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# Shifted developments of power functions

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**Abstract**

General expressions are derived for the coefficients which appear when a power function $k^r$ is developed into a sum consisting either of shifted powers of the form $(k-t)^j$, or of shifted factorials of the form $(k-t)_j$. The numerical values of the coefficients are listed for integers r and t not exceeding 5.

## 1. Introduction

It may happen that in a lengthy evaluation an algebraic rearrangement is needed which, although of an elementary nature, is nevertheless quite tedious, especially if it occurs repeatedly. To ensure the reliability of the final result of the evaluation - and not for the possible mathematical interest of the problems involved - one may be led to look at such questions in detail and to solve them once for ever.

In evaluating sums of the type

$$\sum_{k=t}^{\infty} \frac{\mu^k k^r}{(k-t)!} \tag{1}$$

one is confronted with the problem of expressing a power in terms of "shifted" factorials, thus to find the coefficients $\beta_j(r,t)$ in the development

$$k^r = \sum_{j=0}^{r} \beta_j(r,t)\,(k-t)_j \,, \tag{2}$$

where k, r and t are natural numbers and

$$(n)_j \equiv n\,(n-1)\,(n-2)\,...\,(n-j+1) \tag{3}$$

is a falling factorial, for n integer.

A knowledge of $\beta_j(r,t)$ would greatly simplify the evaluation of (1) which then becomes

$$\sum_{k=t}^{\infty} \frac{\mu^k}{(k-t)!} \sum_{j=0}^{r} \beta_j(r,t)\,(k-t)_j = \mu^t \sum_{k=0}^{\infty} \mu^k \sum_{j=0}^{r} \beta_j(r,t)\,\frac{(k)_j}{k\,!}$$

$$= \mu^t \sum_k \mu^k \sum_j \beta_j(r,t)\,\frac{1}{(k-j)\,!} = \mu^t \sum_{j=0}^{r} \beta_j(r,t)\,\mu^j \sum_{k=0}^{\infty} \frac{\mu^k}{k\,!}$$

$$= e^{\mu} \sum_{j=0}^{r} \beta_j(r,t)\,\mu^{t+j}\ . \tag{4}$$

The decomposition (2) may be compared with a similar problem. Replacement of the factorial by a power (both of order j) yields

$$k^r = \sum_{j=0}^{r} \alpha_j(r,t)\,(k-t)^j\ . \tag{5}$$

When there is no danger of ambiguity, the arguments may be dropped; so in what follows we simply write $\alpha_j$ and $\beta_j$.

The special case with no shift is worth considering. This is trivial for (5), where obviously

$$\alpha_j(r,0) = \delta_{0,r-j} \quad , \quad \text{for } t = 0\ . \tag{6}$$

For the development of (2) into factorials, the answer is also well known, with

$$\beta_j(r,0) = S(r,j) \quad , \quad \text{for } t = 0\ , \tag{7}$$

where $S(r,j)$ is a Stirling number of the second kind [1], with $S(r,0) = \delta_{r,0}$. For an extended table see [2].

This does not, however, solve (2) or (5) for the general case where $t > 0$. The aim of this report is therefore to arrive at general expressions for $\alpha_j(r,t)$ and for $\beta_j(r,t)$.

## 2. Shifted powers

Since (5) is easier to treat than (2), we begin with this decomposition. By applying a binomial expansion, (5) can be written as

$$k^r = \sum_{j=0}^{r} \alpha_j(r,t) \sum_{i=0}^{j} \binom{j}{i} k^i (-t)^{j-i} \ . \tag{8}$$

In order to obtain the coefficients $\alpha_j(r,t)$, we now assemble all the terms which belong to a given power $k^i$.

For $\underline{i = r}$, and thus $j = r$, we readily find from (8) that

$$k^r = \alpha_r(r,t) \binom{r}{r} k^r (-t)^0 \ ,$$

hence

$$\alpha_r(r,t) = 1 \ , \tag{9}$$

as expected.

For $\underline{i < r}$, and hence $j \geq i$, we obtain from (8) the condition

$$\sum_{j=i}^{r} \alpha_j \sum_{i=0}^{j} \binom{j}{i} k^i (-t)^{j-i} = 0 \ . \tag{10}$$

By putting $i = r-f$, with $0 < f \leq r$, this leads to

$$\sum_{j=r-f}^{r} \alpha_j \binom{j}{r-f} (-t)^{j-r+f} = 0 \ ,$$

since $k^{r-f} = 0$ ,

and we can write likewise

$$\alpha_{r-f} = - \sum_{j=r-f+1}^{r} \alpha_j \binom{j}{r-f} (-t)^{j-r+f} = - \sum_{j=1}^{f} \alpha_{r-f+j} \binom{r-f+j}{j} (-t)^j \ . \tag{11}$$

In the hope of finding a simpler form for (11) we insert for f some of its lowest possible values. This leads successively to the following expressions :

- for $\underline{f = 1}$: $\alpha_{r-1} = - \alpha_r \binom{r}{1} (-t)^1 = rt$ ,

- for $\underline{f = 2}$: $\alpha_{r-2} = - [\alpha_{r-1} \binom{r-1}{1} (-t) + \alpha_r \binom{r}{2} t^2] = \frac{r(r-1)}{2} t^2(2-1) = \binom{r}{2} t^2$ ,

- for $\underline{f = 3}$: $\alpha_{r-3} = \alpha_{r-2} \binom{r-2}{1} t - \alpha_{r-1} \binom{r-1}{2} t^2 + \alpha_r \binom{r}{3} t^3 = \binom{r}{3} t^3$ ,

- for $\underline{f = 4}$: $\alpha_{r-4} = \alpha_{r-3} \binom{r-3}{1} t - \alpha_{r-2} \binom{r-2}{2} t^2 + \alpha_{r-1} \binom{r-1}{3} t^3 - \alpha_r \binom{r}{4} t^4 = \binom{r}{4} t^4$ ,

etc.

This leads to the supposition that the general result is

$$\alpha_{r-f}(r,t) = \binom{r}{f} t^f , \quad \text{for } f = 0, 1, 2, \ldots, r . \tag{12}$$

In order to prove this, we substitute (12) in (10). If the assumption is correct, we should obtain an identity; otherwise not. This leads to

$$\sum_{j=i}^{r} \binom{r}{r-j} t^{r-j} \binom{j}{i} (-t)^{j-i} = (-1)^i t^{r-i} \sum_{j=i}^{r} (-1)^j \binom{r}{j} \binom{j}{i} \stackrel{?}{=} 0 \ . \tag{13}$$

In the inexhaustible collection of surprising formulae in Riordan's book [3], one can find, among the "inverse relations",

$$\delta_{n,k} = \sum_{j=k}^{n} (-1)^{j+k} \binom{n}{j} \binom{j}{k} \ . \tag{14a}$$

For n > k this gives

$$\sum_{j=k}^{n} (-1)^j \binom{n}{j} \binom{j}{k} = 0 \ , \tag{14b}$$

which is the identity looked for. Thereby (13) is confirmed and, in consequence, the general relation (12) is recognized as valid.

## 3. Shifted factorials

We now return to the original problem which is given by (2). Since the decomposition of $k^r$ into shifted powers is now known to be

$$k^r = \sum_{j=0}^{r} \binom{r}{j} t^{r-j} (k-t)^j \ , \tag{15}$$

a link with the form given in (2) will be established if we can express $(k-t)^j$ in terms of $(k-t)_j$. This is simple since we already know that

$$k^r = \sum_{j=0}^{r} S(r,j)\, (k)_j \ , \tag{16a}$$

hence also

$$(k-t)^r = \sum_{j=0}^{r} S(r,j)\,(k-t)_j \ . \tag{16b}$$

Therefore

$$k^r = \sum_{j=0}^{r} \beta_j\,(k-t)_j = \sum_{j=0}^{r} \binom{r}{j} t^{r-j} \sum_{i=0}^{j} S(j,i)\,(k-t)_i \ . \tag{17}$$

For any given value of $i \leq r$ we have

$$\sum_{j=i}^{r} \beta_j\,(k-t)_j = \sum_{j=i}^{r} \binom{r}{j} t^{r-j} \sum_{i=0}^{j} S(j,i)\,(k-t)_i$$

or, with $i = r\text{-}f$ ,

$$\sum_{j=r-f}^{r} \beta_j\,(k-t)_j = \sum_{j=r-f}^{r} \binom{r}{j} t^{r-j}\, S(j,r-f)\,(k-t)_{r-f} \ .$$

This leads to the general formula

$$\beta_{r-f}\,(r,t) = \sum_{j=r-f}^{r} \binom{r}{j} t^{r-j}\, S(j,r-f) \ . \tag{18}$$

If the coefficients $\alpha_j(r,t)$ are known, the new coefficients $\beta_j(r,t)$, as a result of (12), can be obtained numerically by forming the sum

$$\beta_{r-f}\,(r,t) = \sum_{j=r-f}^{r} \alpha_j\,(r,t)\, S(j,r-f) \ . \tag{19}$$

For the lowest values of f this leads to the following explicit expressions

- for $\underline{f = 0}$: $\beta_r \;=\; \alpha_r\, S(r,r)$

- for $\underline{f = 1}$: $\beta_{r-1} \;=\; \alpha_{r-1}\, S(r-1,r-1) + \alpha_r\, S(r,r-1)$ ,

- for $\underline{f = 2}$: $\beta_{r-2} \;=\; \alpha_{r-2}\, S(r-2,r-2) + \alpha_{r-1}\, S(r-1,r-2) + \alpha_r\, S(r,r-2)$ ,

- for $\underline{f = 3}$: $\beta_{r-3} \;=\; \alpha_{r-3}\, S(r-3,r-3) + \alpha_{r-2}\, S(r-2,r-3) + \alpha_{r-1}\, S(r-1,r-3) + \alpha_r\, S(r,r-3)$ , (20)

etc.

While (19) is no doubt the simplest general form, its use requires knowledge of the Stirling numbers. The explicit expressions given above suggest that the numbers needed, which are of the form S(n,n-f), with f = 0, 1, 2, ... , could be written in another form. This is indeed the case as they can be replaced by sums over binomial coefficients. According to [4] the relations are

$$
\begin{aligned}
S(n,n) &= 1 , \\
S(n,n-1) &= \binom{n}{2} , \\
S(n,n-2) &= 3\binom{n}{4} + \binom{n}{3} , \\
S(n,n-3) &= 15\binom{n}{6} + 10\binom{n}{5} + \binom{n}{4} , \\
S(n,n-4) &= 105\binom{n}{8} + 105\binom{n}{7} + 25\binom{n}{6} + \binom{n}{5} , \\
S(n,n-5) &= 945\binom{n}{10} + 1\,260\binom{n}{9} + 490\binom{n}{8} + 56\binom{n}{7} + \binom{n}{6} ,
\end{aligned}
\tag{21}
$$

etc.

By substituting (12) and (21) into (19) we arrive, after some rearrangement, at the expressions

$$\beta_r = 1 ,$$

$$\beta_{r-1} = \binom{r}{2} + rt ,$$

$$\beta_{r-2} = \frac{1}{4}\binom{r}{3}[3r-5 + 12t] + \binom{r}{2}t^2 ,$$

$$\beta_{r-3} = \frac{1}{2}\binom{r}{4}[r^2-5r+6 + (6r-16)t + 12t^2] + \binom{r}{3}t^3 ,$$

$$\beta_{r-4} = \frac{1}{48}\binom{r}{5}\{15r^3-150r^2+485r-502 + 120[(r^2-7r+12)t + (3r-11)t^2 + 4t^3]\} + \binom{r}{4}t^4 , \quad (22)$$

etc.

As the expressions listed in (21) rapidly become quite complicated, only the first two or three are of practical use; the formulae (18) or (19) are easier to apply.

## 4. A numerical example

There are at least three ways to determine the necessary coefficients $\alpha_j$ and/or $\beta_j$. The first is a direct, successive evaluation for a given application with specific values of r and t. This implies in most cases rather long and quite uninteresting calculations, as is evident from the numerical example given below.

Use of the formulae (12) and (18) allows us to do this with less effort. It is still simpler, however, just to look up the relevant coefficients in a table, and this is why all the coefficients $\alpha_j$ and $\beta_j$ have been assembled in tabular form for values r and t up to 5.

To show how tedious the direct evaluation is, we consider the simple case with r = 3 and t = 2: we want to determine the coefficients $\beta_j$ which appear in the decomposition

$$k^3 = \beta_3(k-2)_3 + \beta_2(k-2)_2 + \beta_1(k-2)_1 + \beta_0(k-2)_0.$$

Since $(k-2)_3 = (k-2)\,(k-3)\,(k-4) = k^3 - 9k^2 + 26k - 24$, we obtain, as expected, $\beta_3 = 1$.

From $(k-2)_2 = (k-2)\,(k-3) = k^2 - 5k + 6$, and by assembling the terms proportional to $k^2$, we have

$$-9k^2 + \beta_2 k^2 = 0, \quad \text{thus } \beta_2 = 9.$$

Likewise, we find for the terms with k

$$26k - 9 \times 5k + \beta_1 k = 0, \quad \text{hence } \beta_1 = 19.$$

Finally, since $(k-2)_0 = 1$, we obtain for the constant terms the condition

$$-24 + 9 \times 6 - 19 \times 2 + \beta_0 1 = 0, \quad \text{thus } \beta_0 = 8.$$

With the tables available in the next section, all the values for $\beta_j(3,2)$ can be obtained by a glance at the appropriate line. It is obvious that for $r > 3$ the chain of conclusions illustrated by the above example would not only become longer, but also more prone to error. It is the purpose of this report to show how this danger can be avoided.

## 5. **Tabulation of the coefficients**

In order to make the decomposition of the powers $k^r$ to one of the shifted forms as easy as possible, that is, to allow

$$k^r = \sum_{j=0}^{r} \alpha_j(r,t)\ (k-t)^j$$

$$\text{or} \qquad k^r = \sum_{j=0}^{r} \beta_j(r,t)\ (k-t)_j \tag{23}$$

to be expressed in terms of the coefficients $\alpha_j(r,t)$ and $\beta_j(r,t)$, values for the coefficients are listed below in tabular form for integer values of r and t not exceeding 5.

The tabulation shows that the coefficients can be foreseen easily only for $t = 1$, when

$$\alpha_j(r,1) = \binom{r}{j}$$

$$\text{and} \qquad \beta_j(r,1) = S(r+1,j+1)\ . \tag{24}$$

Obviously, and for any value of t, we also have

$$\text{- for } j = 0 : \quad \alpha_0(r,t) = \beta_0(r,t) = t^r\ ,$$

$$\text{- for } j = r : \quad \alpha_r(r,t) = \beta_r(r,t) = 1\ . \tag{25}$$

a) The coefficients $\alpha_j(r,t)$

- for t = 1:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 1 | 1 | | | | |
| 2 | 1 | 2 | 1 | | | |
| 3 | 1 | 3 | 3 | 1 | | |
| 4 | 1 | 4 | 6 | 4 | 1 | |
| 5 | 1 | 5 | 10 | 10 | 5 | 1 |

- for t = 2:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 2 | 1 | | | | |
| 2 | 4 | 4 | 1 | | | |
| 3 | 8 | 12 | 6 | 1 | | |
| 4 | 16 | 32 | 24 | 8 | 1 | |
| 5 | 32 | 80 | 80 | 40 | 10 | 1 |

- for t = 3:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 3 | 1 | | | | |
| 2 | 9 | 6 | 1 | | | |
| 3 | 27 | 27 | 9 | 1 | | |
| 4 | 81 | 108 | 54 | 12 | 1 | |
| 5 | 243 | 405 | 270 | 90 | 15 | 1 |

- for t = 4:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 4 | 1 | | | | |
| 2 | 16 | 8 | 1 | | | |
| 3 | 64 | 48 | 12 | 1 | | |
| 4 | 256 | 256 | 96 | 16 | 1 | |
| 5 | 1 024 | 1 280 | 640 | 160 | 20 | 1 |

- for t = 5:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 5 | 1 | | | | |
| 2 | 25 | 10 | 1 | | | |
| 3 | 125 | 75 | 15 | 1 | | |
| 4 | 625 | 500 | 150 | 20 | 1 | |
| 5 | 3 125 | 3 125 | 1 250 | 250 | 25 | 1 |

b) The coefficients $\beta_j(r,t)$

- for t = 1:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 1 | 1 | | | | |
| 2 | 1 | 3 | 1 | | | |
| 3 | 1 | 7 | 6 | 1 | | |
| 4 | 1 | 15 | 25 | 10 | 1 | |
| 5 | 1 | 31 | 90 | 65 | 15 | 1 |

- for t = 2:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 2 | 1 | | | | |
| 2 | 4 | 5 | 1 | | | |
| 3 | 8 | 19 | 9 | 1 | | |
| 4 | 16 | 65 | 55 | 14 | 1 | |
| 5 | 32 | 211 | 285 | 125 | 20 | 1 |

- for t = 3:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 3 | 1 | | | | |
| 2 | 9 | 7 | 1 | | | |
| 3 | 27 | 37 | 12 | 1 | | |
| 4 | 81 | 175 | 97 | 18 | 1 | |
| 5 | 243 | 781 | 660 | 205 | 25 | 1 |

- for t = 4:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 4 | 1 | | | | |
| 2 | 16 | 9 | 1 | | | |
| 3 | 64 | 61 | 15 | 1 | | |
| 4 | 256 | 369 | 151 | 22 | 1 | |
| 5 | 1 024 | 2 101 | 1 275 | 305 | 30 | 1 |

- for t = 5:

| | j=0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| r=1 | 5 | 1 | | | | |
| 2 | 25 | 11 | 1 | | | |
| 3 | 125 | 91 | 18 | 1 | | |
| 4 | 625 | 671 | 217 | 26 | 1 | |
| 5 | 3 125 | 4 651 | 2 190 | 425 | 35 | 1 |

**References**


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[3] J. Riordan: "Combinatorial Identities" (Wiley, New York, 1968), p. 44

[4] C. Jordan: " Calculus of Finite Differences" (Chelsea, New York, 1965), p. 171 ff.


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