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The source-pulser method revisited

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Abstract: We reexamine the source-pulser method of measuring dead time, which is an interesting variant of the well-known two-source method. A description based on the underlying statistical processes is given which leads to an improved expression for the dead time to be determined. Thereby use has to be made of a correction factor μ discussed previously. The results permit to extend the application of the method and to check the validity of the simple approximate formula used till now.

1. Introduction

In the last two or three decades, a number of methods have been proposed to measure the dead time of a counting system. For a recent review of this field see [1] or [2]. The well-known two-source method has given rise to several variants, one of which is the source-pulser method where one source is replaced by a train of periodic pulses. The merits of this version have been described in some detail by Baerg [3] (see also [4]).

One of the virtues of this method lies in the fact that the dead time can be obtained by applying a simple formula which involves only directly measured quantities, namely the oscillator frequency ν, the experimental source rates r and r_ν, measured without and with the periodic pulse train superimposed, respectively. For a non-extended dead time, its value is then determined [3, 4] by

$$\tau_o = \frac{1}{r}\left[1 - \sqrt{\frac{r_\nu - r}{\nu}}\right] . \tag{1}$$

The derivation of this formula relies on a number of assumptions on the stochastic nature of the superimposed process and the survival probabilities for the pulses which are difficult to justify in detail. On the other hand, careful experiments have verified at various laboratories that (1) gives excellent results for a large range of parameters which are of practical interest. Only when ν exceeds a value of about $(3\tau)^{-1}$, some irregularities begin to appear in calculated numerical values for the dead times [5] which are attributed to the approximate nature of (1), although the exact origin of the trouble could not be determined.

In view of this somewhat unclear situation it seemed worthwhile to try to arrive at an exact description of the underlying stochastic processes, in particular for the case of a non-extended dead time. Such an approach is presented in what follows.

2. The four interval densities involved

In order to derive the observable count rate of the superposition, we have to determine the dead-time losses for both components of the process. For the sake of simplicity these will be called ρ - and ν-pulses for those originating from the source and the pulse generator respectively. The calculation is then based on the simple idea of the balance equation [6]. This relation states that the new count rate is equal to the old one minus the dead-time losses, and it has to be applied twice.

For evaluating the loss ℓ produced by the dead time τ of a given registered pulse, we have to know the (total) density D(t) of the following events (of a certain type) in the original sequence (i.e. without dead time), since

$$\ell_x = \int_0^\tau D_x(t)\, dt , \tag{2}$$

where t is the time interval between the moment a pulse was registered (at t = 0) and the arrival of a subsequent event. The index x characterised the "start" and "stop" pulses. As there are two types of events in the superposition considered, four different interval densities and losses have to be distinguished, i.e. x may stand for "$\rho\rho$", "$\rho\nu$", "$\nu\rho$" or "$\nu\nu$".

Let us consider briefly these four cases for $D_x(t)$, namely

a) density of original ρ-pulses following a registered ρ-pulse,
b) " " " ν- " " " " ρ- " ,
c) " " " ρ- " " " " ν- " ,
d) " " " ν- " " " " ν- " .

As the original sequence of pulses from the source is assumed to form a Poisson process (with count rate ρ), which is known to be "memoryless", we find for a) and c), for the conditional (integral) densities,

$$D_{\rho\rho}(t) = D_{\nu\rho}(t) = \rho \sum_{k=1}^{\infty} \frac{(\rho t)^{k-1}}{(k-1)!} \cdot e^{-\rho t} \cdot U(t) = \rho \cdot U(t) , \quad (3)$$

i.e. a constant value ρ for all intervals $t > 0$.

Similarly, case d) gives rise to a simple result, namely

$$D_{\nu\nu}(t) = \sum_{k=1}^{\infty} \delta(t - kT) , \qquad \text{with } T = 1/\nu , \quad (4)$$

which is a series of regularly spaced δ-functions (also known as Dirac comb).

As for case b) one might feel tempted to assume complete independence between the ρ- and ν-pulses. In this situation we would expect

$$D'_{\rho\nu}(t) = \sum_{k=1}^{\infty} \frac{1}{T} \cdot U(t - [k-1]T) \cdot U(kT - t) = \nu \cdot U(t) .$$

However, the assumed independence holds only for the <u>original</u> ρ- and ν-pulses. The fact that a certain ρ-pulse has been registered (at $t = 0$) allows us to draw some conclusion about the possible position of the preceding ν-event. Thus, for instance, this pulse cannot have arrived within the time interval from $-\tau$ to 0 (unless it was eliminated by another counted ρ-pulse), since such a pulse would have suppressed the ρ-event at the time origin. The real interval distribution for case b) is therefore more complicated. This problem has been treated recently in [7] where the density considered, called D(t), was evaluated by a recursive numerical method. The corresponding loss due to the dead time of the ρ-pulse is given by

$$\ell_{\rho\nu} = \int_0^{\tau} D(t)\, dt = \mu \cdot \nu\tau . \quad (5)$$

μ is a correction factor which depends e.g. on the two ratios $\rho' = \rho/\nu$ and $\tau' = \tau/T$. As a result of the correlation described above, we have always $\mu \gg 1$. The correction factor $\mu(\rho', \tau')$ has been extensively tabulated in [7].

The losses due to the dead time of a registered pulse can therefore be described as indicated in Table 1.

Table 1 - Dead-time losses produced by a registered pulse for the two types of events.

$[[\nu\tau]]$ is the largest integer below $\nu\tau$ and therefore vanishes for $\tau \leq T$.

Type of registered pulse	case	Losses within τ - for type ρ	case	- for type ν
ρ	a)	$\rho\tau$	c)	$\mu \cdot \nu\tau$
ν	b)	$\rho\tau$	d)	$[[\nu\tau]]$

3. Sketch of the new evaluation

It is practical to introduce the notion of transmission factors in what follows. For ρ-pulses, for instance, this quantity is defined by the ratio

$$T_\rho = \frac{\text{observed count rate of } \rho\text{-pulses}}{\text{original count rate of } \rho\text{-pulses}} , \qquad (6)$$

and likewise for the ν-pulses or for the superposition.

Apart from section 5, the dead times involved are always assumed to be of the non-extended type. For the sake of simplicity, however, the index n or e (on the left) will always be omitted for the transmission factors when no confusion is likely to happen.

The derivation of the transmission factors is first done for an arbitrary value of τ. After having reached some general conclusions, the formulae will be specified for the case where τ does not exceed $1/\nu$.

a) The general case

Taking into account Table 1, the balance equation for the count rates becomes

- for ρ-pulses:

$$\rho T_\rho = \rho - \left[\rho T_\rho \cdot \rho\tau + \nu T_\nu \cdot \rho\tau\right]$$

$$= \rho - \rho\tau(\rho T_\rho + \nu T_\nu) ,$$

where the expression in brackets represents the losses due to the dead time of registered ρ- and ν-pulses. Hence

$$T_\rho = 1 - \tau(\rho T_\rho + \nu T_\nu) . \qquad (7)$$

Similarly, it becomes

- for ν-pulses:

$$\nu T_\nu = \nu - \left[\rho T_\rho \cdot \mu \nu \tau + \nu T_\nu \cdot K \right] ,$$

where $K \equiv [\,[\nu \tau]\,]$.

Hence

$$T_\nu = \frac{1 - \mu \rho \tau T_\rho}{1 + K} . \tag{8}$$

When (8) is applied to (7) we obtain, after a simple rearrangement,

$$T_\rho = \frac{1 - \frac{\nu \tau}{1 + K}}{1 + (1 - \frac{\mu \nu \tau}{1 + K}) \rho \tau} . \tag{9}$$

Together with (8) this gives

$$T_\nu = \frac{1}{1 + K} \cdot \frac{1 - (\mu - 1) \rho \tau}{1 + (1 - \frac{\mu \nu \tau}{K + 1}) \rho \tau} . \tag{10}$$

By using the abbreviations

$$\rho \tau \equiv x , \qquad \nu \tau \equiv z ,$$
$$1 + K \equiv K' \qquad \text{and} \qquad \frac{\mu z}{K'} \equiv z' , \tag{11}$$

the partial transmission factors can also be written as

$$T_\rho = \frac{1}{K'} \frac{K' - z}{1 + (1 - z') x} , \tag{9'}$$

$$T_\nu = \frac{1}{K'} \frac{1 - (\mu - 1) x}{1 + (1 - z') x} . \tag{10'}$$

For the total process, superimposed by ρ- and ν-pulses, the observed count rate r_ν is now readily obtained by means of (9') and (10') as

$$r_\nu = \rho T_\rho + \nu T_\nu = \frac{K' \rho + (1 - \mu x) \nu}{K' \left[1 + (1 - z') x \right]} , \tag{12}$$

since $\rho z = \nu x$.

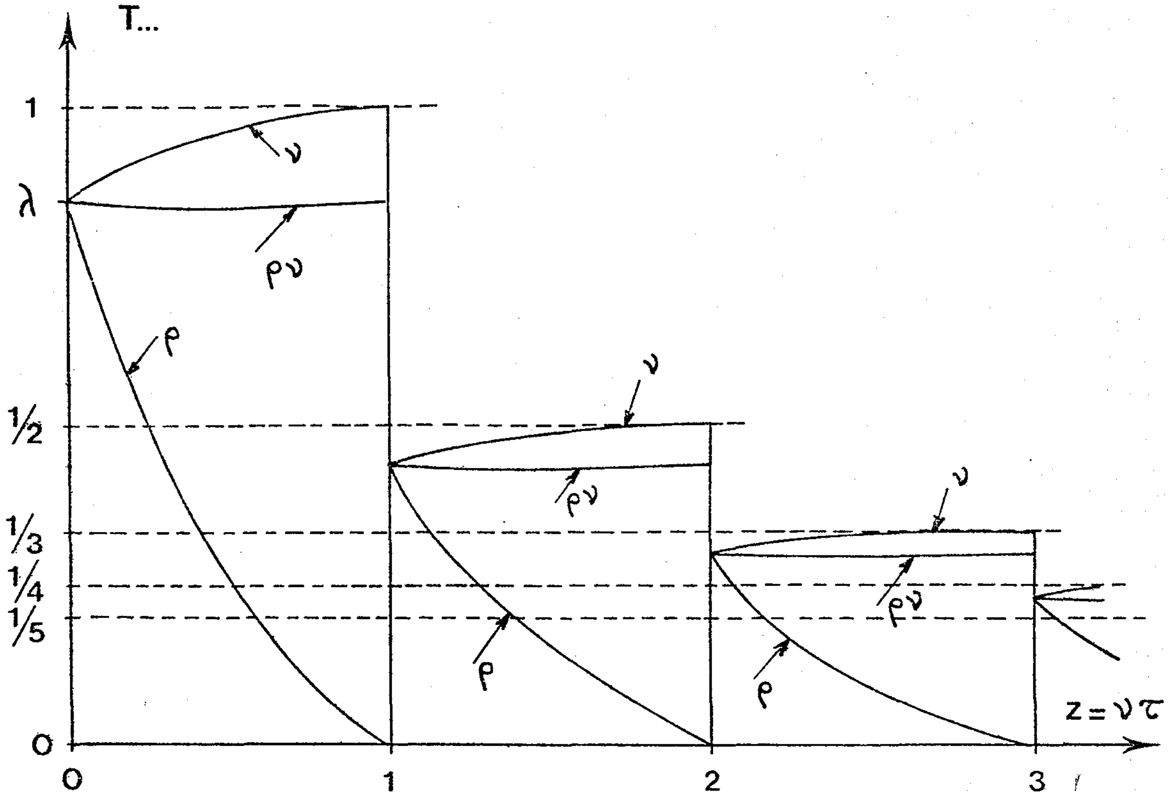

Figure 1.- Schematic behaviour of the transmission factors T_ρ, T_ν and $T_{\rho\nu}$ (for a given non-extended dead time τ) as a function of the oscillator frequency ν. The drawing assumes $x \simeq 0.25$.

The overall transmission factor $T_{\rho\nu}$ is therefore

$$T_{\rho\nu} = \frac{r_\nu}{\rho + \nu} = \frac{K'\rho + (1 - \mu x)\nu}{K'(\rho + \nu)\left[1 + (1 - z')x\right]} \quad . \tag{13}$$

Let us briefly discuss the discontinuities of the transmission factors at the points where $\nu\tau$ is equal to a positive integer k. Since there $\mu = 1$, we find

- for $\nu\tau = k-$ (i.e. limit "from the left"), since then $K' = k$ and $z' = 1$:

$$T_\rho = 0, \qquad T_\nu = 1/k \quad \text{and} \qquad T_{\rho\nu} = 1/(k+x) \; ; \tag{14}$$

- for $\nu\tau = k+$ (i.e. limit "from the right"),
since now $K' = k+1$ and $z' = k/(k+1)$:

$$T_\rho = T_\nu = T_{\rho\nu} = 1/(k+1+x) \,, \tag{15}$$

as can be readily verified. Here $k = 0$ is also allowed; the transmission is then given by $(1+x)^{-1} = \lambda$.

We may conclude from (14) and (15) that the transmission factors vary as a function of the pulser frequency ν roughly as sketched in Fig. 1. Whereas both T_ρ and T_ν change markedly with ν (and in opposite directions), the overall transmission $T_{\rho\nu}$ remains approximately constant for a given value K', at least for $\rho \ll \nu$.

The values of the transmission factors on the left-hand side of the discontinuities in Fig. 1 can be easily interpreted. Thus, with $\tau = 1/\nu$, for instance, the dead time following a registered ν-event suppresses all ρ-pulses, but as it stops immediately before the next oscillator pulse arrives, this event will be counted too. Hence, this situation leads at the same time to a complete extinction of the source pulses ($T_\rho = 0$) and to an assured survival of all pulser events ($T_\nu = 1$). Similar explanations are valid for $\nu\tau = 2, 3, \ldots$.

b) The region of practical interest

In what follows we shall restrict ourselves mainly to the domain $0 < \nu\tau < 1$, i.e. $K' = 1$. Here the general formulae given above for the transmission factors may be simplified to

$$T_\rho = \frac{1 - z}{1 + (1 - \mu z)\, x} \,, \tag{16}$$

$$T_\nu = \frac{1 - (\mu - 1)\, x}{1 + (1 - \mu z)\, x} \quad \text{and} \tag{17}$$

$$T_{\rho\nu} = \frac{1 - \dfrac{\mu x z}{x + z}}{1 + (1 - \mu z)\, x} \,, \tag{18}$$

where we may recall that $x = \rho\tau$ and $z = \nu\tau$.

The formulae (16) and (17) are somewhat different from the transmissions used previously ([3 - 5]). In particular, it can be seen that they are not identical for both types of pulses. As the expressions (16) to (18) form the basis for all the subsequent calculations, they are also responsible for any difference between the new formulae and the earlier relationships for r_ν and τ .

We note in passing that (17) has been used previously to calculate the correction factor μ ([7], eq. 9).

c) Determination of the dead time

The next task is to derive an expression which allows the determination of the dead time involved, τ, which is still assumed to be non-extended. Starting from (12) and using only directly measurable count rates, i.e. replacing ρ by $r/(1 - r\tau)$, we find

$$r_\nu = \frac{K'r + (1 - r\tau)\nu - \mu\nu \cdot r\tau}{K'\left[1 - r\tau + (1 - z')\, r\tau\right]} = \frac{K'r + \nu - (1 + \mu)\, rz}{K' - \mu z \cdot r\tau} . \qquad (19)$$

When written as a power series in τ, we obtain (remembering that $z = \nu\tau$)

$$\tau^2 \cdot r_\nu\, r \cdot \mu\nu - \tau \cdot r\nu\,(1 + \mu) - r_\nu K' + rK' + \nu = 0 .$$

The solution of this quadratic equation is

$$\tau = \frac{1 + \mu}{2\mu r_\nu}\left[1 - \sqrt{1 + \frac{4\mu r_\nu}{(1 + \mu)^2} \cdot \frac{K'(r_\nu - r) - \nu}{r\nu}}\right] . \qquad (20)$$

This gives for $0 < \tau < 1/\nu$, the practical range that is of most interest (and at present the only one where tabulated values for the correction factors μ are available), the formula

$$\tau = \frac{1 + \mu}{2\mu r_\nu}\left[1 - \sqrt{1 - \frac{4\mu}{(1 + \mu)^2} \cdot \frac{r_\nu}{r} \cdot \frac{r + \nu - r_\nu}{\nu}}\right] , \qquad (21)$$

which is the main result of the present study and replaces the approximate expression (1).

In practice, (21) has to be used by iteration, using first some approximate value for τ when looking up the appropriate value of μ. For the moment, a similar problem may also arise for ρ, but this will disappear once the new tabulation of μ with $r' = r/\nu$ (instead of ρ') is available.

Alternative forms of (21) are known, as e.g.

$$\tau = \frac{1 + \mu}{2\mu r_\nu}\left[1 - \frac{2}{1 + \mu}\sqrt{\left(\frac{\mu - 1}{2}\right)^2 + \mu\,\frac{(r_\nu - r)(r_\nu - \nu)}{r \cdot \nu}}\right] . \qquad (21a)$$

The reader will easily verify the identity and perhaps establish other useful variants.

It should be noted that (21) does not reduce to (1) if we put $\mu = 1$. In fact, the corresponding formula would be wrong, as has been noted before [8]. The reason is that some of the changes produced by the correction factor μ had been taken into account explicitly in the derivation of (1). This can be seen by a detailed comparison with [3] and [5] .

4. Discussion

An obvious question is to ask how the numerical values of the dead time compare when determined according to (1) or (21). Such a comparison can be readily made in a simple way. From (12) we get for K' = 1

$$r_\nu = \frac{\rho + \nu - \mu \cdot \nu x}{1 + (1 - \mu \cdot \nu\tau)x} .$$

Since $r = \frac{\rho}{1 + x}$, the insertion of these expressions into (1) leads to

$$\tau_o = \frac{1 + x}{\rho}\left[1 - \sqrt{\frac{\rho}{\nu} + 1 - \mu x - \frac{\rho/\nu}{1+x}}\right]$$

and hence for the ratio to

$$\frac{\tau_o}{\tau} = \frac{1+x}{x}\left[1 - \sqrt{\frac{1 + \rho' - \mu x}{1 + (1 - \mu\tau')x} - \frac{\rho'}{1 + x}}\right] , \quad (22)$$

with the abbreviations

$$\rho' = \rho/\nu \qquad \text{and} \qquad \tau' = z = \nu\tau .$$

This ratio is represented graphically in Fig. 2 for some values of the parameter ρ'. For the range $\tau' \lesssim 0.35$ and for $\rho' \leq 1$ (see Fig. 2a), the ratio τ_o/τ seldom deviates by more than 10^{-3} from unity, and usually much less, demonstrating that the simple formula (1) is fully adequate in this region. Also for values of ρ' as high as 5 the differences are within 1%. The situation is quite different for higher values of τ', in particular for $\tau' \gtrsim 0.5$, as can be seen from Fig. 2b. The approximation (1) then breaks progressively down, even for $\rho' \longrightarrow 0$. It should be mentioned, however, that cautious users had always refrained from applying the formula in this domain.

In his interesting second paper on the periodic pulse method, Baerg ([5], eq. 5) suggests a method to compare the "observed" and theoretical transmission by forming the quantity

$$R_B = \frac{r_\nu - r}{\nu(1 - r\tau)^2} \quad (23)$$

which should have unit value at the oscillator frequencies used to determine τ . His measurements, reported graphically, show that this expectation is very well supported as long as $\nu\tau < 1/3$. For higher frequencies - at least for the parameters used - Baerg's ratio R_B may deviate from unity. In particular, for $\nu\tau > 1/2$ it seems to increase very rapidly. The curious behaviour of R_B has never been fully understood, nor its exact relation to the corresponding dead-time measurement. It seems that the new approach can shed some light on this problem too.

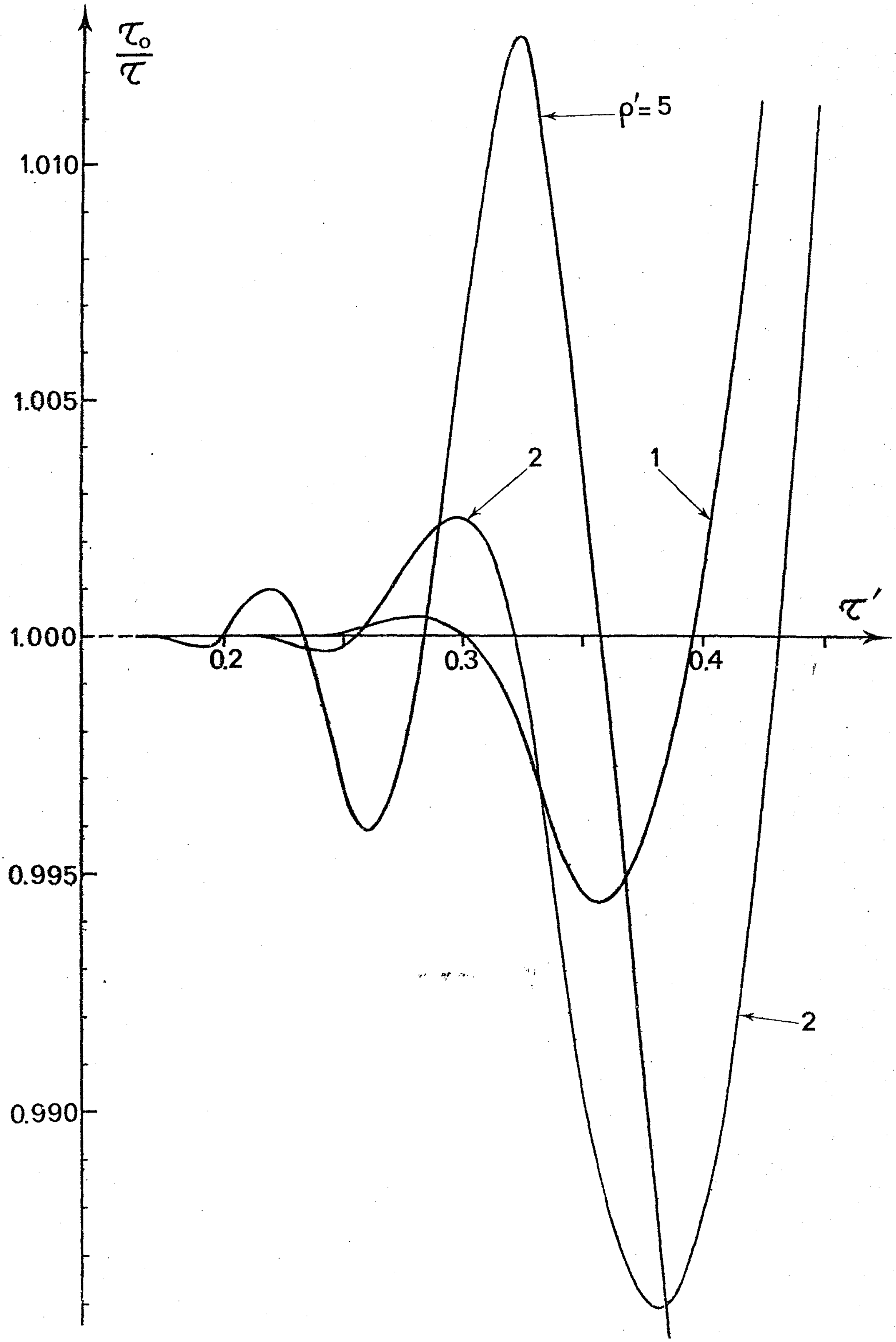

Figure 2a - Graphical représentation of the ratio τ_o/τ, for $\tau' < 0.4$ and some values of ρ'. This plot permits to determine the accuracy of the approximate formula (1).

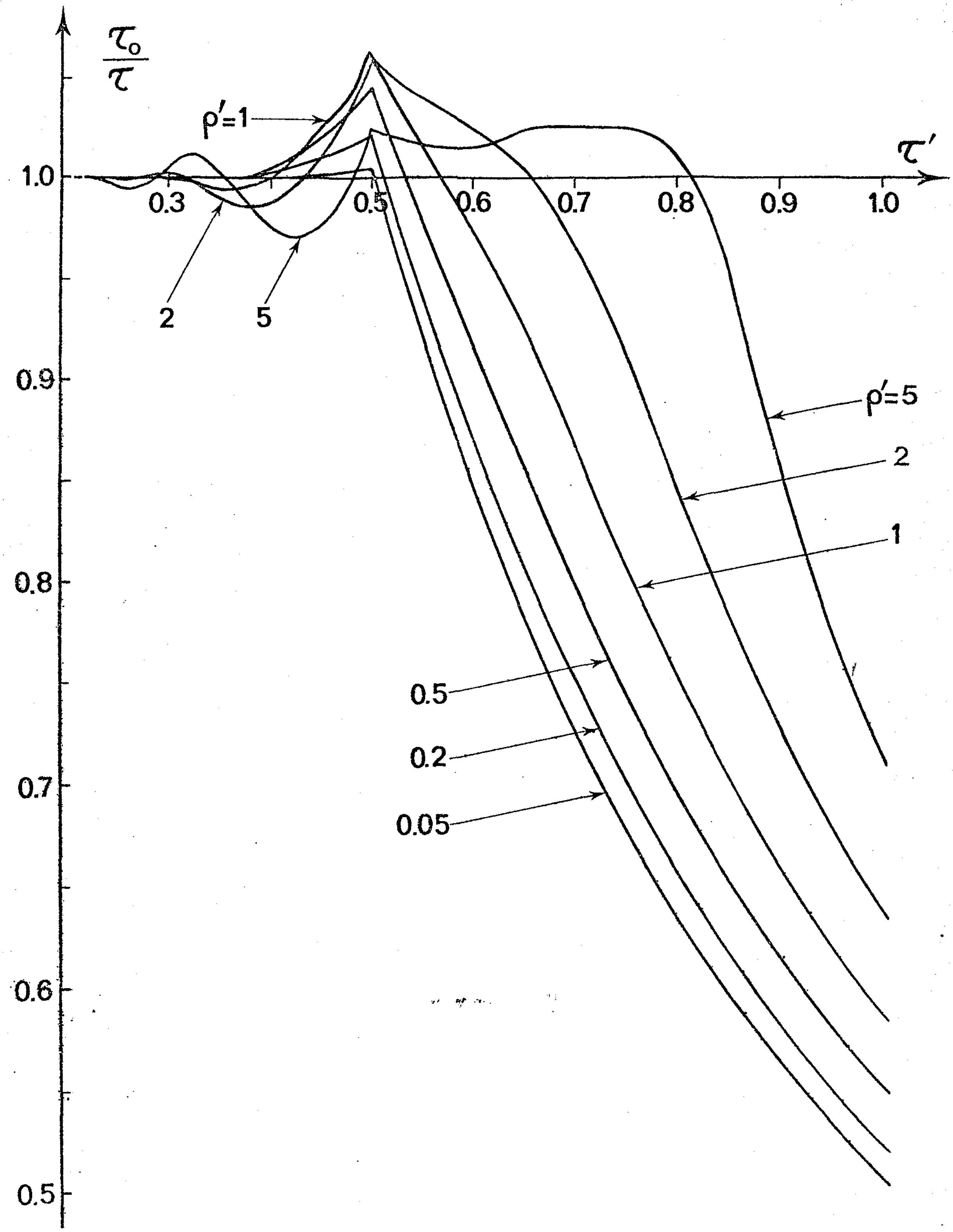

Figure 2b - As in Fig. 2a, but for $\tau' > 0.3$. Note the different scale used for the ordinate.

Inserting again the expressions given previously for r_ν and r into (23), we arrive at

$$R_B = \frac{\dfrac{\rho + \nu - \mu \cdot \nu x}{1 + (1 - \mu \cdot \nu \tau)x} - \dfrac{\rho}{1 + x}}{\nu \left(1 - \dfrac{x}{1 + x}\right)^2}$$

$$= \frac{(\rho' + 1 - \mu x)(1 + x)^2}{1 + (1 - \mu \tau')x} - \rho'(1 + x) , \qquad (24)$$

with $x = \rho\tau = \rho'\tau'$. Figure 3 shows the theoretical behaviour of R_B for values of the parameters corresponding to Baerg's experimental conditions where $\rho \simeq 1\,000\ s^{-1}$ and $\tau \simeq 250\ \mu s$, hence $x \simeq 0.25$. We note that for a graph like Fig. 3 with x constant, τ' is the only independent variable since $\rho' = x/\tau'$. It is obvious that the experimental findings are very well reproduced, giving thereby additional support to the theory as outlined above. For the critical minimum at $\tau' = 0.5$ we obtain with (24) the value $R_B = 0.978$ while from the plot given in [5] one can read $R_B = 0.977 \pm 0.002$*. For $\tau' = 1$, R_B reaches the value $1 + x$.

5. The situation with an extended dead time

As a matter of fact, the case of an extended dead time τ essentially was solved several years ago and we have little to add. After determining the interval density for the superposition - for details we refer to [9] - a simple integration gives for the overall losses [10]

$$1 - {}_eT_{\rho\nu} = 1 - e^{-x}\left(1 - \frac{\rho\nu}{\rho + \nu}\tau\right). \qquad (25)$$

This implies for the experimental count rate of the superimposed process

$${}_er_\nu = (\rho + \nu)\ {}_eT_{\rho\nu} = (\rho + \nu - \rho\nu\tau)\, e^{-x} . \qquad (26)$$

It has been pointed out by Baerg [5] and by Taylor [11] that this may also be written in the form

$${}_er_\nu = \rho \cdot (1 - \nu\tau)\, e^{-x} + \nu \cdot e^{-x} , \qquad (26a)$$

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* Note added in proof: Dr. A.P. Baerg informs me that the overshoot around $1/\tau' \simeq 2.8$ has well been seen in his measurements and that there is also very good quantitative agreement, the maximum reaching indeed a value of about $R_B = 1.001\,5$.

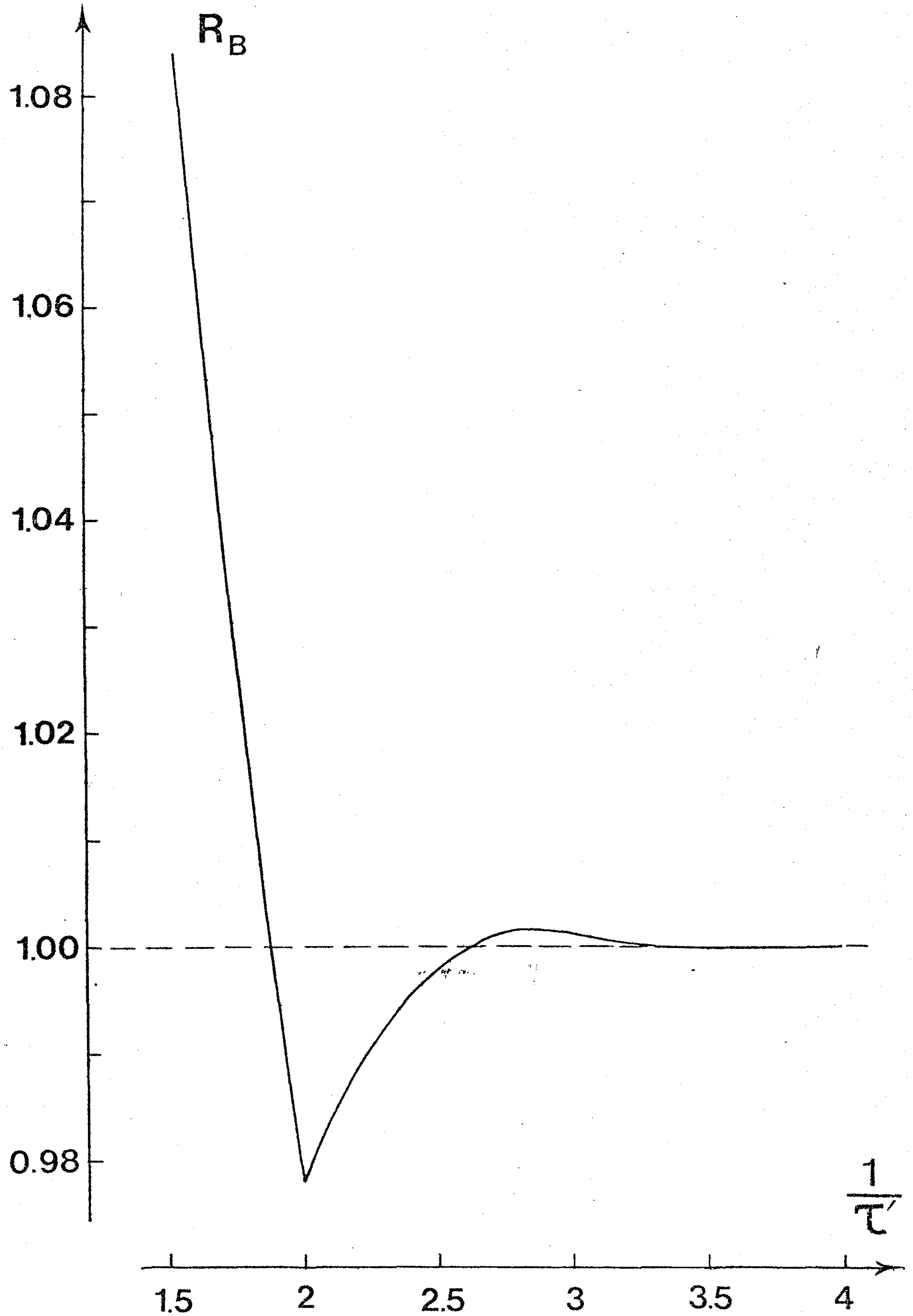

Figure 3 - Baerg's transmission ratio R_B, calculated according to eq. (24) for $x = 0.25$. There is excellent agreement with the measurements reported in [5] .

from which one can readily infer the partial transmissions as

$$T_\rho = (1 - \nu\tau)\, e^{-x} \quad \text{and} \qquad\qquad (27)$$
$$T_\nu = e^{-x} ,$$

valid for an extended dead time in the source-pulser process. Obviously (27) holds only for $\nu < 1/\tau$, whereas $T_\rho = T_\nu = 0$ for $\nu \geq 1/\tau$.

The partial transmission factors can also be easily derived directly, as has been done by previous workers. As a result of the periodicity imposed by the ν-pulses (which are renewal points [7]), any interval of length $1/\nu$ which starts with a ν-event may be taken as characteristic for the whole process. Thereby it is immaterial whether the ν-pulse has been registered or not as it is invariably followed by a dead time. Whether the next ν-event is counted or not (for $\tau > 1/\nu$ there is complete paralysis) depends only on the ρ-pulses: its survival requires ν to be preceded by a gap in the original ρ-series of minimal length τ , hence

$$T_\nu = \text{Prob}\,(\text{no } \rho \text{ in } \tau) = e^{-x}.$$

The condition for a ρ-pulse to be counted is twofold: ρ should neither fall into the dead time of the ν-pulse (at the beginning of the interval), nor should it be too close to a preceding ρ-pulse; therefore

$$T_\rho = (1 - \frac{\tau}{T}) \cdot e^{-x} = (1 - z) \cdot e^{-x} .$$

We note that (25) can also be written in the form

$$T_{\rho\nu} = (1 - \frac{x}{1 + \rho'})\, e^{-x} . \qquad (25')$$

A schematic plot is given in Fig. 4 and we point out in particular the surprising fact that T_ν is independent of the frequency of the oscillator pulses. A comparison with Fig. 1 is instructive.

A possible way to determine τ is offered by the relation [5]

$$\frac{r_\nu - r}{\nu} = (1 - x) \cdot e^{-x} , \qquad (28)$$

from which the root x - and hence also τ , since ρ is sufficiently well known - can be deduced numerically. An approximate analytical solution has been suggested recently by Taylor [11] .

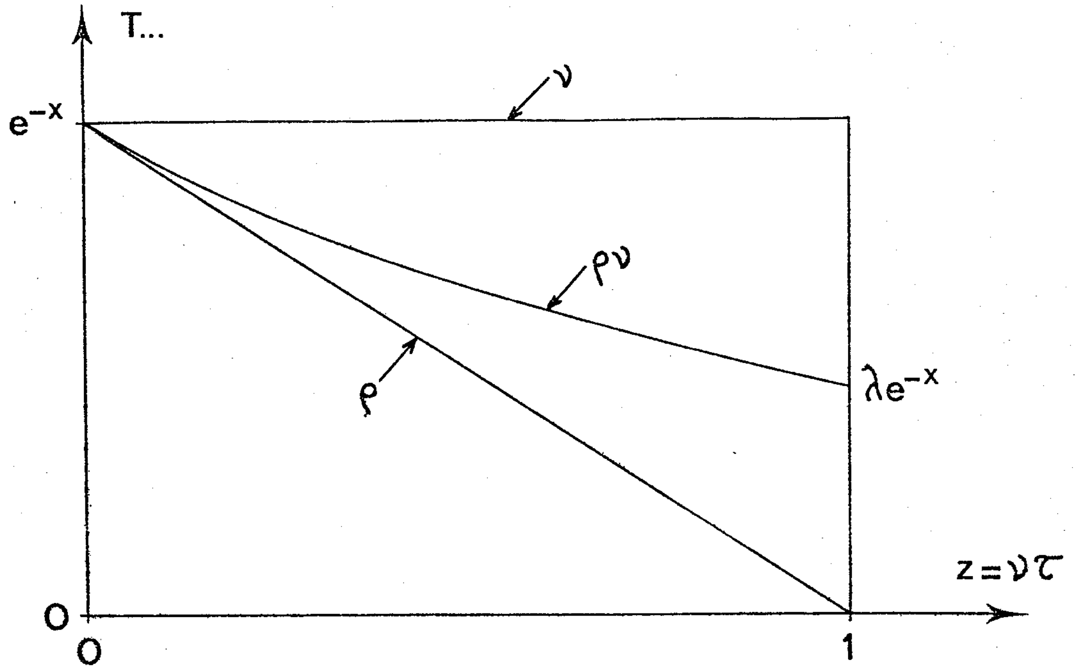

Figure 4 - Schematic representation of the transmission factors T_ρ, T_ν and $T_{\rho\nu}$ for an extended dead time τ as a function of the oscillator frequency ν . (Drawing similar to Fig. 1, but for $x \simeq 1$).

Among the items not treated here there remains the problem of the interval between successive registered events. This distribution might be of interest because it is easy to measure with a time-amplitude converter. This interval density is likely to be very informative for the type of dead time involved. Whereas the case of an extended τ could probably be solved analytically, the non-extended type does not seem amenable to an explicit solution.

Some further consequences as well as a proposal for a simplified experimental arrangement will be described in another report.

It is a pleasure to acknowledge my indebtedness to some Canadian friends and colleagues. Abe Baerg (NRC, Ottawa) as well as Janet Merritt and John Taylor (both at AECL, Chalk River) provided me with most useful comments on a draft version and their continued interest has been an invaluable support and encouragement.

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