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Alternate moments and parity moments

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Abstract

The recently developed parity method suggests that it may be helpful to define two new concepts, called alternate moments and parity moments. These quantities may then be applied to the cases of Poissonian and binomial variates, enabling some new relations to be derived.

1. General considerations

The so-called parity method, a novel way of measuring the rate of true coincidences, with interesting applications in the absolute determination of activities, is based on a particular approach to counting statistics. The normal counting of random events simply implies a measurement of the number k of events (normally pulses) which occur in a given time interval t. In the parity approach, the quantity we are interested in is not k, or the probability P_k of registering exactly k events in t, for some well-specified experimental conditions. Rather, it concerns a seemingly minor feature of the registered numbers k, namely their property of being even or odd. Thus, the parity function, which is characteristic of a particular measurement, is defined by the quantity

$$\Pi = \text{prob (k=odd)} = \sum_{j=0}^{\infty} P_{2j+1} \,. \tag{1}$$

Whereas for strictly periodic phenomena Π can take any value between 0 and 1, it is normally restricted to the region from 0 to ½ for events occurring "at random". The two cases can be exemplified by pulses from an oscillator or from the decay of a radioactive source.

A measurement of Π is of interest only if its value can be related through some theoretical expectation to a quantity of physical interest. It happens, however, that the direct evaluation of Π from a given probability distribution P_k is virtually always a complicated matter, especially for cases of practical interest. The question, therefore, arises whether the tedious algebra could be simplified by tackling the problem in a different way. One possibility is to examine appropriate moments of k rather than its probability distribution. The idea may merit detailed study, but it is necessary first to establish the relations involved since the utility of the procedure will depend on their algebraic form.

It is possible that such relations, even if they are of limited help for the evaluation of Π, may be interesting in their own right, especially when applied to specific cases. In this way, we may be able to find new relations involving binomial coefficients.

2. Alternate moments

It turns out to be useful to consider first, for a discrete random variable k governed by the probability of occurrence P_k, the alternate sum defined by

$$m^{\pm} = \sum_{k=0}^{\infty} (-1)^k P_k . \tag{2}$$

This can readily be generalized to an alternate, ordinary moment of order r, written as

$$m_r^{\pm} = \sum_{k=0}^{\infty} (-1)^k k^r P_k , \quad \text{for } r = 0, 1, 2, \ldots \tag{3}$$

Apart from the alternating signs, this corresponds to the usual moment of order r. Among the traditional moments, it is well known that the factorial moment of order r, defined by

$$(m)_r = \sum_{k=0}^{\infty} (k)_r P_k , \tag{4}$$

where $(k)_r = k\,(k-1)\,(k-2) \ldots (k-r+1)$ is a falling factorial, plays a prominent role for discrete distributions.

The notion of a factorial can be combined with (3) to yield an alternate factorial moment of order r

$$(m^{\pm})_r = \sum_{k=0}^{\infty} (-1)^k (k)_r P_k . \tag{5}$$

Powers and factorials are linked by the well-known expression

$$k^r = \sum_{j=0}^{r} S(r,j)\,(k)_j , \tag{6}$$

where S are Stirling numbers of the second kind, with the convention $S(r,0) = \delta_{r,0}$.

This allows us to arrive at the following relation between alternate ordinary moments and factorial moments

$$m_r^{\pm} = \sum_{j=0}^{r} S(r,j)\,(m^{\pm})_r . \tag{7}$$

The reason for often preferring factorial moments to ordinary ones is that they usually lead to simpler expressions, as will be seen later.

3. Parity moments

According to (1), the parity function Π is given by

$$\Pi = \sum_{j=0}^{\infty} P_{2j+1} .$$

We can thus define parity moments of order r by

$$\Pi_r = \sum_{j=0}^{\infty} (2j+1)^r P_{2j+1} , \tag{8}$$

and likewise factorial parity moments

$$(\Pi)_r = \sum_{j=0}^{\infty} (2j+1)_r P_{2j+1} . \tag{9}$$

The main question now is whether the quantity Π, which is of direct experimental importance, or perhaps even its generalization $(\Pi)_r$, can be expressed in terms of moments like those mentioned above. If such a relation exists, there still remains the problem of evaluating the moments for a situation involving practical measurements.

4. Establishment of the link

The required relation can readily be found if we look at the development of the two moments considered before in (4) and (5), namely

$$(m)_r = (0)_r P_0 + (1)_r P_1 + (2)_r P_2 + (3)_r P_3 + \dots ,$$

$$(m^{\pm})_r = (0)_r P_0 - (1)_r P_1 + (2)_r P_2 - (3)_r P_3 \pm \dots ,$$

and form the difference

$$(m)_r - (m^{\pm})_r = 0 + 2\,(1)_r P_r + 0 + 2\,(3)_r P_3 + \dots .$$

This is seen to be just twice the factorial parity moment $(\Pi)_r$; therefore

$$\begin{aligned} (\Pi)_r &= \sum_{j=0}^{\infty} (2j+1)_r P_{2j+1} \\ &= \frac{1}{2} \left[\sum_{k=0}^{\infty} (k)_r P_k - \sum_{k=0}^{\infty} (-1)^k (k)_r P_k \right] \\ &= \frac{1}{2} \left[(m)_r - (m^{\pm})_r \right] . \end{aligned} \tag{10}$$

This is the general relation of the form sought, with the parity a simple function of the moments of the measured variable k.

The ordinary parity moments defined in (8) are readily obtained from (10) by means of (6), as noted before for the alternate moments. Indeed

$$\Pi_r = \sum_{j=0}^{r} S(r,j)\,(\Pi)_j = \frac{1}{2}\sum_{j=0}^{r} S(r,j)\left[(m)_j - (m^{\pm})_j\right]. \tag{11}$$

Once again, the factorial form (10) of the moment turns out to be simpler than the ordinary one (11).

5. Applications

Let us apply (10) to two of the most important discrete distributions, namely the Poisson law and the binomial distribution. The first case can serve as a check, while the second will lead to some new results.

a) *Poisson distribution*

For a variable k which follows the Poisson law, i.e. for which

$$P_k = P(\mu,k) \equiv \frac{\mu^k}{k!}e^{-\mu}, \tag{12}$$

we already know that the factorial moment is

$$(m)_r = \sum_{k=0}^{\infty} (k)_r\, P(\mu,k) = \mu^r. \tag{13}$$

For the alternate factorial moment we find that

$$(m^{\pm})_r = \sum_{k=0}^{\infty} (-1)^k (k)_r\, P(\mu,k) = e^{-\mu}\sum_{k=r}^{\infty}\frac{(-\mu)^k}{(k-r)!}$$

$$= e^{-\mu}(-\mu)^r\sum_{j=0}^{\infty}\frac{(-\mu)^j}{j!} = (-\mu)^r e^{-2\mu}. \tag{14}$$

Hence, we now obtain from (10)

$$(\Pi)_r = \frac{1}{2}\left[\mu^r - (-\mu)^r e^{-2\mu}\right] = \frac{\mu^r}{2}\left[1 - (-1)^r e^{-2\mu}\right], \tag{15}$$

which is in agreement with the result given in [1].

By substituting (14) into (7) we can also find the alternate ordinary Poisson moment

$$m_r^{\pm} = \sum_{k=0}^{\infty} (-1)^k k^r\, P(\mu,k) = e^{-2\mu}\sum_{j=0}^{r} S(r,j)\,(-\mu)^j. \tag{16}$$

b) *Binomial distribution*

The binomial distribution is an interesting case which yields some new, and occasionally rather unexpected, relations.

The probability distribution is now

$$P_k = B(n,k) \equiv \binom{n}{k} p^k q^{n-k}, \tag{17}$$

with $n \geq k$ and $q = 1 - p$.

Whereas the ordinary moments are known to become rather involved for r beyond 2 (for exact formulae see [2] or [3]), the factorial moments are given by the simple expression [2]

$$(m)_r = \sum_{k=0}^{n} (k)_r \, B(n,k) = (n)_r \, p^r . \tag{18}$$

For the alternate moments no general expression seems to be known. It can, however, be found in the following way. From the definition

$$(m^{\pm})_r = \sum_{k=0}^{n} (-1)^k (k)_r \, B(n,k) \tag{19}$$

we obtain, as $(k)_r = k!/(k-r)!$,

$$(m^{\pm})_r = \left(\frac{-p}{q}\right)^r \sum_{k=r}^{n} \frac{n!}{(n-k)!\,(k-r)!} (-p)^{k-r} q^{n-k+r}$$

$$= \left(\frac{-p}{q}\right)^r \sum_{j=0}^{n-r} \binom{n-r}{j} (n)_r (-p)^j q^{n-j} .$$

Since, according to the binomial theorem,

$$(-p + q)^{n-r} = \sum_{j=0}^{n-r} \binom{n-r}{j} (-p)^j q^{n-r-j} ,$$

this allows us, with $- p + q = 1 - 2p$, to bring the expression into the simple form

$$(m^{\pm})_r = (n)_r (-p)^r (1-2p)^{n-r} . \tag{20}$$

This is the general formula for alternate factorial moments of a binomial variate.

For $p = ½$, (19) and (20) lead to an interesting relation involving binomial coefficients. In particular we obtain, for $n > r$,

$$\sum_{k=0}^{n} (-1)^k (k)_r \binom{n}{k} = 0 . \tag{21}$$

By virtue of (6) we also have

$$\sum_{k=0}^{n} (-1)^k k^r \binom{n}{k} = \sum_{j=0}^{1} S(r,j) \sum_{k=0}^{n} (-1)^k (k)_j \binom{n}{k} = 0 . \tag{22}$$

Both (21) and (22) generalize formulae given in [4] for $r = 0$ or 1.

The parity moments, for which the general relation is given by (10), can now also be obtained for a binomial variate. From relations (18) and (20) we find that

$$\begin{aligned}(\Pi)_r &= \frac{1}{2}\left[(n)_r\, p^r - (n)_r\, (-p)^r\, (1-2p)^{n-r} \right] \\ &= \frac{1}{2}\, (n)_r\, p^r \left[1 - (-1)^r\, (1-2p)^{n-r} \right] .\end{aligned} \tag{23}$$

For the special case $p = ½$ this leads to the relation

$$\sum_{k=0}^{[n/2]} (2k+1)_r \binom{n}{2k+1} = (n)_r\, 2^{n-r-1} . \tag{24}$$

With the help of (6) we also arrive at the formula

$$\sum_{k=0}^{[n/2]} (2k+1)^r \binom{n}{2k+1} = \sum_{j=0}^{r} S(r,j) \sum_{k=0}^{[n/2]} (2k+1)_j \binom{n}{2k+1} = \sum_{j=0}^{r} S(r,j)\, (n)_j\, 2^{n-j-1} . \tag{25}$$

The general relations (24) and (25), valid for $n > r$, are new; the listing in [4] gives only the case for $r = 0$, i.e.

$$\sum_{k} \binom{n}{2k+1} = 2^{n-1} .$$

References

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