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A test for judging the presence of additional scatter in a Poisson process

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Summary

The effect of additional scatter on a Poisson process is studied. Possible causes for such fluctuations are insufficient stability of the detection efficiency or of the associated electronics. It is shown with a simple model that the presence of fluctuations results in a characteristic broadening of the counting distribution. Comparison of the observed distribution with the one expected for a Poisson process with the same mean value will show three different regions, each with predictable sign of the deviation; the presence of scatter can thus be decided upon by a sign test. Experimental results are in excellent agreement with this expectation.

1. Introduction

In the experimental measurement of a process which, for theoretical reasons, is assumed to follow a simple Poisson law, complete agreement with this distribution can be expected only if the methods used to detect and register the "events" in question do not introduce any measurable distortion. The effect of a finite resolving or dead time, which results in a more regular spacing of the events and hence in a reduction of the variance-to-mean ratio, has been treated some time ago [1] and the results will not be repeated here.

In some sense the situation we wish to describe in what follows is just the opposite of the case treated before: we look for the possible presence of effects which augment the variance of the observed process. In the field of counting nuclear events, the parameter μ of the Poisson distribution corresponds to some expectation value $\mu = \rho t$, where t is the measuring time and ρ denotes the (observed) mean count rate. Since ρ is related to a more fundamental quantity like the source activity N_o by means of an experimental parameter which is of the nature of a detection efficiency ε , we have in the simplest situation a relation of the type

$$\mu = N_o \cdot t \cdot \mathcal{E} \ , \tag{1}$$

neglecting decay or other possible complications. Whereas N_o is a constant and t can usually be measured with negligible uncertainty, this is not necessarily the case for $\mathcal{E}$. Various causes for possible fluctuations of $\mathcal{E}$ as a function of time can be imagined, depending on the nature of the detector and the associated electronics. Thus for a proportional counter, for instance, it may prove difficult or impossible to ensure sufficient stability in the high tension or uniformity in the composition and pressure of the counting gas to render these effects negligible [2] . In all these cases, the result would be a corresponding increase in the observed variance, exceeding thereby the value expected for a pure Poisson process.

While the very knowledge of the possible presence or absence of effects augmenting the scatter of the experimental data is already an interesting piece of information, the neglect of such effects would result in serious errors for quantities which are directly derived from the value of the observed variance, as it is for instance the case for dead times determined on the basis of variance-to-mean ratios alone [3] .

2. The negative binomial distribution as a model

The exact way an experimental distribution may differ from the Poisson law obviously depends on the detailed mechanism of the interfering effects. If we wish to arrive at a general description, a model has to be chosen which is both reasonably flexible and sufficiently simple for an exact mathematical treatment. As a matter of fact, there is an embarassingly rich choice of possible generalizations of the Poisson law; for a good general review see e.g. [4] . Considering the experimental origin of the distortion, it may be natural in our case to look for such a generalization where the constant detection efficiency $\mathcal{E}$ is replaced by a positive random quantity. Perhaps the simplest choice consistent with the criteria given above is the family of gamma densities, i.e.

$$f(\mu) = \frac{c^r}{\Gamma(r)} \mu^{r-1} e^{-c\mu} , \qquad \text{for} \qquad \mu > 0 , \tag{2}$$

with $r > 0$ and $c > 0$.

It can be shown that the relative spread of μ is then given by

$$\frac{\sigma(\mu)}{m_1(\mu)} = \frac{1}{\sqrt{r}} \ . \tag{2a}$$

In the original Poisson distribution

$$P_{\mu}(k) = \frac{\mu^k}{k!} e^{-\mu} , \tag{3}$$

which indicates the probability that exactly k events are observed if the expectation value is μ, we now consider μ as a random quantity which is described by (2). The corresponding new probabilities W(k) resulting from a superposition or mixture is then given by

$$W(k) = \int_0^{\infty} P_{\mu}(k) \cdot f(\mu)\, d\mu$$

$$= \int_0^{\infty} \frac{\mu^k e^{-\mu}}{k!} \frac{c^r}{\Gamma(r)} \mu^{-r-1} e^{-c\mu} d\mu$$

$$= \frac{c^r}{k!\,\Gamma(r)} \int_0^{\infty} \mu^{k+r-1} e^{-\mu(c+1)} d\mu .$$

Putting $\mu(c+1) = \lambda$, i.e. $d\mu = d\lambda/(c+1)$, we obtain

$$W(k) = \frac{c^r}{(c+1)^{k+r} k!\,\Gamma(r)} \int_0^{\infty} \lambda^{k+r-1} e^{-\lambda} d\lambda$$

$$= \frac{\Gamma(k+r)}{k!\,\Gamma(r)} \frac{c^r}{(c+1)^{k+r}} . \tag{4}$$

This result and its derivation have been well known for long, of course; they can be found in the textbooks under the headings negative binomial, Pólya or Pascal distribution.

For many applications somewhat different expressions for W(k) are often more convenient. Thus, since

$$\frac{\Gamma(k+r)}{k!\,\Gamma(r)} = \binom{k+r-1}{r-1} = \binom{k+r-1}{k} ,$$

the use of the abbreviation

$$p = \frac{c}{c+1} ,$$

hence also

$$q \equiv 1 - p = \frac{1}{c+1},$$

permits to bring (4) into the equivalent form

$$W(k) = \binom{k+r-1}{k} p^r q^k$$

$$= \binom{k+r-1}{r-1} p^r q^k, \qquad \text{where} \quad 0 \leqslant p \leqslant 1 . \tag{5a}$$

As for any real r there exists the identity

$$\binom{-r}{k} = (-1)^k \binom{k+r-1}{k}, \qquad \text{for } k \text{ integer and non-negative,}$$

another possible form is

$$W(k) = \binom{-r}{k} p^r (-q)^k , \tag{5b}$$

always with $k = 0, 1, 2, \ldots$ This form well explains the name "negative binomial" given to the distribution.

Finally, a further useful alternative can be derived from (4) by observing that

$$\Gamma(k+r) = (k+r-1)(k+r-2) \ldots (k+r-k) \; \Gamma(r)$$

$$= r(r+1)(r+2) \ldots (r+k-1) \; \Gamma(r)$$

$$= 1 \left(1+\frac{1}{r}\right)\left(1+\frac{2}{r}\right) \ldots \left(1+\frac{k-1}{r}\right) \Gamma(r) \; r^k .$$

If we put

$$b = \frac{1}{r},$$

(4) can be brought into the form

$$W(k) = \frac{r^k}{k!} \prod_{j=1}^{k-1} (1+jb) \left[\frac{c}{c+1}\right]^{1/b} \frac{1}{(c+1)^k}$$

$$= \left[\frac{1}{b(c+1)}\right]^k \frac{1}{k!} \left[\frac{1}{1+1/c}\right]^{1/b} \prod_{j=1}^{k-1} (1+jb) .$$

With the additional abbreviation

$$\mu = \frac{1}{bc},$$

we arrive at the expression

$$W(k) = \frac{\mu^k}{k\,!} \left[\frac{1}{1 + b\mu}\right]^{k+1/b} \prod_{j=1}^{k-1} (1 + jb) \ . \qquad (6)$$

Whichever form may be preferred, it will be noted that the distribution has two independent parameters (which are μ and b in the case of eq. 6), and it is this feature which guarantees its additional flexibility compared to the simple Poisson law.

For the moments see the Appendix.

3. The Poisson approximation of the Pólya distribution

For obvious practical reasons, our main interest in the Pólya (or negative binomial) distribution is for the case where it differs little from the Poisson law. The aim will therefore be to derive an expression where W(k) is represented by a Poisson probability multiplied by a correction which might be a power series development in a suitable parameter like b. If such a form were available, we could then hope to proceed further by applying a reasoning similar to the one used already in [1] . The parameter b seems to be a good choice indeed, since it follows easily from (6) that

$$\lim_{b \to 0} W(k) = P_{\mu}(k) \ . \qquad (7)$$

Hence, our next task is to derive an approximation for W(k) which includes a correction term linear in b. Let us consider the different factors in (6) separately. Two of them are easy to treat and they give readily

$$(1 + b\mu)^{-k} \cong 1 - k\, b\mu \ , \qquad (8)$$

$$\begin{aligned} \prod_{j=1}^{k-1} (1 + jb) &= (1 + b)\,(1 + 2b) \ \ldots \ (1 + [k-1]\, b) \\ &\cong 1 + (b + 2b + 3b + \ \ldots \ + [k-1]\, b) \\ &= 1 + \frac{b}{2}\, k\, (k-1) \ . \end{aligned} \qquad (9)$$

The handling of the term

$$W(0) = \left(\frac{1}{1 + b\mu}\right)^{1/b}$$

is a bit more delicate. Putting $1/b = n$, we can write

$$[W(0)]^{-1} = (1+\frac{\mu}{n})^n = \sum_{i=0}^{n} \binom{n}{i} (\frac{\mu}{n})^i$$

$$= 1 + n\frac{\mu}{n} + \frac{n}{2}(n-1)(\frac{\mu}{n})^2 + \frac{n}{6}(n-1)(n-2)(\frac{\mu}{n})^3 + \dots$$

$$= 1 + \mu + \frac{1}{2}(1-\frac{1}{n})\mu^2 + \frac{1}{6}(1-\frac{1}{n})(1-\frac{2}{n})\mu^3 + \frac{1}{24}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n})\mu^4 + \dots$$

$$= 1 + \mu + \frac{\mu^2}{2!}(1-b) + \frac{\mu^3}{3!}(1-b)(1-2b) + \frac{\mu^4}{4!}(1-b)(1-2b)(1-3b) + \dots$$

$$= 1 + \mu + \sum_{i=2}^{n} \frac{\mu^i}{i!} \prod_{k=1}^{i-1} (1-kb)$$

$$\cong 1 + \mu + \sum_{i=2}^{\infty} \frac{\mu^i}{i!} \left[1 - i(i-1)\frac{b}{2}\right]$$

$$= e^{\mu} - \sum_{i=2}^{\infty} \frac{\mu^i}{i!} i(i-1)\frac{b}{2} = e^{\mu} - \frac{b}{2} \sum_{i=2}^{\infty} \frac{\mu^i}{(i-2)!}$$

$$= e^{\mu} - \frac{b}{2}\mu^2 \sum_{k=0}^{\infty} \frac{\mu^k}{k!} = e^{\mu} - \frac{b}{2}\mu^2 e^{\mu}$$

$$= e^{\mu}(1 - \frac{b}{2}\mu^2);$$

hence

$$W(0) \cong e^{-\mu}(1 + \frac{b}{2}\mu^2). \tag{10}$$

Substitution of (8), (9) and (10) into (6) leads to the approximation looked for, namely

$$W(k) = \frac{\mu^k}{k!} e^{-\mu} (1 + \frac{b}{a}\mu^2)(1 - bk\mu)(1 + \frac{b}{a}k[k-1])$$

$$\cong P_{\mu}(k) \left\{ 1 + b\left[\frac{\mu^2}{2} - k\mu + \frac{k}{2}(k-1)\right] \right\}$$

$$= P_{\mu}(k) \left\{ 1 + \frac{b}{2}\left[(\mu - k)^2 - k\right] \right\}. \tag{11}$$

Before drawing some conclusions from this approximation which is of the desired form, let us check whether (11) leads to the correct first moments. For explicit expressions of the ordinary moments $\sum_{k=0}^{\infty} k^r P_\mu(k)$ of a Poisson distribution see e.g. [4]. Dropping the arguments k and μ for brevity in the summations, we obtain

- for the normalization:

$$N = \sum W = \sum P + \frac{b}{2}\left[\mu^2 \sum P - (2\mu+1)\sum kP + \sum k^2 P\right]$$

$$= 1 + \frac{b}{2}\left[\mu^2\, 1 - (2\mu+1)\,\mu + \mu\,(\mu+1)\right] = 1\,, \tag{12a}$$

- for the first moment:

$$m_1(k) = \sum kW = \sum kP + \frac{b}{2}\left[\mu^2 \sum kP - (2\mu+1)\sum k^2 P + \sum k^3 P\right]$$

$$= \mu + \frac{b}{2}\left[\mu^2\,\mu - (2\mu+1)\,\mu\,(1+\mu) + \mu(1+3\mu+\mu^2)\right] = \mu\,, \tag{12b}$$

- for the second moment:

$$m_2(k) = \sum k^2 W = \sum k^2 P + \frac{b}{2}\left[\mu^2 \sum k^2 P - (2\mu+1)\sum k^3 P + \sum k^4 P\right.$$

$$\left.= \mu(1+\mu) + \frac{b}{2}\left[\mu^2\mu(1+\mu) - (2\mu+1)\mu(1+3\mu+\mu^2) + \mu(1+7\mu+6\mu^2+\mu^3)\right]\right.$$

$$= \mu(1+\mu) + b\,\mu^2\,, \tag{12c}$$

which leads for the variance to

$$\sigma^2(k) = m_2(k) - m_1^2(k) = \mu(1+b\mu)\,. \tag{12d}$$

Since the results (12) agree with the values given in Table A1 of the Appendix for the moments of the Pólya distribution, we can be assured that the approximation (11) is a valid one. It follows from (11) that W(k) coincides with P(k) if the term in the curly brackets vanishes, hence for

$$(\mu - k)^2 - k = 0\,,$$

which has the two solutions

$$k_{1,2} = \mu + \frac{1}{2} \mp \sqrt{\mu + \frac{1}{4}}\,. \tag{13}$$

This is quite an interesting result. First we note that the crossing of the two curves occurs at exactly the same points we have found previously for a Poisson process disturbed by a (small) dead time (see eq. 8 in [1]). As for the deviations between W(k) and P(k), however, the signs are different: if the original Poisson distribution has been somehow (slightly) perturbed so that its variance is increased, this should reflect itself in the sign of the differences W(k) - P(k), which is now expected to be negative for values of k lying between the limits k_1 and k_2 determined by (13), but positive for all k values outside this range. For a given set of measurements, the question whether additional scatter can be detected or not may therefore be decided upon e.g. by a sign test, as it has been shown previously.

We note in passing that it is no doubt more than a mere coincidence that the result (13) agrees with the confidence limits determined by van der Waerden [5] for the parameter μ of a Poisson process on the basis of a measured value k (taking g = 1). The practical usefulness of such "asymmetrical errors" has been shown in [6].

4. How to simulate a Pólya process

In order to show the feasibility of the approach sketched above, a practical application will no doubt be helpful; in addition, it would be more convincing if actual recordings rather than simulations were used. Finally, the quantitative measurement of some quantity characterizing the size of the distortion detected would be welcome, especially if this value can be compared with a prediction.

By using a traditional counting arrangement, it seems difficult to meet all these requirements. Distortions introduced are usually hard to control in an independent way; even their stability in time and the absence of any long-term drift are not readily guaranteed. For this reason, a different approach has been chosen. A look at (1) reveals that it is not necessary to introduce scatter by means of the detection efficiency ε, but that the same effect can also be produced by influencing the measuring time t. This is much simpler to perform and it can be easily controlled.
A particularly elegant solution then consists in using the arrival times of an independent Poisson process. If we define the measuring interval t by the time it takes to count S pulses from a radioactive source (realized by the intervals between registrations after a scale of S), then t is a random quantity which is known to have a density

$$f(t) = \left\{ \tilde{\rho}\, e^{-\tilde{\rho} t} \right\}^{*S} = \frac{\tilde{\rho}^{S}}{(S-1)!}\, t^{S-1}\, e^{-\tilde{\rho} t}, \qquad \text{for } t \geq 0, \tag{14}$$

where $\tilde{\rho}$ is the count rate of the source.

This function is exactly of the same form as the one used in (2) for μ in constructing the model which leads to the Pólya distribution. Since S now takes the role of $r = 1/b$, it follows from Table A1 that the relative variance V will be given by

$$V \equiv \frac{\sigma^2(k)}{m_1(k)} = 1 + \frac{m_1}{S} , \tag{15}$$

a prediction which can be checked with actual data. In a series of measurements made by P. Bréonce, the relation (15) could be very well verified. As an example, the frequencies F(k) obtained in a run are reproduced in Table 1. The count rate of the gamma pulses used in this experiment was only about 430 s^{-1}, which ensures a negligible dead-time effect. The mean time interval t, after a scale factor of S = 1 000, was close to 0.1 s. The expected frequencies G(k) for a Poisson process were calculated from (3) by

$$G(k) = N \; P \; m_1(k), \tag{16}$$

where $N = 385\,411$ and $m_1 \cong 41.02$,

with both parameters derived from the measurements. Details of the electronic equipment used to accumulate the data on a multiscaler will be given shortly in a separate report [7] .

The two "critical" values of k derived from (13) are $k_1 \cong 35.1$ and $k_2 \cong 47.9$ and a look at Table 1 shows that the prediction of the signs for the differences F - G is very good indeed: there are only five exceptions and three of them appear at the "crossing points" while the other two are in the tail of the distribution ; they are all well within the statistical precision of the individual points. It is obvious that a correct prediction of the sign in 49 out of 54 cases cannot be a matter of pure chance and that even a smaller perturbation could have been detected in the available data by a sign test. For the shortcomings of a "symmetrical" test such as the well-known chi-square test, see our earlier remarks in [1]. The relative variance V expected on the basis of (15), i.e.

$$V \cong 1 + 41.02/1\,000 \cong 1.041 ,$$

is clearly in excellent agreement with the observed value 1.041 ± 0.003.

k	Frequencies F(k)	G(k)	Sign of difference F - G
≤11	0	0	
12	2	0	
13	1 } 6	0 } 3	+
14	1	0	
15	0	1	
16	2	2	
17	6	4	+
18	13	10	+
19	24	22	+
20	47	44	+
21	104	86	+
22	182	161	+
23	361	287	+
24	561	491	+
25	927	806	+
26	1 359	1 271	+
27	2 071	1 931	+
28	3 129	2 828	+
29	4 183	4 000	+
30	5 645	5 469	+
31	7 486	7 236	+
32	9 553	9 274	+
33	11 636	11 527	+
34	14 149	13 905	+
35	16 217	16 295	-
k_1			
36	18 548	18 565	+
37	20 424	20 580	-
38	21 933	22 214	-
39	22 849	23 362	-
40	23 322	23 955	-
41	23 307	23 964	-
42	23 005	23 402	-
43	22 106	22 322	-
44	20 323	20 808	-

k	Frequencies F(k)	G(k)	Sign of difference F - G
45	18 548	18 966	-
46	16 625	16 911	-
47	14 791	14 758	+
k_2			
48	12 648	12 610	+
49	10 694	10 555	+
50	8 786	8 659	+
51	7 253	6 964	+
52	5 605	5 493	+
53	4 414	4 251	+
54	3 546	3 229	+
55	2 605	2 408	+
56	1 912	1 763	+
57	1 332	1 269	+
58	1 017	897	+
59	698	624	+
60	462	426	+
61	348	287	+
62	235	190	+
63	136	123	+
64	103	79	+
65	55	50	+
66	27	31	-
67	28	19	+
68	16	11	+
69	6	7	-
70	4	4	(zero)
71	1	2	
72	3 } 5	1 } 5	(zero)
73	1	1	
≥74	0	1	

N = 385 411

The experimental moments are

$m_1(k) = 41.016 \pm 0.011,$

$\sigma^2(k) = 42.71 \pm 0.10,$

$\mu_3(k) = 44.7 \pm 1.2 \ .$

Table 1 - Comparison between experimental frequencies F(k) and those expected for a Poisson process, G(k). The observed signs of the differences F - G follow very well the predicted pattern. For details on this and also on the experimental generation of a Pólya process see text.

5. Some distinctions between related processes

The preceding discussion once more showed that the Poisson process is not only a most useful distribution in itself, but that it can often serve as a model or for comparison. It may be interesting, therefore, to have some simple criteria at hand which could help to decide when the use of the Poisson law is justified or, in the negative case, which alternative could possibly be applied as a better approximation. It will be obvious that in practice such a decision is only of real interest in those cases where a possible departure from the Poisson distribution is relatively small. Apart from the sign test mentioned above, other simple means for distinguishing between similar processes exist, some of which will be sketched in what follows.

Among the many possible "competitors" to the Poisson law, we confine ourselves (rather arbitrarily) to the binomial and the Pólya distribution. A first characteristic distinction is given by the value of the relative variance V defined in (15). Another typical feature concerns the ratio of successive probabilities, i.e. the quantity

$$Q(k) \equiv \frac{W(k)}{W(k-1)}, \qquad k = 1, 2, \ldots . \tag{17}$$

This will be quickly derived for the three distributions in question.

a) Binomial distribution

Here we have the well-known expression

$$W(k) = \binom{n}{k} p^k q^{n-k},$$

where mean and variance are given by

$$m_1(k) = n p, \qquad \sigma^2(k) = n p q,$$

hence $V = q$.

The ratio of successive probabilities leads to the recursion formula

$$Q(k) = \frac{\binom{n}{k}}{\binom{n}{k-1}} \frac{p}{q} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{n(n-1)(n-2)\ldots(n-k+2)} \frac{(k-1)!}{k!} \frac{p}{q}$$

$$= \frac{p}{q} \frac{n-k+1}{k} = \frac{1}{k} \frac{p(n+1)}{q} - \frac{p}{q}. \tag{18}$$

The reason for preferring the (apparently more complicated) last form will become obvious later.

b) Poisson distribution

From $W(k) = \frac{\mu^k}{k!} e^{-\mu}$, with

$m_1(k) = \sigma^2(k) = \mu$, thus $V = 1$,

we get readily also

$$Q(k) = \frac{\mu^k}{\mu^{k-1}} \frac{(k-1)!}{k!} = \frac{\mu}{k} . \tag{19}$$

c) Pólya distribution

From (5b) and Table A1, i.e.

$$W(k) = p^r \binom{-r}{k} (-q)^k$$

and

$$m_1(k) = \frac{rq}{p} \quad \text{and} \quad \sigma^2(k) = \frac{rq}{p^2} , \quad \text{thus} \quad V = \frac{1}{p} ,$$

one obtains

$$Q(k) = \frac{\binom{-r}{k}}{\binom{-r}{k-1}} \frac{(-q)^k}{(-q)^{k-1}} = \frac{(-r-k+1)}{k} (-q) = \frac{1}{k} q (r-1) + q . \tag{20}$$

It follows from (18) to (20) that for all the three cases considered $Q(k)$ can be brought into the form

$$Q(k) = A + B/k . \tag{21}$$

The corresponding coefficients A and B are given in Table 2, together with the values for V and the special case Q(1). It is easy to show that in the graphical plot of Q(k) all the three straight lines have a common intersection (see Fig. 1). If we approximate Q(k) by the experimentally available ratios F(k)/F(k-1), then, provided that the scatter in the frequencies is not too large, a first decision on the probability distribution to be used can possibly be based on such a plot of the data.

Distribution	V	A	B	Q(1)
Binomial	$q < 1$	$\frac{-p}{q} < 0$	$\frac{p\,(n+1)}{q} > \mu$	$\frac{\mu}{q} > \mu$
Poisson	1	0	μ	μ
Pólya	$\frac{1}{p} > 1$	$q > 0$	$q\,(r-1) < \mu$	$p\mu < \mu$

Table 2 - Summary of some characteristics of the binomial, Poisson and Pólya distributions, with $V = \sigma^2(k)/m_1(k)$. The quantities A and B are the coefficients appearing in (21) for $Q(k) = W(k)/W(k-1)$.

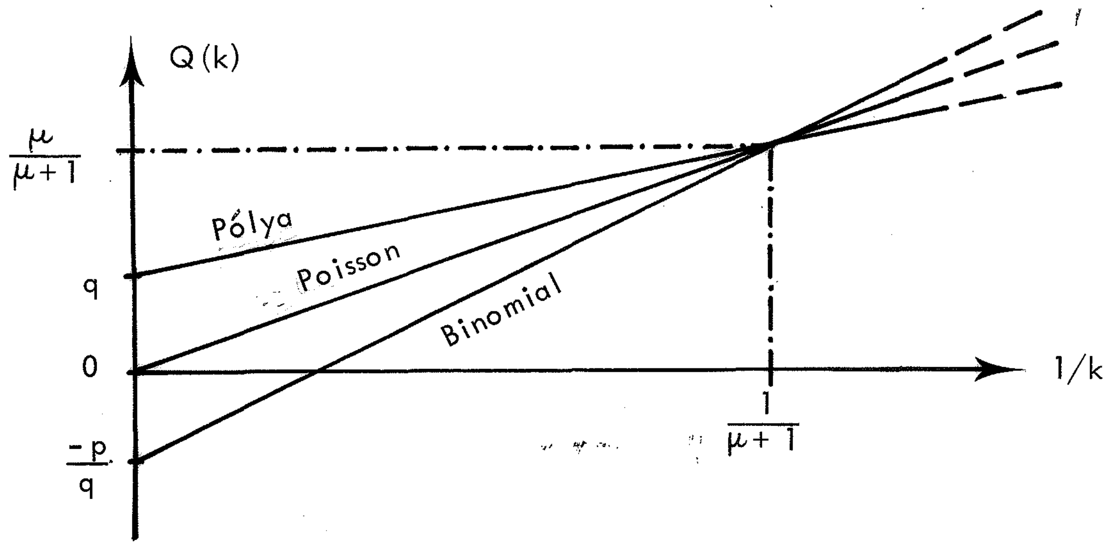

Figure 1 - Schematic plot of the ratios $Q(k) = W(k)/W(k-1)$ as a function of $1/k$, for a Pólya, a Poisson and a binomial distribution with the common expectation value μ.

APPENDIX

Some remarks on the moments

For the sake of completeness, we want to indicate also briefly the first three moments; the corresponding results are assembled in Table A1.

Perhaps the simplest way to obtain these moments without any real effort of calculation is as follows. We start from (5b) and put

$$\frac{-q}{p} = \tilde{P} \ , \qquad \text{i.e.}$$

$$\tilde{Q} = 1 - \tilde{P} = \frac{p+q}{p} = \frac{1}{p} \ , \tag{A1}$$

and $\quad r = -N$.

With these variables (5b) takes the form

$$W(k) = \binom{N}{k} p^{-N} (p\tilde{P})^k = \binom{N}{k} p^{k-N} \tilde{P}^k$$

$$= \binom{N}{k} \tilde{P}^k \tilde{Q}^{N-k} \ . \tag{A2}$$

This corresponds to a binomial distribution for which the first moments are known to be

$$m_1(k) = N\tilde{P} \ , \quad \sigma^2(k) = N\tilde{P}\tilde{Q} \quad \text{and} \quad \mu_3(k) = N\tilde{P}\tilde{Q}(\tilde{Q} - \tilde{P}) \ . \tag{A3}$$

Hence, by substitution of (A1) we get for the moments of (5b)

$$m_1(k) = \frac{rq}{p} \ , \quad \sigma^2(k) = \frac{rq}{p^2} \quad \text{and} \quad \mu_3(k) = \frac{rq}{p^3}(1+q) \ . \tag{A4}$$

It is possible that this heuristic derivation will not look very convincing to everybody. In this case, a more satisfactory approach can be based for instance on the characteristic function

$$\varphi(t) = E\left[e^{itk}\right] = \left(\frac{p}{1 - q\,e^{it}}\right)^r \ , \tag{A5}$$

from which the moments are obtained in the well-known way by forming the derivations at the origin.

	for the variables used in		
Moment	eq. (4)	eq. (5)	eq. (6)
$m_1(k)$	$\frac{r}{c}$	$\frac{rq}{p}$	μ
$m_2(k)$	$\frac{r}{c^2}(r+c+1)$	$\frac{rq}{p^2}(1+rq)$	$\mu(1+\mu+b\mu)$
$m_3(k)$	$\frac{r}{c^3}\left[r^2+c^2+3(r+rc+c)+2\right]$	$\frac{rq}{p^3}\left[1+(3r+1)q+r^2q^2\right]$	$\mu\left[1+3\mu+\mu^2+3b\mu(1+\mu)+2b^2\mu^2\right]$
$\sigma^2(k)$	$\frac{r}{c^2}(c+1)$	$\frac{rq}{p^2}$	$\mu(1+b\mu)$
$\mu_3(k)$	$\frac{r}{c^3}(c^2+3c+2)$	$\frac{rq}{p^3}(1+q)$	$\mu(1+3b\mu+2b^2\mu^2)$
$V=\sigma^2/m_1$	$1+\frac{1}{c}$	$\frac{1}{P}$	$1+b\mu$
$W=\mu_3/m_1$	$(1+\frac{1}{c})(1+\frac{2}{c})$	$\frac{1+q}{p^2}=\frac{1}{p}(\frac{2}{p}-1)$	$(1+b\mu)(1+2b\mu)$

Table A1 - The first three moments of the Pólya distribution expressed in the different variables used in eqs. (4) to (6)

After a number of rearrangements, a general expression is obtained for the ordinary moments of order $n = 0, 1, 2, \ldots$ which can be brought (for the variables used in eq. 5) into the form

$$m_n(k) = \sum_{j=0}^{n} S(n, j) \cdot r^{(j)} \cdot \left(\frac{q}{p}\right)^j , \tag{A6}$$

where $r^{(j)} \equiv r(r+1)(r+2) \ldots (r+j-1)$ is a "rising" factorial

and $S(n, j)$ are the Stirling numbers of the second kind, to which we have to add the supplementary condition that

$$r^{(0)} = 1 \qquad \text{and} \qquad S(n, 0) = \delta_{n,0} .$$

We note in passing that a corresponding formula given by Fisz [8] is wrong (except for $n = 1$ and 2).

Likewise a general expression can be found for the central moments of the Pólya distribution which reads

$$\mu_n(k) = \sum_{t=0}^{n} \left(-\frac{rq}{p}\right)^t \binom{n}{t} \sum_{j=0}^{n-t} S(n-t, j) \cdot r^{(j)} \cdot \left(\frac{q}{p}\right)^j . \tag{A7}$$

Direct use of equations (4), (5) and (6) for obtaining the moments is possible, but may occasionally lead to intermediate expressions which call for a rather acrobatic skill in reducing them to simpler forms; those who feel attracted by this kind of challenge may test their abilities. The transition to another set of variables is no problem, of course. Thus, for the heuristic approach sketched above, expressions of the moments in the variables used in (6) can be obtained directly from those valid for a binomial distribution if the substitutions

$$N\mathcal{P} = \mu \qquad \text{and} \qquad \widetilde{\mathcal{P}} = -b\mu \tag{A8}$$

are used instead of (A1).

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