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Sums of alternate powers - an empirical approach

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Abstract

By looking more closely at the numerical properties of alternate powers of integers, an explicit expression for the general formula for their sum can be obtained. This has a structure which is strongly reminiscent of the well-known Bernoulli formula, valid for positive terms, but without being just a simple variant of it. In particular, there appears a series of (integer) numbers which play a role similar to those of Bernoulli.

1. Introduction

Recently, in studying the parity method, we met the problem of determining sums of the form

$$ {}_rZ = \sum_{n=1}^{\infty} p_n \sum_{j=1}^{n} (-1)^j j^r , \qquad r = 1, 2, ..., \tag{1} $$

where p_n is a Poisson probability. Let us concentrate in what follows on the evaluation of

$$ {}_rZ_n \equiv \sum_{j=1}^{n} (-1)^j j^r . \tag{2} $$

Apart from the alternate sign, this looks like the old problem of summing over powers of natural numbers, for which Jakob Bernoulli has provided his well-known formula [1].

One may be tempted to try to rearrange (2) in such a way that his solution can somehow be applied, but this idea seems to be rather difficult to use. Instead, we adopt a purely empirical approach, hoping that the appearance of some regular pattern will guide our further steps.

2. Some elementary considerations

Let us first consider the simplest cases with r = 1, 2 and 3, for which some numerical results are assembled in Table 1.

Table 1. Numerical values for ${}_1Z_n$, ${}_2Z_n$ and ${}_3Z_n$, for $n \leq 10$.

n	$(-1)^n\, n$	${}_1Z_n$	$(-1)^n\, n^2$	${}_2Z_n$	$(-1)^n\, n^3$	${}_3Z_n$
1	-1	-1	-1	-1	-1	-1
2	2	1	4	3	8	7
3	-3	-2	-9	-6	-27	-20
4	4	2	16	10	64	40
5	-5	-3	-25	-15	-125	-81
6	6	3	36	21	216	135
7	-7	-4	-49	-28	-343	-208
8	8	4	64	36	512	304
9	-9	-5	-81	-45	-729	-425
10	10	5	100	55	1 000	575

Thus, for instance, the alternate quadratic sum (for r = 2) becomes

$${}_2Z = [-1\, p_1 - 6\, p_3 - 15\, p_5 - 28\, p_7 - \ldots] + [3\, p_2 + 10\, p_4 + 21\, p_6 + 36\, p_8 + \ldots]\,, \tag{3}$$

where we have separated the negative from the positive contributions, which result from the odd and the even values of n. This reminds us that the problem has its origin in the parity method.

We rewrite (1) in the form

$${}_rZ = \sum_{t=1}^{\infty} {}_rA_t\, p_{2t-1} + \sum_{j=1}^{\infty} {}_rB_t\, p_{2t}\,. \tag{4}$$

Obviously the problem now is the determination of the coefficients

$${}_rA_t \equiv {}_rZ_{2t-1} \quad \text{and} \quad {}_rB_t \equiv {}_rZ_{2t}\,, \quad \text{with } t = 1, 2, \ldots\,.$$

Let us proceed empirically. For the case <u>r = 2</u> we make the ansatz

$${}_2A_t = {}_2\mu_0 + {}_2\mu_1\, t + {}_2\mu_2\, t^2\,,$$

and likewise

$${}_2B_t = {}_2\nu_0 + {}_2\nu_1\, t + {}_2\nu_2\, t^2\,. \tag{5}$$

From Table 1 it follows that, for t = 1 to 3,

$${}_2A_1 = {}_2\mu_0 + {}_2\mu_1 + {}_2\mu_2 = -1\,,$$
$${}_2A_2 = {}_2\mu_0 + 2\,{}_2\mu_1 + 4\,{}_2\mu_2 = -6\,,$$
$${}_2A_3 = {}_2\mu_0 + 3\,{}_2\mu_1 + 9\,{}_2\mu_2 = -15\,.$$

One can readily find that ${}_2\mu_0 = 0$, ${}_2\mu_1 = 1$ and ${}_2\mu_2 = -2$. Hence, the required expression is

$$ {}_2A_t = t - 2\,t^2 . \tag{6a}$$

Similarly we find

$$ {}_2B_t = t + 2\,t^2 . \tag{6b}$$

The case r = 3 leads, with Table 1, to the relation

$$ {}_3Z = \sum_{t=1}^{\infty} {}_3A_t\, p_{2t-1} + \sum_{j=1}^{\infty} {}_3B_t\, p_{2t} $$

$$ = \left[-1\,p_1 - 20\,p_3 - 81\,p_5 - 208\,p_7 - \dots \right] + \left[7\,p_2 + 40\,p_4 + 135\,p_6 + 304\,p_8 + \dots \right] . $$

Again we put

$$ {}_3A_t = {}_3\mu_0 + {}_3\mu_1\, t + {}_3\mu_2\, t^2 + {}_3\mu_3\, t^3 . $$

Solving, with t = 1 to 4 and the known values ${}_3A_t$ and ${}_3B_t$, we find

$$ \begin{aligned} {}_3A_t &= 3\,t^2 - 4\,t^3 , \\ {}_3B_t &= 3\,t^2 + 4\,t^3 . \end{aligned} \tag{7}$$

By proceeding likewise for r = 4 and r = 5 we arrive at

$$ \begin{aligned} {}_4A_t &= -t + 8\,t^3 - 8\,t^4 , \\ {}_4B_t &= -t + 8\,t^3 + 8\,t^4 \end{aligned} \tag{8}$$

and

$$ \begin{aligned} {}_5A_t &= -5\,t^2 + 20\,t^4 - 16\,t^5 , \\ {}_5B_t &= -5\,t^2 + 20\,t^4 + 16\,t^5 . \end{aligned} \tag{9}$$

It may be useful to assemble the coefficients obtained till now in tabular form, in the hope that this will give us some more insight.

Table 2. List of the coefficients ${}_r\mu_j$ and ${}_r\nu_j$, for $r \leq 5$. In the case of double signs, minus refers to ${}_r\mu_r$ and plus to ${}_r\nu_r$.

	j = 0	1	2	3	4	5
r = 1	0	∓ 1				
2	0	1	∓ 2			
3	0	0	3	∓ 4		
4	0	- 1	0	8	∓ 8	
5	0	0	- 5	0	20	∓ 16

3. Preliminary conclusions

Inspection of Table 2 leads to the following tentative observations:

a) with reasonable confidence:

$$
\begin{aligned}
{}_r\mu_0 &= {}_r\nu_0 = 0 , \\
{}_r\mu_r &= - {}_r\nu_r = - 2^{r-1} ;
\end{aligned}
\tag{10}
$$

b) with some hesitation:

$$
\begin{aligned}
\sum_{j=1}^{r} {}_r\mu_j &= - 1 , \\
\sum_{j=1}^{r} {}_r\nu_j &= 2^r - 1 .
\end{aligned}
\tag{11}
$$

In addition, Table 2 seems to be arranged in "falling diagonals", some of which have only positive or negative signs (if j = 0 is left out), while others are zero. In order to clarify these issues, further numerical values are needed.

It is rather easy, by continuing with the guessing method described above (followed by rigorous checks), to continue to r = 9. With a computer programme, kindly set up by F. Lesueur, it has subsequently been possible to verify all these values and extend the tabulation to r = 12. The coefficients are listed in Table 3 and they allow us to confirm all the observations made above.

Table 3. Table of the coefficients ${}_r\mu_j$, for $r \leq 12$.

	j = 1	2	3	4	5	6	7	8	9	10	11	12
r = 1	-1											
2	1	-2										
3	0	3	-4									
4	-1	0	8	-8								
5	0	-5	0	20	-16							
6	3	0	-20	0	48	-32						
7	0	21	0	-70	0	112	-64					
8	-17	0	112	0	-224	0	256	-128				
9	0	-153	0	504	0	-672	0	576	-256			
10	155	0	-1 020	0	2 016	0	-1 920	0	1 280	-512		
11	0	1 705	0	-5 610	0	7 392	0	-5 280	0	2 816	-1 024	
12	-2 073	0	13 640	0	-26 928	0	25 344	0	-14 080	0	6 144	-2 048

4. Towards a general solution

Let us look at the values in the various "diagonals". For the diagonal with the values 1, 3, 8, 20, 48, ..., we readily find the general relation

$$ {}_r\mu_{r-1} = 2^{r-3}\, r \,, \quad \text{for } r \geq 2 \,. \tag{12a} $$

For the diagonal with the values -1, -5, -20, -70, ..., we obtain

$$ {}_r\mu_{r-3} = -\, 2^{r-4} \frac{r\,!}{(r-3)\,!\,4\,!} \,, \quad \text{for } r \geq 4 \,. \tag{12b} $$

Likewise, for the other diagonals we have

$$ {}_r\mu_{r-5} = 2^{r-6} \frac{r\,!}{(r-5)\,!\,6\,!}\; 3 \,, \quad \text{for } r \geq 6 \,, \tag{12c} $$

and

$$ {}_r\mu_{r-7} = 2^{r-8} \frac{r\,!}{(r-7)\,!\,8\,!}\; (-17) \,, \quad \text{for } r \geq 8 \,, \tag{12d} $$

respectively. These results also hold for the corresponding coefficients ${}_r\nu_j$.

Particularly relevant is the observation that all coefficients ${}_r\mu_j$ (and ${}_r\nu_j$) admit the simple recursion formula

$$ {}_r\mu_j = {}_{r-1}\mu_{j-1} \left(\frac{2r}{j} \right) , \quad \text{for } 2 \leq j \leq r \,. \tag{13} $$

From this it follows that the coefficients listed in Table 3 can be reconstructed from a few basic elements, namely the numbers in column j = 1. Putting

$$ {}_r\mu_1 = {}_r\nu_1 \equiv M_r \,, \tag{14} $$

we can write, by repeated application of (13),

- for r - j odd:

$$ {}_r\mu_j = \frac{2^{j-1}\, r\,!}{j\,!\,(r-j+1)\,!}\, M_{r-j+1} \,, \quad \text{with } r > j \,; \tag{15a} $$

- for r - j even:

$$ {}_r\mu_j = 0 \,, \quad \text{with } r \geq j\text{-}2 \,. \tag{15b} $$

For r = j, the coefficients are

$$ {}_r\mu_r = -\, {}_r\nu_r = -\, 2^{r-1} \,, \tag{16} $$

as already suggested in (10).

These relations reduce our problem to the determination of the "initial" numbers M_r.

5. Evaluation of the numbers M_r

For the evaluation of M_r we take advantage of the fact that, for t = 1, ${}_rA_1 = {}_rZ_1 = -1$, hence

$$ {}_rZ_1 = \sum_{j=1}^{r} {}_r\mu_j = -1 , \tag{17} $$

which confirms (11).
To show how M_r can be determined, let us , as an example, consider the case r = 8.

With (16) and (17) we find

$$ \begin{aligned} M_8 &= -1 - \sum_{j=2}^{8} {}_8\mu_j \\ &= -1 - ({}_8\mu_3 + {}_8\mu_5 + {}_8\mu_7 + {}_8\mu_8) \\ &= 2^7 - 1 - {}_8\mu_3 - {}_8\mu_5 - {}_8\mu_7 . \end{aligned} $$

Since, with (15),

$$ \begin{aligned} {}_8\mu_3 &= \frac{2^2\, 8\,!}{3\,!\,6\,!} M_6 , \\ {}_8\mu_5 &= \frac{2^4\, 8\,!}{5\,!\,4\,!} M_4 , \\ {}_8\mu_7 &= \frac{2^6\, 8\,!}{7\,!\,2\,!} M_2 , \end{aligned} $$

one obtains

$$ M_8 = 127 - \frac{112}{3} M_6 - 224\, M_4 - 256\, M_2 = -17 . $$

In the general case, we thus have (with r even)

$$ M_r = 2^{r-1} - 1 - \sum_{\substack{j=2 \\ (\text{even})}}^{r-2} \frac{2^j\, r\,!}{(r-j)\,!\,(j+1)\,!} M_{r-j} . $$

Since

$$ \frac{r\,!}{(r-j)\,!\,(j+1)\,!} = \frac{1}{j+1}\binom{r}{j} , $$

we arrive at the general recursion formula

$$ M_r = 2^{r-1} - 1 - \sum_{\substack{j=2 \\ (\text{even})}}^{r-2} \frac{2^j}{j+1}\binom{r}{j} M_{r-j} . \tag{18} $$

This allows us to determine the first numbers M_r. These are given in Table 4.

Table 4. Numerical values of the numbers M_r, for $r \leq 20$.

r	M_r	r	M_r
2	1	12	- 2 073
4	- 1	14	38 227
6	3	16	- 929 569
8	- 17	18	28 820 619
10	155	20	- 1 109 652 905

6. Final results

Let us return to the alternate sums ${}_rZ_n$ defined in (2). In order to arrive at a general formula, we first note that they can be written more explicitly in the form

- for n odd, putting $t = (n+1)/2$:

$$
{}_rZ_n = \sum_{j=1}^{r} {}_r\mu_j\, t^j = \begin{cases} {}_r\mu_2\, t^2 + {}_r\mu_4\, t^4 + \dots + {}_r\mu_r\, t^r, & \text{if r odd,} \\ {}_r\mu_1\, t + {}_r\mu_3\, t^3 + \dots + {}_r\mu_r\, t^r, & \text{if r even;} \end{cases}
$$

- for n even, putting $t = n/2$:

$$
{}_rZ_n = \sum_{j=1}^{r} {}_r\nu_j\, t^j = \begin{cases} {}_r\nu_2\, t^2 + {}_r\nu_4\, t^4 + \dots + {}_r\nu_r\, t^r, & \text{if r odd,} \\ {}_r\nu_1\, t + {}_r\nu_3\, t^3 + \dots + {}_r\nu_r\, t^r, & \text{if r even.} \end{cases}
$$

Since ${}_r\nu_j = {}_r\mu_j$, for $j < r$ and ${}_r\nu_r = -{}_r\mu_r$, we do not have to work any longer with the coefficients ${}_r\nu_j$ and we can write

$$
{}_rZ_n = \sum_{j=1}^{r-1} \sigma_{r,j}\; {}_r\mu_j\, t^j + (-1)^n\, 2^{r-1}\, t^r , \tag{20}
$$

where the "survival factor" $\sigma_{r,j}$, defined by

$$
\sigma_{r,j} = \frac{1}{2}\,[1 - (-1)^{1+j}] , \tag{21}
$$

is 1 or 0. Note that $t \equiv \left[\frac{n+1}{2}\right]$, which denotes the largest integer not exceeding $\frac{n+1}{2}$.

For $r \leq 20$, the coefficients ${}_r\mu_j$ can be taken from Table 3.

Formula (20) is the main result of this study.

7. Examples

To illustrate this formula, we consider two numerical examples.

a) For r = 2 and n = 5, i.e. t = 3, we have with (20)

$$ {}_2Z_5 = \sigma_{2,1}\, {}_2\mu_1\, 3^1 + (-1)^5\, 2^1\, 3^2 = -15\,, $$

since $\sigma_{2,1} = 1$.

b) For r = 5 and n = 8, i.e. t = 4, we have

$$ {}_5Z_8 = \sum_{j=1}^{4} \sigma_{5,j}\, {}_5\mu_j\, 4^j + (-1)^8\, 2^4\, 4^5 = 21\,424\,, $$

since $\sigma_{5,1} = \sigma_{5,3} = 0$.

Both results can easily be verified.

For r beyond 20, the coefficients ${}_r\mu_j$ are not listed in Table 3, but they can be obtained by the application of (15), unless r exceeds 20. If this is the case, one first has to evaluate M_r by the help of (18), and then go back to (15). This is more cumbersome, but powers r beyond 20 are expected to occur rarely in practice.

8. Discussion and conclusion

In view of the similarity of

$$ {}_rZ_n \equiv \sum_{j=1}^{n} (-1)^j\, j^r \tag{2} $$

with the Bernoullian sum

$$ {}_rS_n \equiv \sum_{j=1}^{n} j^r\,, \tag{22} $$

it will be of interest to compare the respective explicit expressions.

For (22), as is well known, the result can be brought into the form

$$ {}_rS_n = \frac{1}{r+1}\, n^{r+1} + \frac{1}{2}\, n^r + \sum_{\substack{j=2 \\ \text{(even)}}}^{J} \frac{1}{j} \binom{r}{j-1} B_j\, n^{r+1-j}\,, \tag{23} $$

where $J \equiv 2[r/2]$ and B_j are the so-called Bernoulli numbers, with $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, etc.

A comparison of (23) with our general formula (20) for alternate sums reveals both a number of interesting similarities and some differences. While in (23) the result is expressed in powers of n, i.e. the last term of the original sum, the alternate sum (20) uses powers of t = [(n+1)/2], which is either n/2 or (n+1)/2, depending on whether n is even or odd. The summation over j in both cases concerns only every second value.

In (20) this is so because of the "survival factor" $\sigma_{r,j}$, in (23) since $B_j = 0$ for all odd j values exceeding 2. It is most interesting that the famous Bernoulli numbers B_j may have found a counterpart in the new numbers M_j. In contrast to B_j, they are integers, but they also alternate in sign and vanish for odd values $j \geq 3$. However, no numerical relation between the two series has been found so far.

A critical reader may rightly object to this report that most of its basic relations are guessed rather than proven. This is true, but in defense I might say that the same actually holds for Bernoulli's treatment. Yet, the weight of the numerical evidence is strong enough (in both cases), that there can be no real doubt as to the correctness of the results. Nevertheless, it is obviously to be hoped that mathematicians will soon transform our loose framework into a solid construction by providing it both with a safer foundation and rigorous connections.

This report is dedicated to Madame Mireille Boutillon (BIPM) for her cutting, but always pertinent, criticism of my occasional excursions in mathematical territory, a field in which she is much more at home than I am. Her remarks go to the core and never concern italics or similar futilities. By protecting me, during more than a quarter of a century, against many ridiculous slips, she has rendered me an invaluable service.

Reference

[1] J. Bernoulli: "Ars Conjectandi" (Basel, 1713), p. 97

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