivation of asymptotic relations for the moments now requires some lengthy power expansions in s. Their common basis is the development

$$\tilde{f}(s) \equiv E\{e^{-st}\} = \sum_{i=0}^{\infty} \frac{(-s)^i}{i!} \cdot m_i \, , \tag{12 = I/14}$$

with $m_i \equiv E\{t^i\}$,

and likewise for the initial density ${}_o f(t)$, the moments of which are denoted by ${}_o m_i(t)$. The derivations are often quite cumbersome and we confine ourselves to giving the results.

As can be seen from (4) or its transform

$$\begin{aligned} {}_m\tilde{\mu}_3(s) = {} & {}_m\tilde{\Psi}_3(s) + 3\, {}_m\tilde{\Psi}_2(s) + {}_m\tilde{\Psi}_1(s) \\ & - 3\mathcal{L}\left\{ {}_m\Psi_1^2(t) \right\} + 2\mathcal{L}\left\{ {}_m\Psi_1^3(t) \right\} - 3\mathcal{L}\left\{ {}_m\Psi_1(t) \cdot {}_m\Psi_2(t) \right\} \, , \end{aligned} \tag{4'}$$

the determination of the third moment requires the evaluation of a number of terms involving the factorial moments ${}_m\Psi_r(t)$ or their transforms which are given by (9). Let us begin with the simplest cases.

a) Asymptotic form of ${}_m\tilde{\Psi}_3(s)$

This term can be evaluated much in the same way as in [1] for ${}_m\Psi_1$ and ${}_m\Psi_2$. If the transformed expressions for the first two factorial moments are written as

$$ {}_m\tilde{\Psi}_1(s) \cong \frac{a_1}{s^2} + \frac{b_1}{s} \tag{13} $$

and

$$ {}_m\tilde{\Psi}_2(s) \cong \frac{a_2}{s^3} + \frac{b_2}{s^2} + \frac{c_2}{s} , \tag{14} $$

the corresponding coefficients, taken from (I/15) and (I/16), are seen to be

$$ a_1 = \frac{1}{m_1} , \qquad b_1 = \frac{1}{2\,m_1^2}\,(m_2 - 2\,{}_o m_1\, m_1) \tag{15} $$

and

$$ \begin{aligned} a_2 &= \frac{2}{m_1^2} , \qquad b_2 = \frac{2}{m_1^3}\,(m_2 - m_1^2 - {}_o m_1\, m_1) , \\ c_2 &= \frac{1}{6\,m_1^4}\,(9\,m_2^2 - 4\,m_1\,m_3 + 6\,{}_o m_2\,m_1^2 \\ &\qquad - 6\,m_1^2\,m_2 - 12\,{}_o m_1\,m_1\,m_2 + 12\,{}_o m_1\,m_1^3) . \end{aligned} \tag{16} $$

We are now looking for the similar development

$$ {}_m\tilde{\Psi}_3(s) \cong \frac{a_3}{s^4} + \frac{b_3}{s^3} + \frac{c_3}{s^2} + \frac{d_3}{s} . \tag{17} $$

As can be seen from (11) this requires the series expansion of the following two terms

$$ \begin{aligned} {}_o\tilde{f}(s)\cdot\tilde{f}(s)^2 \cong\; & 1 - s\,({}_o m_1 + 2\,m_1) + s^2\cdot\frac{1}{2}\,(2\,m_1^2 + 2\,m_2 + 4\,{}_o m_1\,m_1 + {}_o m_2) \\ & - s^3\cdot\frac{1}{6}\,(6\,m_1\,m_2 + 2\,m_3 + 6\,{}_o m_1\,m_1^2 + 6\,{}_o m_1\,m_2 + 6\,{}_o m_2\,m_1 + {}_o m_3) \end{aligned} \tag{18} $$

and

$$ \begin{aligned} \left[1 - \tilde{f}(s)\right]^3 \cong\; & s^3\Big[m_1^3 - s\cdot\frac{3}{2}\,m_1^2\,m_2 + s^2\cdot\frac{1}{4}\,(2\,m_1^2\,m_3 + 3\,m_1\,m_2^2) \\ & - s^3\cdot\frac{1}{8}\,(m_1^2\,m_4 + 4\,m_1\,m_2\,m_3 + m_2^3)\Big] . \end{aligned} \tag{19} $$

Substitution into (11) then yields, after a lengthy division for the coefficients in (17), the values

$$a_3 = \frac{6}{m_1^3}, \qquad b_3 = \frac{3}{m_1^4}(3\,m_2 - 2\,{}_om_1\,m_1 - 4\,m_1^2),$$

$$c_3 = \frac{3}{m_1^5}(4\,{}_om_1\,m_1^3 + {}_om_2\,m_1^2 - 3\,{}_om_1\,m_1\,m_2 + 2\,m_1^4$$
$$- 4\,m_1^2\,m_2 - m_1\,m_3 + 3\,m_2^2) \qquad \text{and} \qquad (20)$$

$$d_3 = \frac{1}{4\,m_1^6}(-24\,{}_om_1\,m_1^5 + 48\,{}_om_1\,m_1^3\,m_2 - 36\,{}_om_1\,m_1\,m_2^2$$
$$+ 12\,{}_om_1\,m_1^2\,m_3 - 24\,{}_om_2\,m_1^4 + 18\,{}_om_2\,m_1^2\,m_2$$
$$- 4\,{}_om_3\,m_1^3 + 12\,m_1^4\,m_2 - 36\,m_1^2\,m_2^2 + 30\,m_2^3$$
$$- 24\,m_1\,m_2\,m_3 + 16\,m_1^3\,m_3 + 3\,m_1^2\,m_4).$$

b) Other asymptotic forms

The asymptotic expressions for the remaining "mixed" terms in (4') call for some additional rearrangements. According to (13) the first factorial moment is

$$_m\Psi_1(t) \cong a_1 \cdot t + b_1 , \qquad (13')$$

hence

$$_m\Psi_1^2(t) \cong a_1^2 \cdot t^2 + 2\,a_1\,b_1 \cdot t + b_1^2 \qquad \text{and}$$
$$_m\Psi_1^3(t) \cong a_1^3 \cdot t^3 + 3\,a_1^2\,b_1 \cdot t^2 + 3\,a_1\,b_1^2 \cdot t + b_1^3 , \qquad (21)$$

and likewise with (14)

$$_m\Psi_2(t) \cong \frac{1}{2}\,a_2 \cdot t^2 + b_2 \cdot t + c_2 . \qquad (14')$$

This gives for the expressions appearing in (4')

$$\mathcal{L}\left\{_m\Psi_1^2(t)\right\} \cong \frac{2\,a_1^2}{s^3} + \frac{2\,a_1\,b_1}{s^2} + \frac{b_1^2}{s},$$

$$\mathcal{L}\left\{ {}_m\Psi_1^3(t) \right\} \cong \frac{6\,a_1^3}{s^4} + \frac{6\,a_1^2\,b_1}{s^3} + \frac{3\,a_1\,b_1^2}{s^2} + \frac{b_1^3}{s} \quad \text{and} \tag{22}$$

$$\mathcal{L}\left\{ {}_m\Psi_1(t) \cdot {}_m\Psi_2(t) \right\} \cong \frac{3\,a_1\,a_2}{s^4} + \frac{2\,a_1\,b_2 + b_1\,a_2}{s^3} + \frac{a_1\,c_2 + b_1\,b_2}{s^2} + \frac{b_1\,c_2}{s} \;.$$

c) Final forms

The expressions (15) and (16) for the coefficients now have to be substituted into (21) and (22); this is obviously rather tedious and will not be reproduced here. Afterwards, the resulting expressions must be used in (4'). The outcome is again of the form

$${}_m\tilde{\mu}_3(s) \cong \frac{A}{s^4} + \frac{B}{s^3} + \frac{C}{s^2} + \frac{D}{s} \;. \tag{23}$$

Lengthy rearrangements finally lead to the following expressions for the coefficients:

$$A = B = 0\,,$$

$$C = \frac{1}{m_1^5}\left(m_1^4 - 3\,m_1^2\,m_2 - m_1\,m_3 + 3\,m_2^2\right) \quad \text{and} \tag{24}$$

$$\begin{aligned} D = \frac{1}{4\,m_1^6}\Big[& -4\,{}_o m_1\,m_1^5 + 2\,m_1^4\left(-6\,{}_o m_2 + m_2 + 6\,{}_o m_1^2\right) \\ & + 4\,m_1^3\left(3\,{}_o m_1\,m_2 - {}_o m_3 + 2\,m_3 + 3\,{}_o m_1\,{}_o m_2 - 2\,{}_o m_1^3\right) \\ & + m_1^2\left(4\,{}_o m_1\,m_3 + 12\,{}_o m_2\,m_2 + 3\,m_4 - 15\,m_2^2 - 12\,{}_o m_1^2\,m_2\right) \\ & + 4\,m_1\left(-3\,{}_o m_1\,m_2^2 - 5\,m_2\,m_3\right) + 22\,m_2^3 \Big] \;. \end{aligned} \tag{25}$$

By using central moments instead of m_2, m_3 and m_4, i.e. with the substitutions [7]

$$m_2 = \sigma^2 + m_1^2\,,$$

$$m_3 = \mu_3 + 3\,m_1\,\sigma^2 + m_1^3\,, \tag{26 $\cong$ I/21}$$

$$m_4 = \mu_4 + 4\,m_1\,\mu_3 + 6\,m_1^2\,\sigma^2 + m_1^4\,,$$

and similarly for the moments ${}_o m_r$ of the initial density ${}_o f(t)$, equations (24) and (25) can be brought into the simpler forms

$$C = \frac{1}{m_1^5} (3\,\sigma^4 - m_1 \mu_3) \qquad \text{and} \tag{24a}$$

$$D = \frac{1}{4\,m_1^6} \Big[22\,\sigma^6 - 4\,m_1\,(3\,{}_o m_1\,\sigma^4 + 5\,\sigma^2\,\mu_3) + m_1^2\,(4\,{}_o m_1\,\mu_3 + 12\,{}_o\sigma^2\,\sigma^2 + 3\,\mu_4 - 9\,\sigma^4) - 4\,m_1^3\,{}_o\mu_3 \Big] . \tag{25a}$$

This obviously leads for the original to the asymptotic expression

$${}_m\mu_3(t) \cong C \cdot t + D\,, \tag{26}$$

with the coefficients given above. This general result for the third central moment of a modified renewal process is the principal outcome of the present study. For most practical applications, however, it will be useful to derive some more specific expressions.

In looking at the above formulae for C and D we note that the moments up to order four of f(t) appear, whereas for ${}_o f(t)$ the third order is sufficient, giving thereby additional support to a corresponding conjecture made in [1]. It is also worth mentioning that C is independent of ${}_o f(t)$.

4. Specific counting processes

The general modified renewal process can now be specialized to the usual three counting processes. The corresponding forms of (26) will only be given in terms of the simpler central moments.

a) Ordinary process

Since here ${}_o f(t) = f(t)$, the index "o" may be dropped for all moments and we then get from (24a) and (25a)

$${}_{or}C = C \qquad \text{and}$$

$${}_{or}D = \frac{1}{4\,m_1^6} \Big[2\,\sigma^2\,(11\,\sigma^4 - 10\,m_1\mu_3) - 3\,m_1^2\,(3\,\sigma^4 - \mu_4) \Big] . \tag{27}$$

b) Equilibrium process

In addition to the expressions given in [1] for ${}_0m_1$ and ${}_0\sigma^2$ we can also obtain from (I/29), after some elementary algebra and using the correspondences listed in [7], the relation

$$ {}_0\mu_3 = \frac{1}{4\, m_1^3} \left[\mu_4\, m_1^2 - 2\,\mu_3 m_1 (\sigma^2 - m_1^2) + \sigma^2\, (\sigma^4 - 3\,\sigma^2\, m_1^2 + m_1^4) \right] . \qquad (28) $$

With their help (25a) is finally transformed into

$$ {}_{eq}C = C \qquad \text{and} $$

$$ {}_{eq}D = \frac{1}{2\, m_1^6} \left[m_1^2\, (\mu_4 - 3\,\sigma^4) + 6\,\sigma^2\, (\sigma^4 - m_1 \mu_3) \right] . \qquad (29) $$

c) Free-counter process

For this case, as mentioned in [1], a more explicit expression of D can only be given for a specified original counting process.

5. Original Poisson process

As it is of great importance for our applications, the special case of an original Poisson process (with count rate ρ) deserves special attention. We shall further assume that the perturbation is caused by a dead time of known type. This will then yield the formulae with which the measured values of $\mu_3(t)$ can be compared directly.

In addition to the moments m_1, σ^2 and μ_3 for the corresponding interval densities already given in Table 2 of [1], we need also an expression for the fourth moment. For a non-extended dead time the interval density f(t) is a simple exponential*, the moments of which are known to be $m_r = r!/\rho^r$. With the correspondences given in [7] this leads immediately to

$$ \mu_4 = 9/\rho^4 . \qquad (30a) $$

This result is also applicable to a free-counter process.

For extended dead times, a method described previously (see section 5 of [9]) can be applied. We find

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* The shift by τ is irrelevant for the central moments.

$$m_4 = 24\,R(R^3 - 3\,R^2\tau + 2\,R\tau^2 - \tau^3/6)\,, \qquad \text{with } R = y/\rho\,.$$

This then corresponds to

$$\mu_4 = \frac{y}{\rho^4}\left[9\,y^3 - 36\,xy\,(y-x) - 4\,x^3\right]\,. \tag{30b}$$

The expressions for the first four moments have now to be inserted into the formulae for $\mu_3(t)$. After a number of rearrangements this leads for the two types of dead times to the final asymptotic results for the coefficients C and D listed in Table 1.

Process	$C \cdot t$	D
- for τ non-extended :		
ordinary	$\lambda^4(3\lambda - 2)\cdot\rho t$	$\frac{1}{2}\lambda^4(11\lambda^2 - 20\lambda + 9)$
equilibrium	$\lambda^4(3\lambda - 2)\cdot\rho t$	$3\,\lambda^4(\lambda - 1)^2$
free counter	$\lambda^4(3\lambda - 2)\cdot\rho t$	$\frac{1}{2}\lambda^3(11\lambda^3 - 26\lambda^2 + 19\lambda - 4)$
- for τ extended :		
ordinary	$\frac{1}{y^3}(y - 3x)^2\cdot\rho t$	$\frac{-x}{y^3}(y^2 - 9\,xy + 17\,x^2)$
equilibrium	$\frac{1}{y^3}(y - 3x)^2\cdot\rho t$	$\frac{3\,x^2}{2\,y^3}(2\,y - 7\,x)$
free counter	$\frac{1}{y^3}(y - 3x)^2\cdot\rho t$	$\frac{1}{y^3}\left\{y\left[3 - y(1+x) + 3\,x(2+3x)\right] - 2 - x(17x^2 + 9x + 6)\right\}$

Table 1 - Coefficients for the asymptotic third central moment $\mu_3(t) \cong C \cdot t + D$ when an original Poisson process (with count rate ρ) has been distorted by a non-extended or an extended dead time τ.

The abbreviations used are $x = \rho\tau$, $\lambda = \frac{1}{1+x}$ and $y = e^x$.

It can be shown by series expansion that for $x \ll 1$ the coefficient D, independently of the type of dead time, reduces to

$$ {}_{or}D \cong -x \, , \qquad {}_{eq}D \cong 3\,x^2 \quad \text{and} \quad {}_{fr}D \cong \frac{7}{2}\,x^2 \, , \tag{31a} $$

whereas for the time-dependent part we always get in this approximation

$$ C \cdot t \cong (1 - 7\,x) \cdot \rho t \; . \tag{31b} $$

6. The ratio W

By analogy with the variance-to-mean ratio given in (2) we can form a similar quantity which involves the third central moment by defining

$$ W \equiv \mu_3(t) / \hat{k}(t) \; . \tag{32} $$

Expressing W in a more explicit form by means of the various asymptotic expressions given in Table 1 for $\mu_3(t)$ would be straightforward, although rather cumbersome. Instead, we restrict ourselves to simplified cases where some characteristic features can be more readily seen.

Since all our relations are asymptotic ones, the measuring time t will have to be relatively long. In particular, if we choose t such that D can be neglected with respect to $C \cdot t$, the formulae for $\mu_3(t)$ become simple. We then get for non-extended (n) or extended (e) dead times the expressions

$$ \begin{aligned} {}_n\mu_3(t) &\cong \lambda^4\,(3\,\lambda - 2) \cdot \rho t \, , \\ {}_e\mu_3(t) &\cong \frac{1}{y^3}\,(y - 3\,x)^2 \cdot \rho t \; . \end{aligned} \tag{33} $$

Similarly we have, according to [1], for the expectation values

$$ \begin{aligned} {}_n\hat{k}(t) &\cong \lambda \cdot \rho t \, , \\ {}_e\hat{k}(t) &\cong \frac{1}{y} \cdot \rho t \; . \end{aligned} \tag{34} $$

Hence, the ratio W turns out to be

$$ \begin{aligned} {}_nW &\cong \lambda^4\,(1 - 2\,x) \, , \\ {}_eW &\cong \frac{1}{y^2}\,(y - 3\,x)^2 \; . \end{aligned} \tag{35} $$

If in addition we can assume that $x \ll 1$, a series development is possible which gives

$$
\begin{aligned}
{}_n\mu_3(t) &\cong \rho t \cdot (1 - 7x + 25x^2) , \\
{}_e\mu_3(t) &\cong \rho t \cdot (1 - 7x + \frac{43}{2}x^2)
\end{aligned}
\tag{33'}
$$

and

$$
\begin{aligned}
{}_n\hat{k}(t) &\cong \rho t \cdot (1 - x + x^2) , \\
{}_e\hat{k}(t) &\cong \rho t \cdot (1 - x + \frac{1}{2}x^2) ;
\end{aligned}
\tag{34'}
$$

thus also

$$
\begin{aligned}
{}_nW &\cong 1 - 6x + 18x^2 , \\
{}_eW &\cong 1 - 6x + 15x^2 .
\end{aligned}
\tag{35'}
$$

If we restrict ourselves to first order in x, the type of the dead time is no longer of importance and we have the common expression

$$W \cong 1 - 6x . \tag{36}$$

Comparing (36) with the corresponding approximate form for V (see e.g. eq. 46 in [1]) which is

$$V \cong 1 - 2x , \tag{37}$$

we recognize that the influence of a dead time is about three times stronger for W than for V. However, as we shall see later, this welcome feature goes along with a reduced precision of W.

In any event, it should be clear already that both with V and W the determination of the type of dead time will not be a simple matter. In the practical application of this approach to measured frequency distributions, we have in fact preferred to use the best approximations available for the moments (hence for μ_3 the coefficients listed in Table 1) rather than the approximate formulae given above for V and W.

7. Problems of perturbation and precision

By using the model of a Pólya process, we have recently shown [10] how the possible presence of any additional scatter will modify the experimental moments of the observed counting distribution. If these fluctuations (which are assumed to be small) are such that they produce for instance for the counting efficiency ε a relative spread of

$$\frac{\sigma(\varepsilon)}{\varepsilon} = \sqrt{b} \quad , \tag{38}$$

then Table A1 in [10] shows that for an original Poisson process with expectation value ϱt both the variance and the third central moment are augmented. This reflects itself in the values of V and W which, instead of being unity, are then (to first order in the perturbation b)

$$V_b = 1 + b \cdot \varrho t$$

and (39)

$$W_b \cong 1 + 3b \cdot \varrho t \; .$$

If this perturbation is superimposed on the distortion due to the dead time which is described roughly by (36) and (37), the measured (m) values are expected to be given approximately by

$$V_m \cong 1 + b \cdot \varrho t - 2 \cdot \varrho\tau \cong 1 - 2\hat{k}(\vartheta - b/2)$$

and (40)

$$W_m \cong 1 + 3b \cdot \varrho t - 6 \cdot \varrho\tau \cong 1 - 6\hat{k}(\vartheta - b/2) \; ,$$

$$\text{with } \vartheta = \tau/t \; .$$

Hence, if we use V_m or W_m for determining the dead time τ involved, but suppose - in ignorance of b - that (36) and (37) can be applied, the resulting value τ_m will be systematically low. As can be seen from the relation

$$\tau \cong \tau_m + \frac{b}{2} \cdot t \; , \tag{41}$$

the possible influence of b can in principle be eliminated by taking data with different measuring times t and extrapolating τ_m to $t = 0$. In practice, however, this procedure may be difficult to apply for reasons of precision (see below) and because asymptotic relations should not be used for small time intervals.

Let us finally tackle the problem of the precision with which V and W (and hence τ) can be determined. As a rough estimate is all we need, approximate methods will be used. If we define the central sample moments of order r for a random variable x by

$$M_r = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^r \; , \qquad r = 0, 1, 2, \ldots , \tag{42}$$

where $\bar{x} = (1/n) \sum_i x_i$ is the mean value of the n measurements x_i, then its variance can be shown [11] to be given approximately (for $n \gg 1$)* by

$$\sigma^2(M_r) \cong \frac{1}{n} (M_{2r} - 2r \cdot M_{r-1} \cdot M_{r+1} - M_r^2 + r^2 \cdot M_2 \cdot M_{r-1}^2) \; . \tag{43}$$

Since $M_1 = 0$, we have in particular

$$\sigma^2(M_2) \cong \frac{1}{n} (M_4 - M_2^2) \tag{44}$$

and

$$\sigma^2(M_3) \cong \frac{1}{n} (M_6 - 6\, M_2\, M_4 - M_3^2 + 9\, M_2^3) \; . \tag{45}$$

As for $n \gg 1$ we can replace the sample moments M_r by the population moments μ_r and since, neglecting the perturbation, the respective moments are still approximately given by those of a Poisson process with mean $\rho\, t \equiv \bar{\lambda}$ (see for instance [13] for explicit expressions), we may also write for the variances of the second and third central moments

$$\begin{aligned} \sigma^2(\mu_2) &\cong \frac{1}{n} (\mu_4 - \mu_2^2) \cong \frac{1}{n} (\bar{\lambda} + 3\,\bar{\lambda}^2 - \bar{\lambda}^2) \\ &= \frac{\bar{\lambda}}{n} (1 + 2\bar{\lambda}) \end{aligned} \tag{46}$$

and

$$\begin{aligned} \sigma^2(\mu_3) &\cong \frac{1}{n} (\mu_6 - \mu_3^2 - 3\,\mu_2 \left[2\,\mu_4 - 3\mu_2^2\right]) \\ &\cong \frac{1}{n} (\bar{\lambda} + 25\,\bar{\lambda}^2 + 15\,\bar{\lambda}^3 - \bar{\lambda}^2 - 3\,\bar{\lambda}\left[2\,\bar{\lambda} + 6\,\bar{\lambda}^2 - 3\,\bar{\lambda}^2\right]) \\ &= \frac{\bar{\lambda}}{n} (1 + 18\,\bar{\lambda} + 6\,\bar{\lambda}^2) \; . \end{aligned} \tag{47}$$

If we neglect the small additional uncertainty due to the experimental mean value, we finally arrive with (46) and (47) at the following approximations for the relative uncertainties of V and W :

$$r_V \equiv \frac{\sigma(V)}{V} \cong \frac{\sigma(\mu_2)}{\mu_2} \cong \sqrt{\frac{1 + 2\,\bar{\lambda}}{n \cdot \bar{\lambda}}} \tag{48}$$

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* Note that the formula indicated in [12] is deformed by a misprint.

and likewise

$$r_W \cong \frac{\sigma(\mu_3)}{\mu_3} \cong \sqrt{\frac{1 + 18\,\bar{\lambda} + 6\,\bar{\lambda}^2}{n \cdot \bar{\lambda}}} \quad . \tag{49}$$

For $\bar{\lambda} \gg 1$ the asymptotic ratio of the relative uncertainties is therefore

$$\lim_{\lambda \to \infty} (r_W / r_V) = \sqrt{3\,\bar{\lambda}} \quad . \tag{50}$$

Some intermediate numerical values are given below in the table.

$\bar{\lambda}$	r_W / r_V	$\bar{\lambda}$	r_W / r_V
1	2.9	10	6.1
1.5	3.2	20	8.2
2	3.5	30	9.9
2.5	3.7	40	11.3
5	4.7	50	12.5

This shows that the relative statistical uncertainty is always appreciably larger for W than for V. However, by choosing a value of $\bar{\lambda}$ in the region of about 1 or 2, the resulting uncertainties for the dead time τ_1 will be roughly the same as a consequence of (36) and (37). This conclusion is well confirmed by our experiments.

A considerable amount of frequency distributions for the number of registered counts in t have been measured by now in order to determine the value and type of the first dead time. However, the detailed description of this experimental part of the work must be deferred to a later report.

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